(Almost)-crystallographic quotients of Artin and surface braid groups and their finite subgroups

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- [GGO1]: Gonçalves, G., Ocampo, A quotient of the Artin braid groups related to crystallographic groups, *J. Algebra* (2017) **474**, 393–423.
- [GGO2]: Gonçalves, G., Ocampo, Almost-crystallographic groups as quotients of Artin braid groups, *J. Algebra* (2019) **524**, 160–186.
- [GGO3]: Gonçalves, G., Ocampo, Embeddings of finite groups in $B_n/\Gamma_k(P_n)$ for $k = 2, 3$, *Ann. Inst. Fourier*, to appear.
- [GGOP]: Gonçalves, G., Ocampo, Pereiro, Crystallographic groups and flat manifolds from surface braid groups, preprint, March 2020.

I. Generalities: braid groups, lower central series...

II. The quotients $B_n/\Gamma_k(P_n)$, $n \geq 3$, $k \in \mathbb{N}$
   - $B_n/\Gamma_k(P_n)$ is (almost) crystallographic
   - Torsion and finite subgroups of $B_n/\Gamma_k(P_n)$, $k \in \{2, 3\}$

III. Recent results for surface braid groups

If M is a (connected) surface and $n \in \mathbb{N}$:

- $F_n(M) = \{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for all } i \neq j\}$, the n^{th} configuration space of M, and S_n acts freely on $F_n(M)$ by permuting coordinates.
- $P_n(M) = \pi_1(F_n(M))$, the n-string pure braid group $P_n(M)$ of M, and $B_n(M) = \pi_1(F_n(M)/S_n)$, the n-string (full) braid group $B_n(M)$ of M (Fox & Neuwirth).
- There is a short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \xrightarrow{\sigma} S_n \longrightarrow 1. \tag{1}$$

- If M is the 2-disc, $B_n(M) = B_n$, $P_n(M) = P_n$, the Artin braid and pure groups.

  $$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,\ \ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \right\rangle$$

  $\sigma\colon B_n \longrightarrow S_n$ is defined by $\sigma(\sigma_i) = (i, i+1)$.

- B_n and P_n are interesting groups with lots of good properties.
- Some quotients of B_n are finite: $B_n/P_n \cong S_n$; $B_n/\langle\langle \sigma_1^m \rangle\rangle$ is finite iff $(m,n) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}$ (Coxeter)

Aim: study some lower central series quotients of B_n.

- If G is a group, $\{\Gamma_i(G)\}_{i\geq 1}$ is its lower central series, where:

$$\Gamma_1(G) = G \text{ and } \Gamma_{i+1}(G) = [\Gamma_i(G), G] \text{ for all } i \geq 1.$$

  We have $\Gamma_{i+1}(G) \lhd \Gamma_i(G)$ and $\Gamma_i(G) \lhd G$.
- $B_n/\Gamma_2(B_n) \cong \mathbb{Z}$, and $\Gamma_2(B_n) = \Gamma_k(B_n)$ for all $k \geq 3$, so the series $\{B_n/\Gamma_k(B_n)\}_{k\in\mathbb{N}}$ isn't interesting.

- $\{B_n/\Gamma_k(P_n)\}_{m\in\mathbb{N}}$ is more interesting: if $m=1$, $B_n/P_n \cong S_n$.
- P_n is residually nilpotent (Falk-Randell, Kohno), so:
$$\bigcap_{k\in\mathbb{N}} \Gamma_k(P_n) = \{1\}, \text{ and } \lim_{k\longrightarrow\infty} B_n/\Gamma_k(P_n) = B_n.$$
- We have $p_{k+1}\colon B_n/\Gamma_{k+1}(P_n) \longrightarrow B_n/\Gamma_k(P_n)$, where:
$$\text{Ker}(p_{k+1}) = \Gamma_k(P_n)/\Gamma_{k+1}(P_n) \cong \mathbb{Z}^{r_{k,n}},\ r_{k,n}\in\mathbb{N} \text{ (Falk-Randell, Kohno)}$$
- If $m<n$, $B_m \hookrightarrow B_n$ induces $B_m/\Gamma_k(P_m) \hookrightarrow B_n/\Gamma_k(P_n)$.
- $p_{k+1}\big|_{\text{torsion elements of } B_n/\Gamma_{k+1}(P_n)}$ is thus injective, and:
$$B_n \longrightarrow\!\!\!\!\!\rightarrow \cdots \xrightarrow{p_{k+2}}\!\!\!\!\!\rightarrow B_n/\Gamma_{k+1}(P_n) \xrightarrow{p_{k+1}}\!\!\!\!\!\rightarrow B_n/\Gamma_k(P_n) \xrightarrow{p_k}\!\!\!\!\!\rightarrow \cdots \xrightarrow{p_3}\!\!\!\!\!\rightarrow B_n/\Gamma_2(P_n) \xrightarrow{p_2}\!\!\!\!\!\rightarrow S_n,$$
where $\text{Ker}(p_i)$ is free Abelian of finite rank.
- Questions: what happens to $B_n/\Gamma_k(P_n)$ between S_n ($k=1$) and B_n ($\lim_{k\longrightarrow\infty} B_n/\Gamma_k(P_n)$)? When does the torsion disappear? What finite subgroups are realised?

- If $n \geq 1$, $P_n = \langle A_{i,j},\ 1 \leq i < j \leq n \rangle$, $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$, subject to the relations (Artin's presentation):

$$A_{r,s} A_{i,j} A_{r,s}^{-1} = \begin{cases} A_{i,j} & \text{if } r < s < i < j \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if } r < i = s < j \\ A_{i,j} A_{s,j} A_{i,j} (A_{i,j} A_{s,j})^{-1} & \text{if } r = i < s < j \\ [A_{r,j}, A_{s,j}] A_{i,j} [A_{r,j}, A_{s,j}]^{-1} & \text{if } r < i < s < j. \end{cases}$$

- By convention, if $i < j$, $A_{j,i} = A_{i,j}$.
- $P_n^{\mathrm{Ab}} = P_n / \Gamma_2(P_n) = \langle A_{i,j},\ 1 \leq i < j \leq n \rangle \cong \mathbb{Z}^{n(n-1)/2}$.
- S_n acts by conjugation on $P_n^{\mathrm{Ab}} \cong \mathbb{Z}^{n(n-1)/2}$ via the action of B_n on P_n.

$$1 \longrightarrow P_n / \Gamma_2(P_n) \longrightarrow B_n / \Gamma_2(P_n) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1.$$

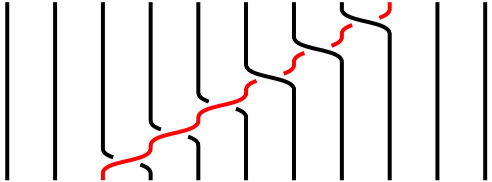
Lemma 1 (GGO1)

If $\alpha \in B_n / \Gamma_2(P_n)$, $\alpha A_{i,j} \alpha^{-1} = A_{(\overline{\sigma}(\alpha))^{-1}(i), (\overline{\sigma}(\alpha))^{-1}(j)}$ in $P_n / \Gamma_2(P_n)$.

If $k, n \geq 3$, k odd, $r + k \leq n$ and $r \geq 0$, define $\delta_{r,k}, \alpha_{r,k} \in B_n/\Gamma_2(P_n)$:

$$\delta_{r,k} = \sigma_{r+k-1} \cdots \sigma_{r+\frac{k+1}{2}} \sigma^{-1}_{r+\frac{k-1}{2}} \cdots \sigma^{-1}_{r+1} \text{ and } \alpha_{r,k} = \sigma_{r+1} \cdots \sigma_{r+k-1}.$$

$\delta_{2,7} =$



$\in B_{11}/\Gamma_2(P_{11})$

Lemma 2

Let $n, k \geq 3$ and $r \geq 0$ be integers such that k is odd and $r + k \leq n$. Then $\delta_{r,k}$ is of order k in $B_n/\Gamma_2(P_n)$, and satisfies:

$$\delta_{r,k} = \left(A_{r+\frac{k+1}{2},r+k} A_{r+\frac{k+3}{2},r+k} \cdots A_{r+k-1,r+k}\right) \alpha^{-1}_{r,k}. \tag{2}$$

Theorem 3 (GGO1,GGO2)

(a) $B_n/\Gamma_2(P_n)$ *(resp.* $B_n/\Gamma_3(P_n)$*) has no element of order* 2 *(resp.* 2 *or* 3*).*

(b) Let $k \in \{2,3\}$, $r \in \mathbb{N}$ *be such that* $\gcd(r, k!) = 1$. *Then* $B_n/\Gamma_k(P_n)$ *has elements of order* r *iff* S_n *does.*

Proof in the case $k = 2$.

(a) Let $\beta \in B_n/\Gamma_2(P_n)$ be of order 2.

Ker($\overline{\sigma}$) torsion free $\Longrightarrow \overline{\sigma}(\beta) = (1,2)\cdots(l,l+1)$, l odd (up to conjugation), and:

$$\beta = \prod_{1\leq i<j\leq n} A_{i,j}^{m_{i,j}} \sigma_1\sigma_3\cdots\sigma_l, \text{ where } m_{i,j} \in \mathbb{Z}.$$

By Lemma 1,

$$\begin{aligned} 1 = \beta^2 &= \left(\prod_{1\leq i<j\leq n} A_{i,j}^{m_{i,j}} \sigma_1\sigma_3\cdots\sigma_l\right)\left(\prod_{1\leq i<j\leq n} A_{i,j}^{m_{i,j}} \sigma_1\sigma_3\cdots\sigma_l\right) \\ &= \left(\prod_{1\leq i<j\leq n} A_{i,j}^{m_{i,j}}\right)\left(\prod_{1\leq i<j\leq n} A_{i,j}^{m_{\sigma(\beta)(i),\sigma(\beta)(j)}}\right) A_{1,2}A_{3,4}\cdots A_{l,l+1}, \end{aligned}$$

The coeff. of $A_{1,2}$ on the rhs is equal to $2m_{1,2}+1$ – contradiction.

Theorem 3 (GGO1,GGO2)

(b) Let $k \in \{2,3\}$, $r \in \mathbb{N}$ be such that $\gcd(r, k!) = 1$. Then $B_n/\Gamma_k(P_n)$ has elements of order r iff S_n does.

Proof in the case $k = 2$.

(b) If $\beta \in B_n/\Gamma_2(P_n)$ is of order r, then so is $\overline{\sigma}(\beta) \in S_n$. Conversely, let $\tau \in S_n$ be of odd order r. Up to conjugation:

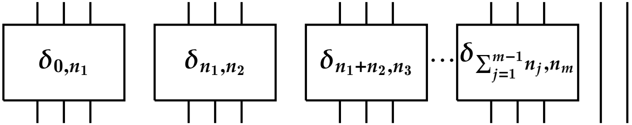
$$\tau = (1, \ldots, n_1)(n_1+1, \ldots, n_1+n_2) \cdots \left(\sum_{i=1}^{m-1} n_i + 1, \ldots, \sum_{i=1}^{m} n_i\right),$$

where $r = \mathrm{lcm}(n_1, \ldots, n_m)$, $n_1, \ldots, n_m$ odd. By (2),

$$\rho = \delta_{0,n_1}\delta_{n_1,n_2}\delta_{n_1+n_2,n_3} \cdots \delta_{\sum_{j=1}^{m-1} n_j, n_m} \in B_n/\Gamma_2(P_n)$$

is of order r (and $\overline{\sigma}(\rho) = \tau$).

$\rho =$



□

Let G be a connected, simply-connected Lie group, C be a maximal, compact subgroup of $\mathrm{Aut}(G)$, and H be a cocompact, discrete subgroup of $G \rtimes C \subset G \rtimes \mathrm{Aut}(G) = \mathrm{Aff}(G)$.

- G Abelian $\Longrightarrow G = \mathbb{R}^n$ for some n, and H is crystallographic.
- G nilpotent $\Longrightarrow H$ is almost crystallographic.
- crystallographic $\Longrightarrow$ almost crystallographic.
- (almost) crystallographic + torsion free = (almost) Bieberbach.

Crystallographic groups are classical objects.

- They are used in the study of the groups of isometries of Euclidean spaces and in crystallography.
- Bieberbach (resp. almost-Bieberbach) groups are precisely the fundamental groups of compact, flat (resp. almost-flat) Riemannian manifolds.
- for fixed n, there are a finite number of crystallographic groups up to conjugacy in $\mathrm{Aff}(\mathbb{R}^n)$.

Proposition 4 (Algebraic characterisation)

A group G is crystallographic iff there exist $n \in \mathbb{N}$, a finite group H (the holonomy group of G) and a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \longrightarrow H \longrightarrow 1 \quad (3)$$

such that the representation $\Theta \colon H \longrightarrow \mathrm{Aut}(\mathbb{Z}^n)$ induced by conjugacy is faithful.

Auslander & Kuranishi (1957): every finite group is the holonomy group of some Bieberbach group.

Theorem 5 (Dekimpe & Igodt)

If G is polycyclic-by-finite, it is almost-crystallographic iff it has a finite-index, nilpotent subgroup and it has no non-trivial finite normal subgroups.

Theorem 6 (GGO1,GGO2)

(a) $B_n/\Gamma_2(P_n)$ *is crystallographic.*

(b) For all $k \geq 3$, $B_n/\Gamma_k(P_n)$ *is almost crystallographic.*

Proof.

By (1), we have a short exact sequence:

$$1 \longrightarrow P_n/\Gamma_k(P_n) \longrightarrow B_n/\Gamma_k(P_n) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1. \tag{4}$$

(a) If $k = 2$, (4) is of the form of the exact sequence (3).

By Lemma 1, the induced action $\phi\colon S_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{n(n-1)/2})$ is injective, so the criterion of Proposition 4 is satisfied.

(b) We use Theorem 5. $P_n/\Gamma_k(P_n)$ is nilpotent (nilpotency class $k-1$) of finite index, torsion free and polycyclic.

Let H be a finite normal subgroup of $B_n/\Gamma_k(P_n)$. The restriction to the torsion of $B_n/\Gamma_l(P_n) \longrightarrow\!\!\!\!\!\rightarrow B_n/\Gamma_{l-1}(P_n)$ is injective for all $l \geq 2$.

So S_n has a normal subgroup $\widetilde{H}$, $\widetilde{H} \cong H$.

Since $B_n/\Gamma_k(P_n)$ has no element of order 2 or 3, H must be trivial. □

- Up to isomorphism, the finite Abelian subgroups of $B_n/\Gamma_2(P_n)$ are the Abelian subgroups of S_n of odd order (proof of Theorem 3).
- The situation in the non-Abelian case is more complicated. In [GGO1], we gave an explicit embedding of the Frobenius group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ in $B_7/\Gamma_2(P_7)$, which is the smallest non-Abelian group of odd order.
- Bieberbach subgroups of $B_n/\Gamma_2(P_n)$ may be constructed by taking the inverse image by $\overline{\sigma}$ of 2-subgroups of S_n.
- Using different techniques, I. Marin generalised the results of [GGO1] to generalised braid groups associated to arbitrary complex reflection groups (Crystallographic groups and flat manifolds from complex reflection groups, *Geom. Dedicata* **182** (2016), 233–247).

The group $P_n/\Gamma_k(P_n)$ is torsion free. By (4), we have:

$$1 \longrightarrow P_n/\Gamma_k(P_n) \longrightarrow B_n/\Gamma_k(P_n) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1.$$

If $H < B_n/\Gamma_k(P_n)$ is finite, then $\overline{\sigma}(H) \cong H$ is a subgroup of S_n.

Questions: let H be a finite group.

- Does H embed in $B_n/\Gamma_k(P_n)$ for some $n \geq 3$ and $k \geq 2$?
- If yes, what is the smallest value of n? And the largest value of k (for n fixed)?

We define:

$$l_k(H) = \min\left\{r \in \mathbb{N} \,\middle|\, H \text{ embeds in } B_r/\Gamma_k(P_r)\right\}$$
$$m(H) = \min\left\{r \in \mathbb{N} \,\middle|\, H \text{ embeds in } S_r\right\}$$

$l_k(H)$ may not exist (e.g. if $|H|$ even and $k \geq 2$). If it exists, $m(H) \leq l_k(H)$.

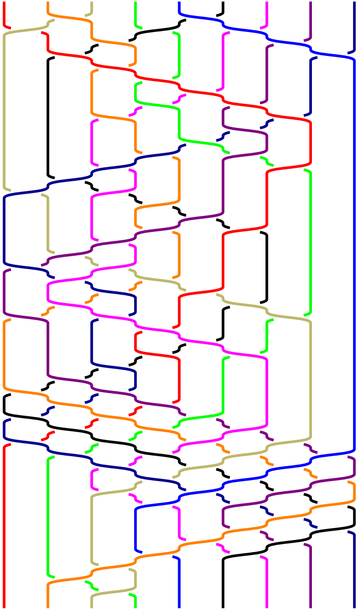
Theorem 7 (GGO3, Cayley-type theorem)

Let H be a finite group and $k \in \{2,3\}$. Then H embeds in $B_{|H|}/\Gamma_k(P_{|H|})$ if and only if $\gcd(|H|, k!) = 1$. Thus:

$$m(H) \leq l_k(H) \leq |H|.$$

By Theorem 7, $m(H) \leq l_k(H) \leq |H|$ if $\gcd(|H|, k!) = 1$ and $k \in \{2, 3\}$.

- The inequality $l_k(H) \leq |H|$ is often strict (e.g. $H = \mathbb{Z}_3 \times \mathbb{Z}_5$).
- If $H = \mathbb{Z}_p$, $p \geq 5$ prime, then $m(H) = l_k(H) = |H|$.
- If $|H|$ is odd then V. Beck and I. Marin proved that $m(H) = l_2(H)$, and if $k = 2$, they obtained other generalisations in the setting of real reflection groups (Torsion subgroups of quasi-abelianized braid groups, *J. Algebra*, to appear). Their techniques don't seem to extend to the case $k = 3$.
- If $\gcd(|H|, 6) = 1$, we think that $m(H) = l_3(H)$.
- We have obtained embeddings of certain semi-direct products, as well as the two non-Abelian groups of order 27 in $B_9/\Gamma_2(P_9)$.



Idea of the proof of Theorem 7 in the case $k = 2$.

Let $|H| = m$ be odd, $\widetilde{H}$ be the embedding of H in S_m of Cayley's Theorem.

If $h \in H$, $h \neq e$, h is represented by a fixed-point free permutation.

Consider the short exact sequence:

$$1 \longrightarrow P_m^{\mathrm{Ab}} \longrightarrow \overline{\sigma}^{-1}(\widetilde{H}) \xrightarrow{\overline{\sigma}\big|_{\overline{\sigma}^{-1}(\widetilde{H})}} \widetilde{H} \longrightarrow 1. \tag{5}$$

$\widetilde{H}$ acts on the basis $\mathcal{B} = \{A_{i,j}\}_{i<j}$ of P_m^{Ab} via Lemma 1. If $\widetilde{h} \in \widetilde{H}$ is such that $\widetilde{h}.A_{i,j} = A_{i,j}$ then $\widetilde{h}$ fixes i and j, thus $\widetilde{h}$ is trivial.

So each orbit is of length m, there are $(m-1)/2$ orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{(m-1)/2}$, and $P_m^{\mathrm{Ab}} \cong \oplus_{i=1}^{(m-1)/2} \langle \mathcal{O}_i \rangle$. Each $\mathcal{O}_i$ may be identified with $\widetilde{H}$, and $\langle \mathcal{O}_i \rangle$ with $\mathbb{Z}[\widetilde{H}]$, thus $P_m^{\mathrm{Ab}} \cong \oplus_{i=1}^{(m-1)/2} \mathbb{Z}[\widetilde{H}]$ as $\mathbb{Z}[\widetilde{H}]$-modules.

Standard results from group cohomology: $H^*(\widetilde{H}, \mathbb{Z}[\widetilde{H}]) = 0$ for $* \geq 1$, in particular $H^2(\widetilde{H}, \mathbb{Z}[\widetilde{H}]) = 0$, and (5) splits.

Since $\overline{\sigma}^{-1}(\widetilde{H}) < B_m/\Gamma_2(P_m)$, H embeds in $B_m/\Gamma_k(P_m)$. □

- The basic ideas for $k = 3$ are similar to those for $k = 2$:

$$1 \longrightarrow \Gamma_2(P_m)/\Gamma_3(P_m) \longrightarrow B_m/\Gamma_3(P_m) \longrightarrow B_m/\Gamma_2(P_m) \longrightarrow 1.$$

- $\Gamma_2(P_m)/\Gamma_3(P_m) \cong \mathbb{Z}^{\binom{m}{3}}$, with basis $\{\alpha_{i,j,k}\}_{i<j<k}$, and $\alpha_{i,j,k} = [A_{i,j}, A_{j,k}]$.
- Main difference: S_n acts on $\{\alpha_{i,j,k}, \alpha_{i,j,k}^{-1}\}_{i<j<k}$, not on $\{\alpha_{i,j,k}\}_{i<j<k}$.
- $B_m/\Gamma_3(P_m)$ has no elements of order 3: if $\widetilde{h}.\,\alpha_{i,j,k} \in \{\alpha_{i,j,k}, \alpha_{i,j,k}^{-1}\}$ then $\widetilde{h} = e$.
- The rest of the proof is similar to that of the case $k = 2$, using the orbits of $\{\alpha_{i,j,k}, \alpha_{i,j,k}^{-1}\}_{i<j<k}$.
- Problem for $k \geq 4$: find a basis of $\Gamma_{k-1}(P_n)/\Gamma_k(P_n)$ on which S_n acts (up to inverses), and understand this action.

Theorem 8 (GGOP)

Let M be a compact, connected surface without boundary. Then $B_n(M)/\Gamma_2(P_n(M))$ is crystallographic iff M is orientable and $M \neq \mathbb{S}^2$.

Proof.

If M is orientable, $M \neq \mathbb{S}^2$, then (1) gives rise to:

$$1 \longrightarrow (P_n(M))^{\mathrm{Ab}} \longrightarrow B_n(M)/\Gamma_2(P_n(M)) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1.$$

Now $(P_n(M))^{\mathrm{Ab}} \cong \mathbb{Z}^{2ng}$, with a basis $\{a_{j,r}\}_{1\leq j\leq n,\, 1\leq r\leq 2g}$, and S_n acts on $(P_n(M))^{\mathrm{Ab}}$ via:

$$\sigma_i a_{j,r} \sigma_i^{-1} = a_{\sigma_i(j),r}.$$

Proposition 4 implies that $B_n(M)/\Gamma_2(P_2(M))$ is crystallographic.

If M is non orientable (resp. $M = \mathbb{S}^2$), $(P_n(M))^{\mathrm{Ab}} \cong \mathbb{Z}_2^r \oplus \mathbb{Z}^s$, where $(r,s) = (n, n(g-1))$ (resp. $(r,s) = (1, n(n-3)/2)$). The $\mathbb{Z}_2^r$-factor yields a finite, non-trivial normal subgroup of $B_n(M)/\Gamma_2(P_n(M))$.

By Theorem 5, $B_n(M)/\Gamma_2(P_n(M))$ is not almost crystallographic, so is not crystallographic. □