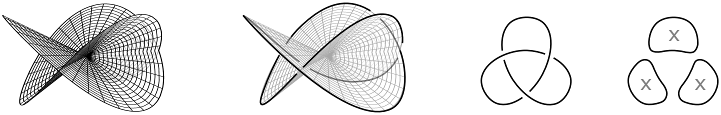
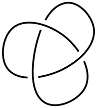
Where are the complex curves in Khovanov homology?



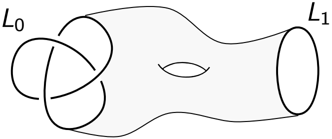
Kyle Hayden (joint with
Isaac Sundberg and Alan Du)

May 5, 2022

What is Khovanov homology?

L  $\rightsquigarrow$ $\mathrm{Kh}(L)$ (bigraded $\mathbb{Z}$-module)

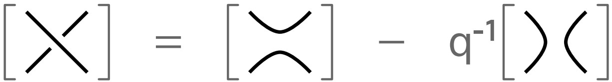
$\Sigma \subset S^3 \times [0,1]$

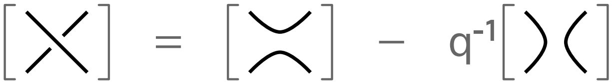


$\rightsquigarrow \mathrm{Kh}(\Sigma) : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$

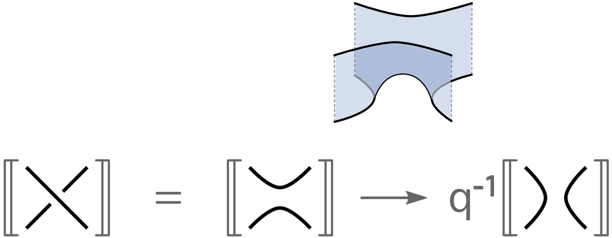
What isn't Khovanov homology?

Unlike gauge-/Floer-theoretic invariants, it doesn't involve geometry!


= − q^{-1}



Categorifies the Kauffman bracket and the Jones polynomial.



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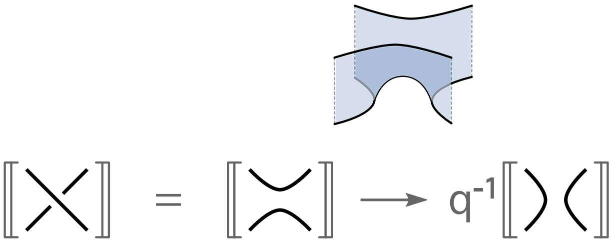


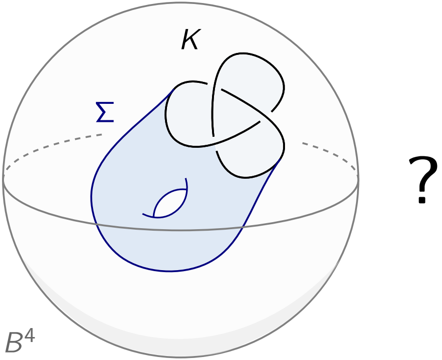
Categorifies the Kauffman bracket and the Jones polynomial.

Credit to David Rose for the schematic: https://icerm.brown.edu/materials/Slides/sp-f19-w2/Webs,_foams,_knot_invariants,_and_representation_theory_]_David_Rose,_University_of_North_Carolina_at_Chapel_Hill.pdf

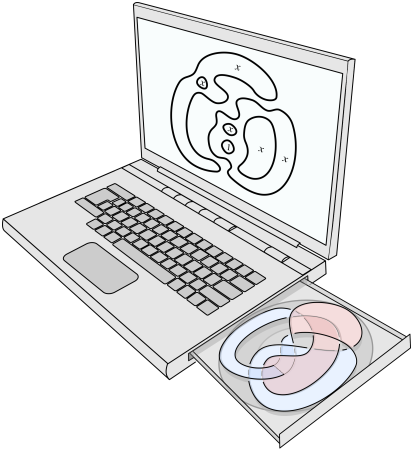
Extended family tree of tools and invariants:

- **Skein modules for 3-manifolds**
  (Turaev '88, Przytycki '91)
- **Khovanov homology for links in thickened surfaces**
  (Asaeda-Przytycki-Sikora '04)
- **Khovanov-Rozansky invariants for 4-manifolds**
  (Morrison-Walker '12, Morrison-Walker-Wedrich '19, Manolescu-Neithalath '20)
- **Slicing obstructions in B^4 and other 4-manifolds**
  (Rasmussen '04, Manolescu-Marengon-Sarkar-Willis '19)

Q1: How can we use the maps $\mathrm{Kh}(\Sigma)$ to study surfaces $\Sigma \subset B^4$?

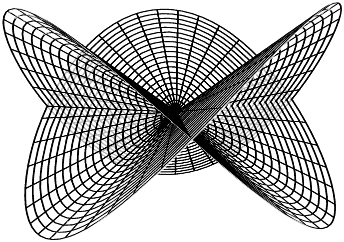


Q2: Can we use computers to study these cobordism maps?



?

Q3: Can we understand these cobordism maps for complex curves?



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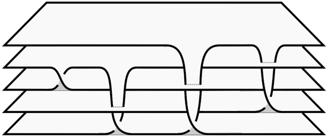
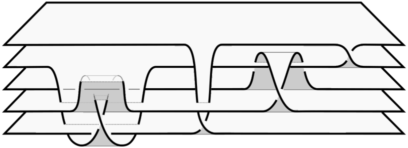
Theorem (H-Sundberg, 2021)

Khovanov homology can distinguish smooth surfaces in B^4 that are isotopic via homeos but not diffeos of B^4 (rel ∂).



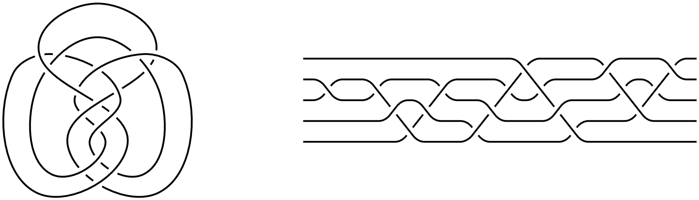


The examples will be holomorphically embedded in $B^4 \subset \mathbb{C}^2$, and we'll use a braid-theoretic perspective to calculate these maps.

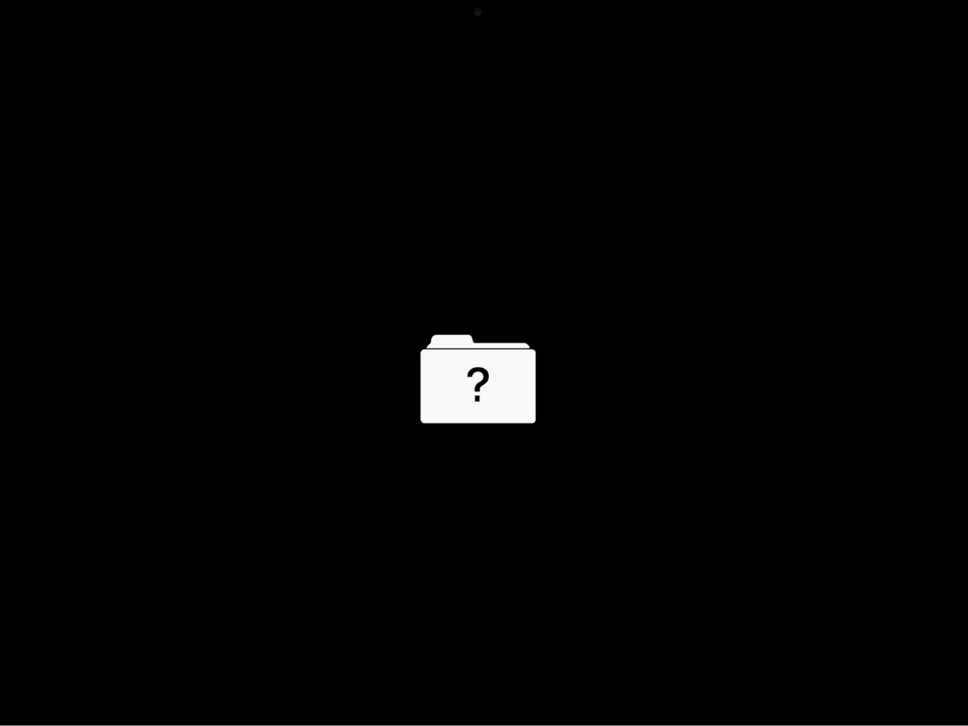
Khovanov homology

Khovanov homology is (relatively) computable.

Example. $J = 17nh_{73}$ (the "positron knot")



Dirk Schuetz's *KnotJob* calculates Kh(J) in <1 sec (on 2013 MacBook).



R.I.P.

August 2013 - April 25, 2022

Performed > 1.6 million KnotJob calculations.

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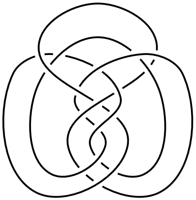

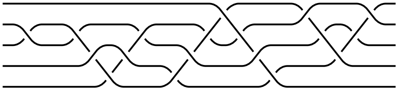

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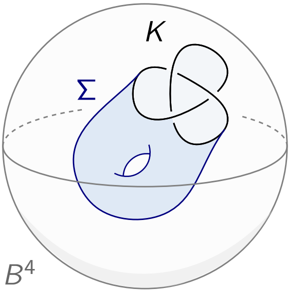
Kh(J)/ Tors

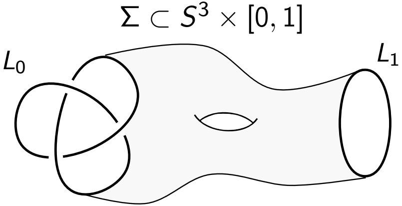
q \ h	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
19														$\mathbb{Z}$
17														
15													$\mathbb{Z}$	
13										$\mathbb{Z}$	$\mathbb{Z}$			
11									$\mathbb{Z}$					
9							$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$				
7							$\mathbb{Z}$	$\mathbb{Z}$						
5					$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$							
3				$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$								
1				$\mathbb{Z}^2$	$\mathbb{Z}$									
-1		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$										
-3														
-5	$\mathbb{Z}$													

How can we use Khovanov homology in dimension 4?

Most 4D applications use Rasmussen's invariant $s(K) \in 2\mathbb{Z}$



$$\Longrightarrow \quad g(\Sigma) \geq |s(K)|/2$$

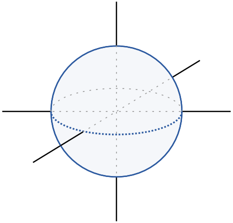


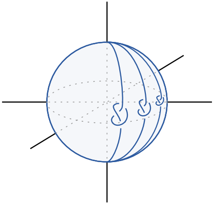
$$\implies \mathrm{Kh}(\Sigma) : \mathrm{Kh}^{h,\boldsymbol{q}}(L_0) \to \mathrm{Kh}^{h,\boldsymbol{q+\chi(\Sigma)}}(L_1)$$

Theorem (Jacobsson, Bar-Natan, Khovanov, Morrison-Walker-Wedrich)

$\mathrm{Kh}(\Sigma)$ is invariant (up to sign) under smooth isotopy rel $\partial\Sigma$.

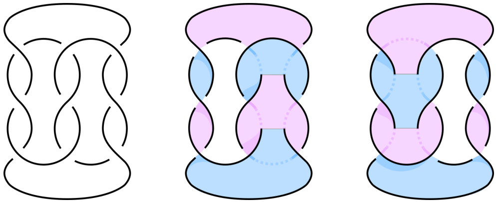
Can these maps distinguish non-isotopic embeddings?



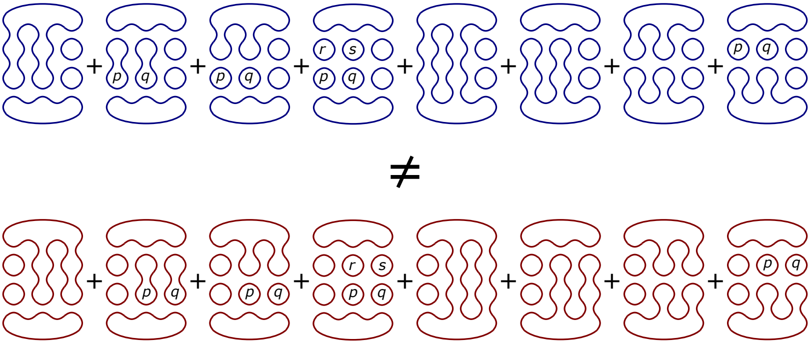


Rasmussen, Tanaka ('05-06): Not if Σ is closed (i.e., $\partial\Sigma = \emptyset$).

Sundberg-Swann ('21): $\mathrm{Kh}(\cdot)$ can distinguish certain disks in B^4.

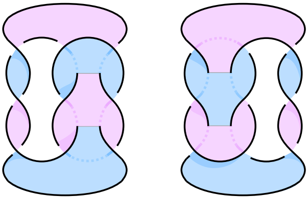


- $D, D' : \emptyset \to K \implies \mathrm{Kh}(D), \mathrm{Kh}(D') : \mathbb{Z} \to \mathrm{Kh}(K)$
- Showed $\mathrm{Kh}(D)(1) \neq \mathrm{Kh}(D')(1)$ in $\mathrm{Kh}(K)$.

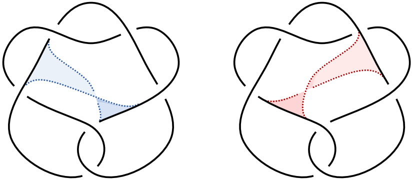


(For code that checks nontriviality of Khovanov homology classes, check out `imsundberg.github.io`)

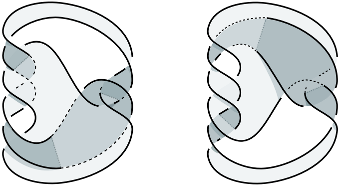
In fact, Khovanov homology distinguishes many pairs of disks.



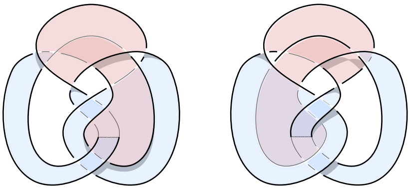
9_{46}



6_1 (stevedore knot)



$15n_{103488}$



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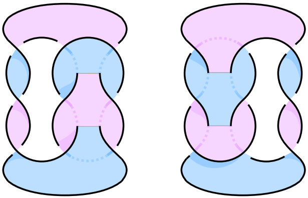

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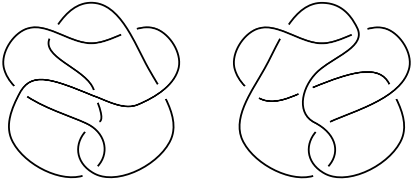

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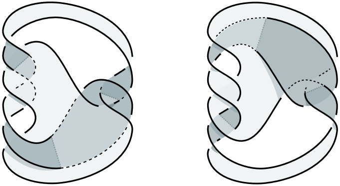

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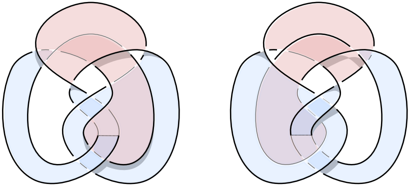

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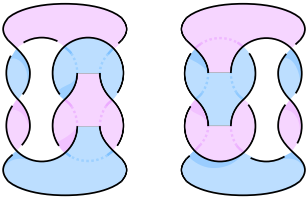

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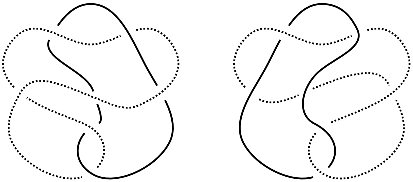

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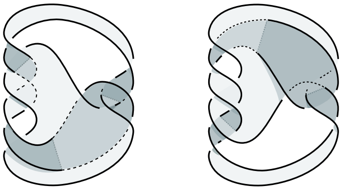

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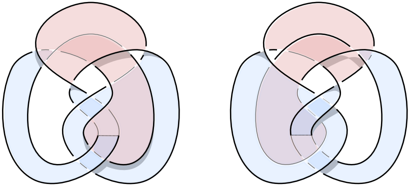
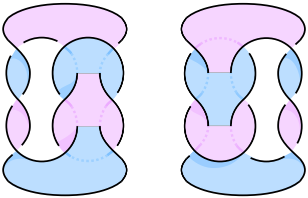


$17nh_{73}$ (positron knot)

Note: Several of these embed holomorphically in $B^4 \subset \mathbb{C}^2$.



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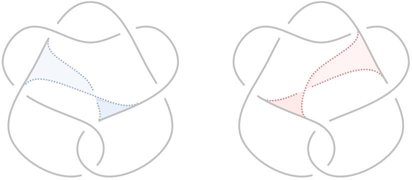

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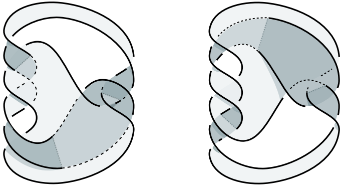

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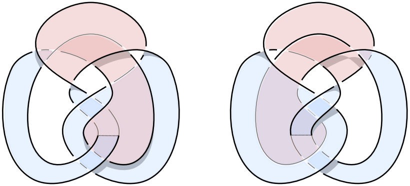
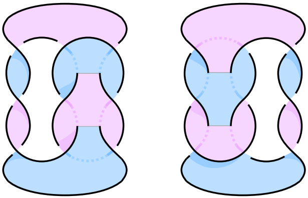


$17nh_{73}$ (positron knot)

Note: Some already distinguished by $\pi_1(S^3 \setminus K) \to \pi_1(B^4 \setminus D)$!



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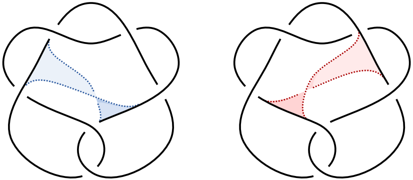

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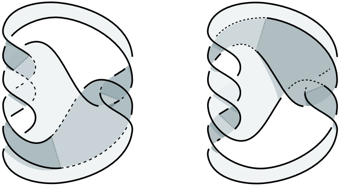

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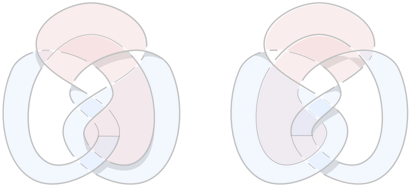

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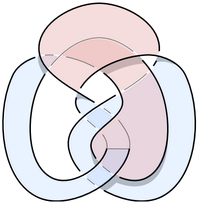
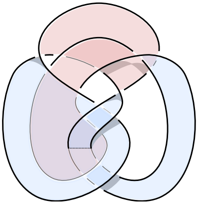
Khovanov homology can distinguish smooth surfaces (of all genera) in B^4 that are isotopic via homeos but not diffeos of B^4 (rel ∂).





Further work:

- **Lipshitz-Sarkar '21:** Mixed Kh-invariant for *non*-orientable surfaces
- **Lipshitz-Sarkar '22:** Distinguish surfaces via spectral sequence relating Kh of equivariant knot and its quotients

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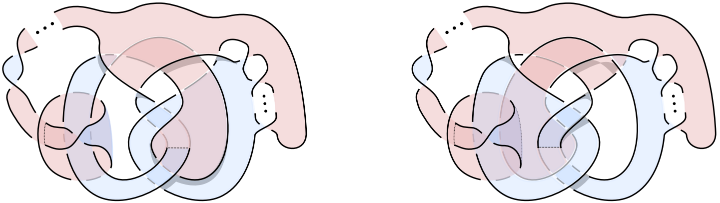

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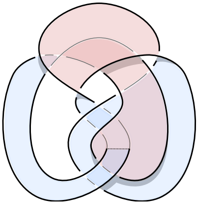

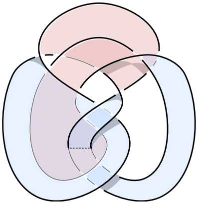

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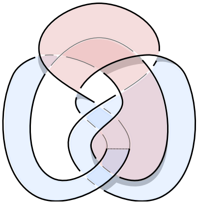

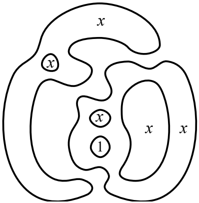

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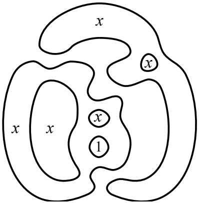

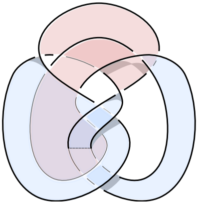

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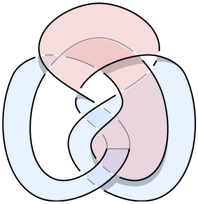

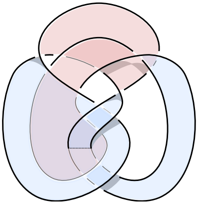

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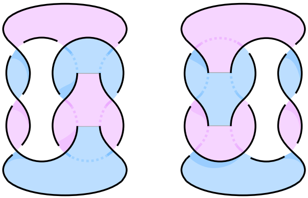
The equivariant perspective works for all of these disks.



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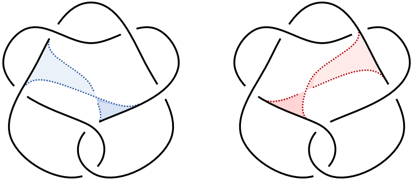
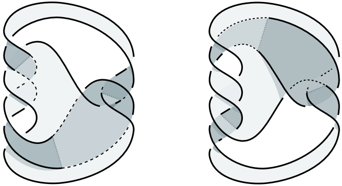
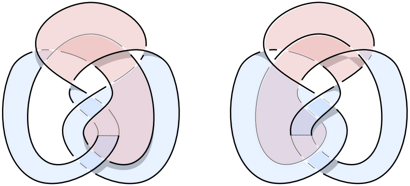


6_1 (stevedore knot)

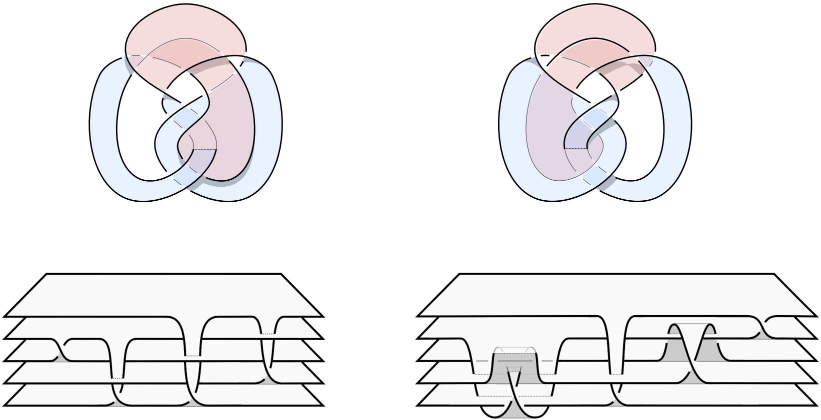


$15n_{103488}$



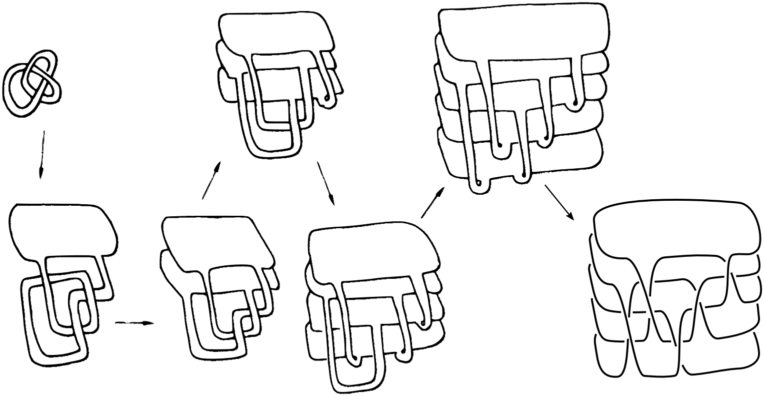
$17nh_{73}$ (positron knot)

Instead, we'll consider braids and braided surfaces.



Braided surfaces

Rudolph '84: Ribbon surfaces in B^4 are isotopic to *braided surfaces*.



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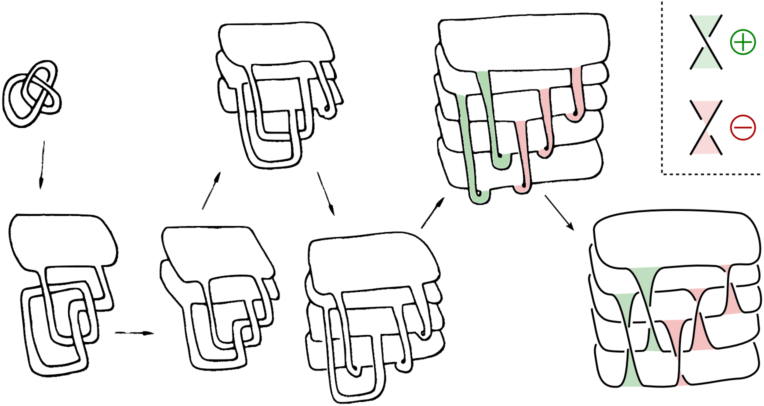
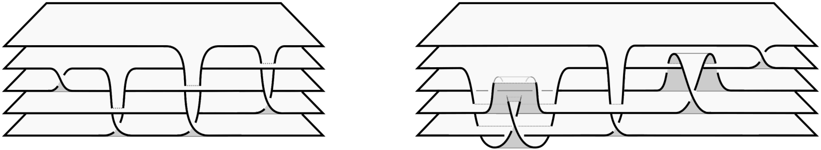


Informal 3D definition

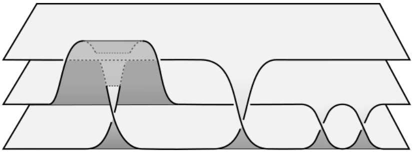
braided surface = a finite collection of parallel disks joined by half-twisted bands (with "ribbon" intersections)

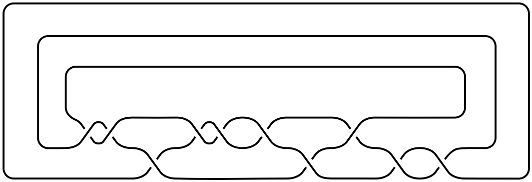


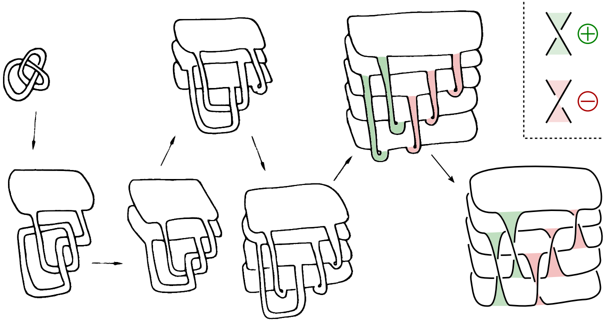
Formal 4D definition

A smooth surface Σ in $B^4 \approx D^2 \times D^2$ is *braided* if the first-coordinate projection $D^2 \times D^2 \to D^2$ restricts to a simple branched covering $\Sigma \to D^2$.

Braided surfaces are encoded by factorizations $\beta = \prod_{k=1}^{\ell} w_k \sigma_{i_k}^{\pm 1} w_k^{-1}$.





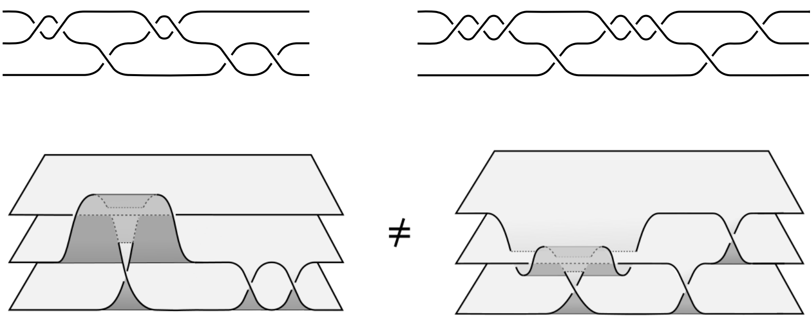


Rudolph ‘83, Boileau-Orevkov ‘01:

$$\left\{\begin{matrix}\text{Bounded complex} \\ \text{curves in } B^4 \subset \mathbb{C}^2\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pos. braided} \\ \text{surfaces}\end{matrix}\right\} \longleftrightarrow \left\{\prod_k w_k \sigma_{i_k}^{+1} w_k^{-1}\right\}$$

Inequivalent factorizations can yield inequivalent surfaces.

Example (Auroux): $(\sigma_2^{-2}\sigma_1\sigma_2^2)\sigma_1^2 = (\sigma_1^{-3}\sigma_1\sigma_3)\sigma_1\sigma_2 \in B_3$



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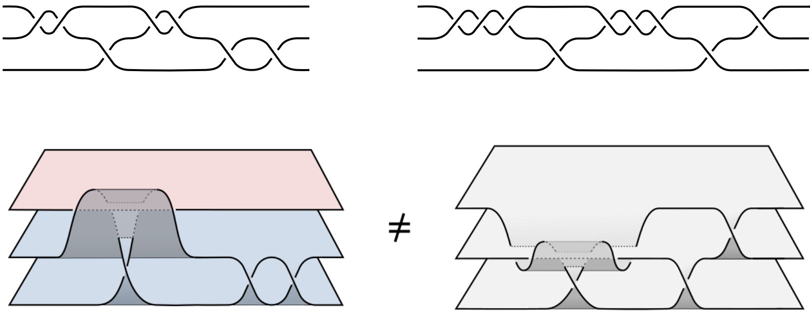
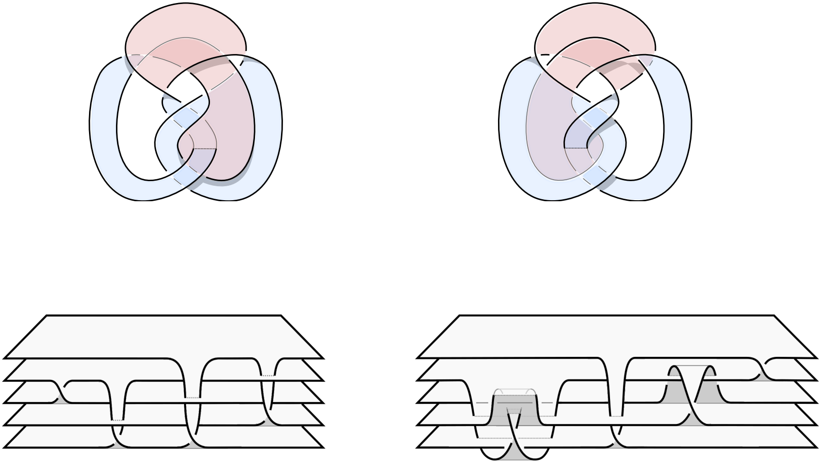
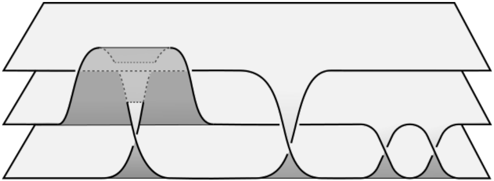
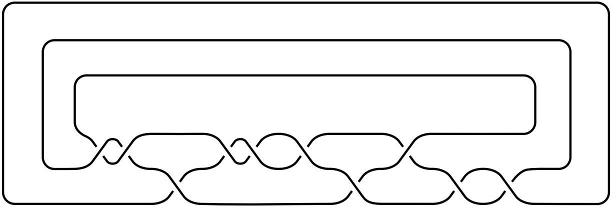
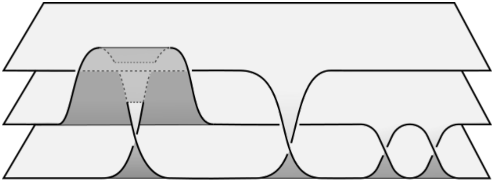


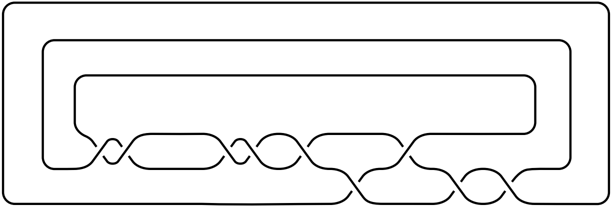


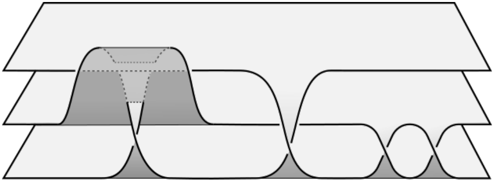
Braided surfaces and Khovanov homology

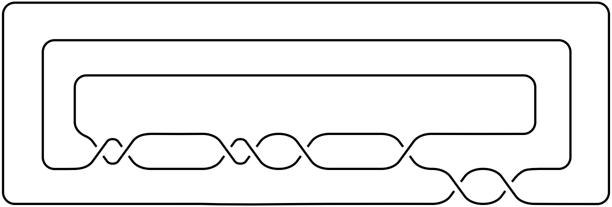


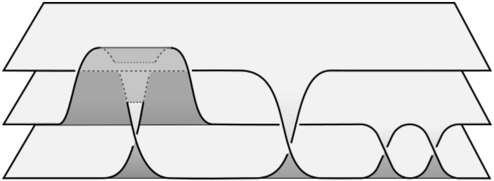


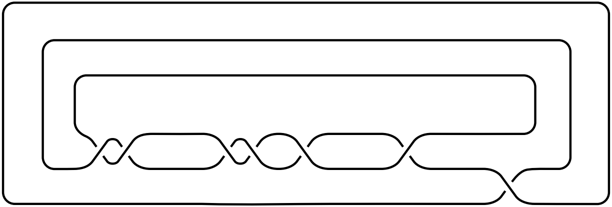


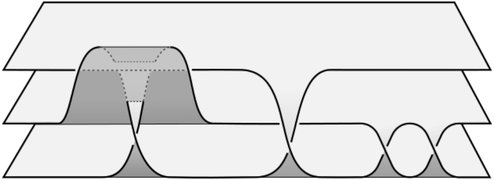


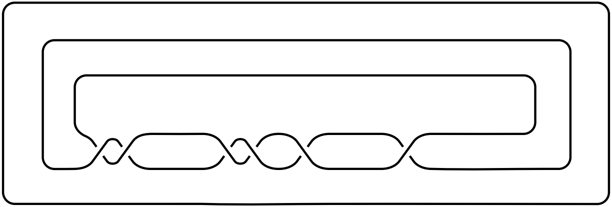


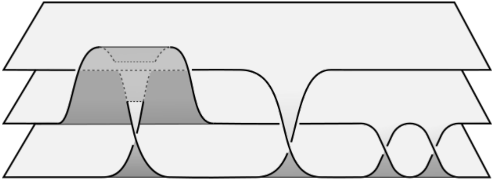


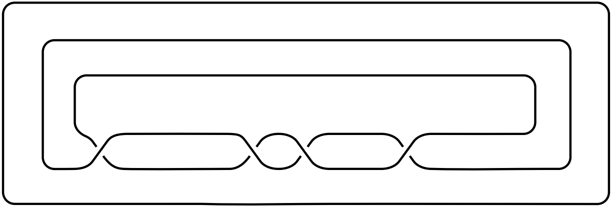


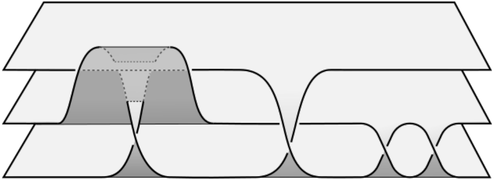


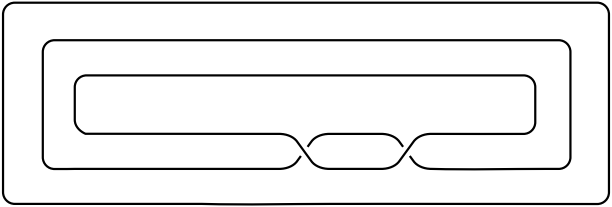


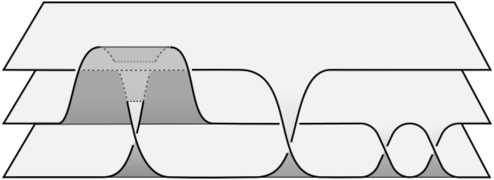


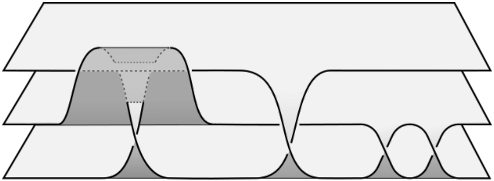


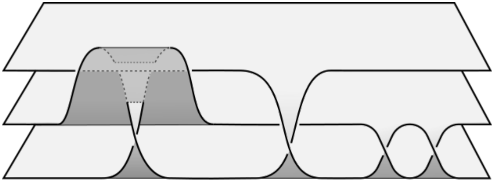


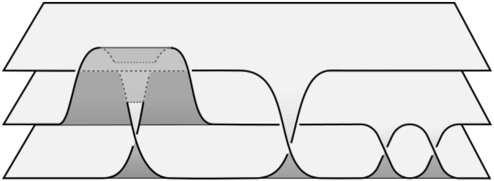












Khovanov homology has elementary maps associated to

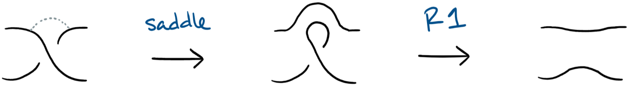
- births/deaths
- saddles
- Reidemeister moves (R1, R2, R3)

Braided surfaces do not require R3.

$\implies$ Easier to work with in theory, practice, and on computer.

[For code that computes cobordism maps for braided surfaces (in the near future), check out Alan Du's github.]

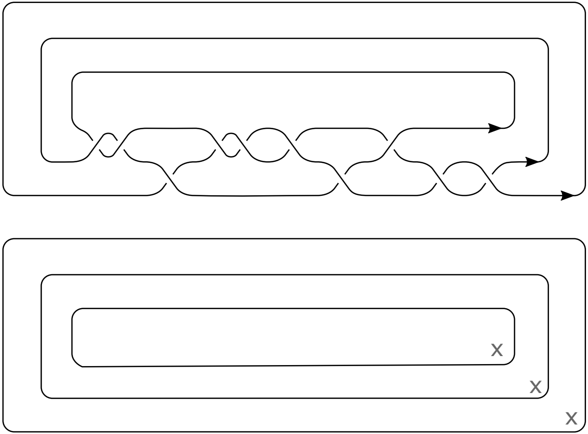
Combine these to get other cobordisms. For example:



Note: The induced map is the "identity" on  but kills)(.

How does Khovanov homology relate to braids / braid factorizations?

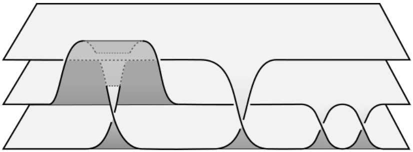
Plamenevskaya'06: Khovanov homology contains a braid invariant.



$\psi(\beta) \in \mathsf{Kh}$

Recipe: Oriented (braid-like) resolution at every crossing, assign every circle the element $x \in \mathcal{A} = \mathbb{Z}[x]/(x^2)$.

Plamenevskaya: If Σ is positively braided, then $\psi(\partial\Sigma) \xmapsto{\mathsf{Kh}(\Sigma)} \pm 1$.

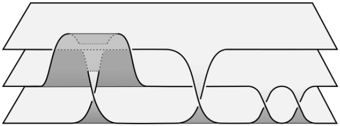


✓ Useful for showing that Kh(Σ) is nonvanishing.

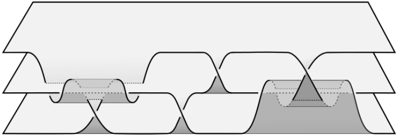
✗ Not useful for *distinguishing* positively braided surfaces.

Does Khovanov homology contain a more sensitive invariant associated to a (quasipositive) *braid factorization*?

Example. The knot 10_{148} bounds ≥ 2 positively braided surfaces:



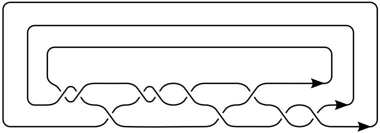
$\neq$

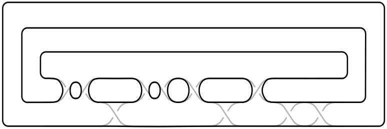


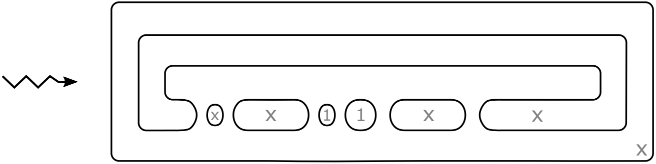
Note: No obvious reason that equivariant considerations could apply here; 10_{148} has no symmetries.

Candidate? Oriented resolutions at core σ_i of each band $w\sigma_i w^{-1}$ and *disoriented* resolutions elsewhere.

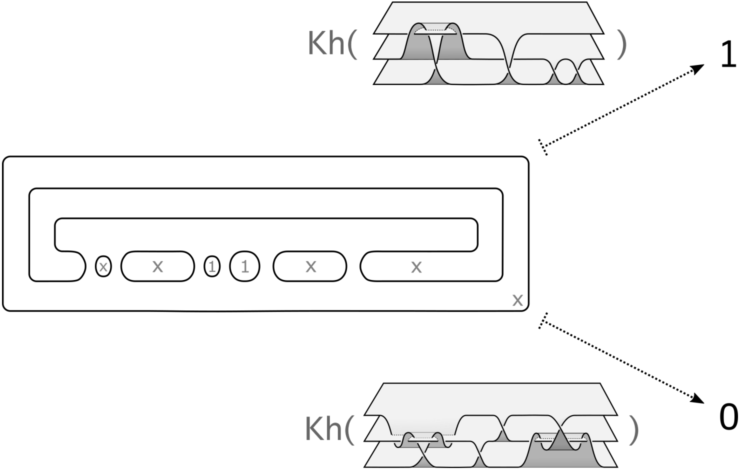
Candidate? Oriented resolutions at core σ_i of each band $w\sigma_i w^{-1}$ and *disoriented* resolutions elsewhere. [And add labels 1, x appropriately...]



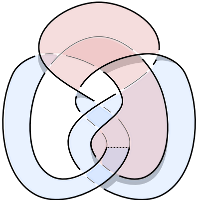


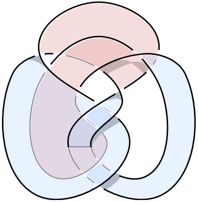


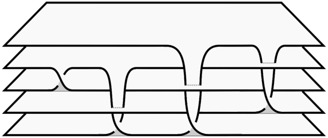
E.g., this element gets sent to ± 1 by $\mathrm{Kh}(\Sigma)$ and to 0 by $\mathrm{Kh}(\Sigma')$. ✓

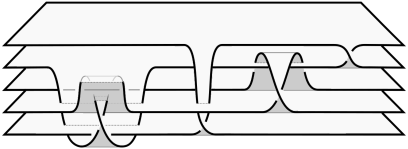


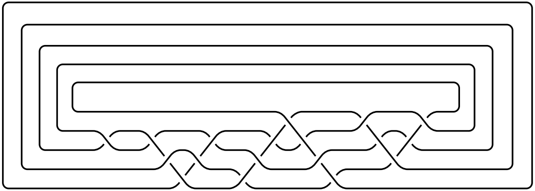
So can we apply this to our desired examples?

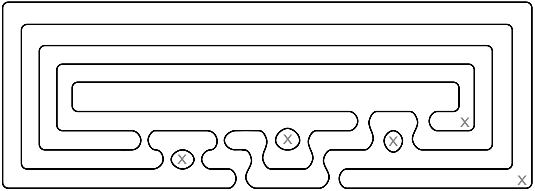




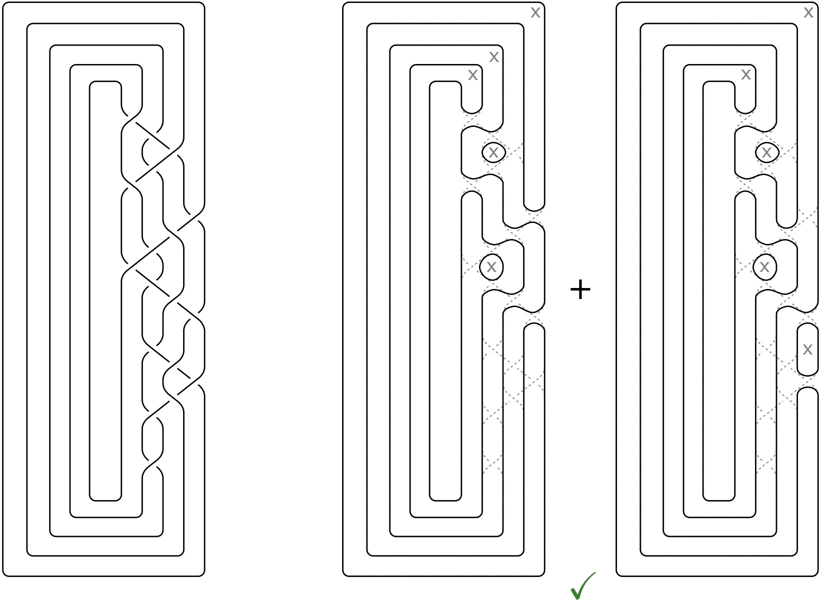






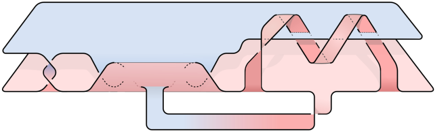


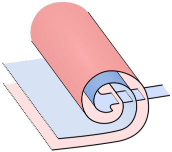
Sadly, such elements $\phi_1 \in \mathsf{CKh}(\beta)$ aren't usually cycles. But can often be "completed" to a cycle by adding some $\phi_0 \in \ker(\mathsf{CKh}(\Sigma))$.


x
x
x
x
x
+
x
x
x
x
x

Lastly, some motivational bad news.

Gompf '91: The knot $12n_{121}$ bounds ∞'ly many knotted tori whose branched covers are distinguished by Seiberg-Witten invariants.





Theorem (Du-H, 2022)

Infinitely many of these induce the the same map $\mathrm{Kh}(12n_{121}) \to \mathbb{Z}$.

Can we enrich Khovanov homology to distinguish surfaces like this?

strong parallel: $\left\{ \begin{array}{c} \text{invariants of surfaces} \\ \text{in 4-manifolds} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{invariants of} \\ \text{4-manifolds} \end{array} \right\}$

Thank you!