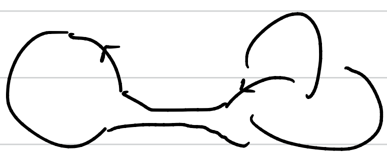
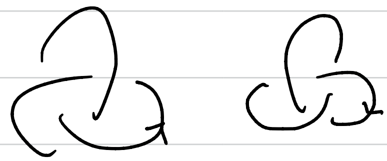
Infinite order rationally slice knots

Jt w/ S. Kang, J. Park, M. Stoffregen



({knots in S^3}, #)

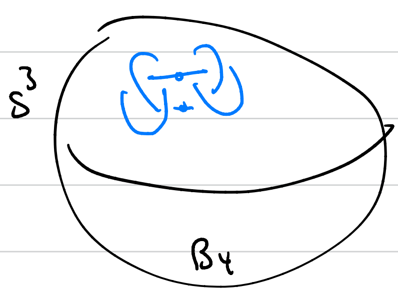
monoid, missing inverses

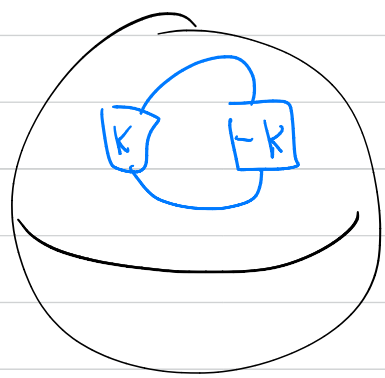


K $\qquad$ $-K$

$K_0 \sim K_1$ concordant if they cobound a smooth properly embedded annulus in $S^3 \times [0,1]$

A knot K is slice if it bounds a smooth, properly embedded disk in B^4





Exercise $K_0 \sim K_1 \iff K_0 \# -K_1$ slice

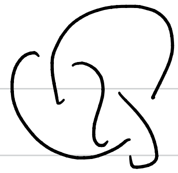
knot concordance group

$$\mathcal{C} = (\{\text{knots in } S^3\}, \#)/\text{concordance} = (\{\text{knots in } S^3\}, \#)/\text{slice knots}$$

id unknot

inverse of K is $-K$

A knot K is $\mathbb{Q}$-slice if it bounds a smooth, properly embedded disk in $\mathbb{Q}H_*B^4$

Ex  is $\mathbb{Q}$-slice (Fintushel–Stern '84) — bounds a disk in a 4-mfld W with $H_1(W;\mathbb{Z}) = \mathbb{Z}/2$

Ex $K_0 \# -K_1$ is $\mathbb{Q}$-slice $\iff$ K_0 and K_1 cobound a smooth, properly embedded annulus in W^4 with

1.) $\partial W^4 = S^3 \amalg S^3$

2.) $H_*(W^4, S^3;\mathbb{Q}) \cong H_*(S^3 \times I, S^3;\mathbb{Q})$

} i.e. W^4 is a $\mathbb{Q}H_*$-cobordism from S^3 to itself

$$C_{\mathbb{Q}} = (\{\text{knots in } S^3\}, \#) \big/ \mathbb{Q}\text{-slice knots}$$

$$0 \to C_{\mathbb{Q}S} \to C \to C_{\mathbb{Q}} \to 0$$

↖ subgroup of $\mathbb{Q}$-slice knots

Ex $K_n =$  $\mathbb{Q}$-slice (Cha '07), order 2 since $K_n = -K_n$

$n=1 \Rightarrow$ figure eight

Thm (H.-Kang-Park-Stoffregen)

$\mathbb{Z}^\infty \leq C_{\mathbb{Q}S}$ generated by $J_{\underbrace{2n+1,1}_{\text{cable}}}$ where $J =$ figure eight

Cor $\mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \leq C_{\mathbb{Q}S}$

- Thm gives first known examples of infinite order knots in $C_{\mathbb{Q}S}$
- false in higher dimensions
- proof relies on Heegaard Floer homology (bordered + involutive)

Recall: algebraic concordance group (Levine '69)

$$\mathcal{A} = (\{\text{Seifert forms}\}, \oplus) \big/ \text{metabolic}$$

← vanish on a $1/2$-dim subspace

$$\cong \underset{\sigma_\omega}{\mathbb{Z}^\infty} \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty$$

$\varphi: \mathcal{C} \to \mathcal{A}$ $\ker \varphi$ non-trivial (Casson-Gordon '75)

isomorphism in higher dim (Levine '69)

$$\mathcal{C}^{2n+1} = \left(\{K^{2n+1} \subset S^{2n+3}\}, \#\right)/_{\text{concordance}}$$

$$n \geq 1$$

K $\mathbb{Q}$-slice $\underset{\text{Cha-Ko}}{\Longrightarrow}$ $\sigma_\omega(K) = 0$ $\underset{\text{Levine}}{\Longrightarrow}$ K is finite order in $\mathcal{C}^{2n+1}$, $n \geq 1$

K is <u>strongly negative amphichiral</u> if $\exists$ orientation reversing involution $\phi: S^3 \to S^3$ such that $\phi(K) = K$

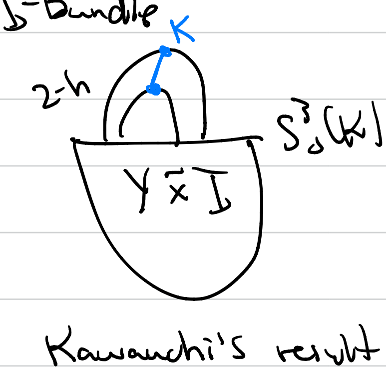
Kawauchi '09 K is concordant to a strongly negative amphichiral knot $\Rightarrow$ K $\mathbb{Q}$-slice

<u>Sketch of pf</u> K is strongly negative amphichiral via involution ϕ induces a fixed pt free involution

$$\tilde{\phi}: S^3_0(K) \circlearrowleft \qquad Y = S^3_0(K)/\tilde{\phi}$$

$Y \tilde{\times} I$ twisted I-bundle

$\partial(Y \tilde{\times} I) \cong S^3_0(K)$



<u>Check</u>
$(Y \tilde{\times} I) \cup 2\text{-h}$ is a $\mathbb{Q}HB^4$

$\boxtimes$

<u>Cor of Thm</u>   Converse to Kawauchi's result is false

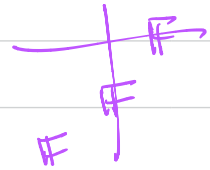
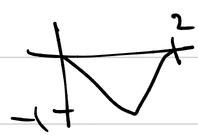
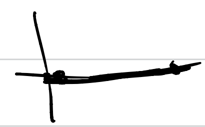
<u>Heegaard Floer homology & knot Floer homology</u>

(Ozsváth-Szabó, Rasmussen)

$K \subset S^3 \rightsquigarrow CFK(K)$ bigraded chain complex over $\mathbb{F}[\mathcal{U}, \mathcal{V}]$

chain homotopy type is an invt of isotopy class of K

Various algebraic operations that lead to more tractable invts:

Invt/type of object	How it's obtained	Properties	RHT	4_1
$\widehat{HFK}$ bigraded vector space	set $U=V=0$, take H_*	categorifies Alexander polynomial	 $t^{-1}-1+t$	 $-t^{-1}+3-t$
HFK^- fin gen graded module over $\mathbb{F}[U]$	set $V=0$, take H_*	$HFK^-(K) \cong \mathbb{F}[U] \oplus \bigoplus_i \mathbb{F}[U]/U^{n_i}$	$\mathbb{F}[U]_{(-1)} \oplus \mathbb{F}_{(0)}$	$\mathbb{F}[U]_{(0)} \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$
$\tau(K) \in \mathbb{Z}$	$\tau(K) = -gr(1)$, $1 \in \mathbb{F}[U] \subseteq HFK^-(K)$	concordance homomorphism $\tau: \mathcal{C} \to \mathbb{Z}$	1	0
$\varepsilon(K) \in \{-1,0,1\}$	...	concordance invt	1	0
$\nu^+(K) \in \mathbb{Z}_{\geq 0}$	...	concordance invt	1	0
$V_0 \in \mathbb{Z}_{\geq 0}$	...	concordance invt	1	0
Υ PL: $[0,2] \to \mathbb{R}$	...	concordance invt		
CFK_{conn} bigraded chain cx over $\mathbb{F}[U,V]$	...	concordance invt determining $\tau, \varepsilon, \nu^+, V_0, \Upsilon$	$CFK_{conn}(RHT) = CFK(RHT)$	$CFK_{conn}(4_1) = CFK_{conn}(\text{unknot}) \cong \mathbb{F}[U,V]$

K slice $\Rightarrow CFK_{conn}(K) \cong CFK_{conn}(\text{unknot}) \Rightarrow \tau(K) = \varepsilon(K) = \nu^+(K) = V_0(K) = 0$, $\Upsilon_K(t) \equiv 0$

$K \rightsquigarrow CFK(K)$

$-K \rightsquigarrow CFK(K)^*$

$K_1 \# K_2 \rightsquigarrow CFK(K_1) \otimes CFK(K_2)$

$$\mathfrak{CFK} = \left(\{CFK(K) \mid K \subset S^3\}, \otimes\right) \Big/ CFK_{conn} \text{ trivial}$$

$\mathcal{C} \to \mathfrak{CFK}$ group homomorphism

$[K] \mapsto [CFK(K)]$

In fact:

<u>Prop</u> (OS) K $\mathbb{Q}$-slice $\Rightarrow$ $CFK_{conn}(K) \cong CFK_{conn}(\text{unknot})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{\mathbb{Q}S} \arrow[r] & \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{C}_{\mathbb{Q}} \arrow[r] \arrow[d] & 0 \\ & & & \mathfrak{CFK} & \end{tikzcd}$$

$\Rightarrow$ Need some additional structure!

Solution: <u>involutive</u> knot Floer homology (Hendricks–Manolescu, –Zemke)

invt of concordances in <u>spin</u> $\mathbb{Q}H_*$-cobordisms

$\rightsquigarrow$ $CFK_{\iota_K\text{-}conn}$

In particular: if K is slice in $\mathbb{Q}HB$ W with $|H_1(W;\mathbb{Z})|$ odd, then

$CFK_{\iota_K\text{-}conn}(K) \cong CFK_{\iota_K\text{-}conn}(\text{unknot})$

<u>But</u> if K is slice in $\mathbb{Q}HB$ W with $|H_1(W;\mathbb{Z})|$ even, then

$CFK_{\iota_K\text{-}conn}(K)$ may be non-trivial

<u>Ex</u> $CFK_{\iota_K\text{-}conn}(\text{figure eight})$ is non-trivial

Ex $CFK_{\iota_K\text{-conn}}(J_{n,1})$ is non-trivial $n=$odd

$J=$ figure eight is trivial for $n=$even

Open Q: Is $J_{2n,1}$ slice? Miyazaki $J_{2n,1}$ not ribbon

$\tilde{\Phi}: S^3_0(K) \circlearrowleft$ $\quad p: S^3_0(K) \longrightarrow S^3_0(K)/\tilde{\Phi}$

involution

mapping cylinder of p

$K_0 \sim K_1$

$f: CFK(K_0) \longrightarrow CFK(K_1)$

isom on $(\mathcal{U},\mathcal{V})^{-1} H_*$

$\{ g \mid g: CFK(K) \circlearrowleft \text{ isom on } (\mathcal{U},\mathcal{V})^{-1} H_* \} =$ self-local equiv

g <u>maximal</u> if $\forall$ self-local equiv f,

$\ker f \subseteq \ker g$

$CFK_{conn}(K) = \operatorname{Im} g \qquad g$ maximal