One stabilization is not enough for exotic contractible 4-manifolds

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Introduction

Given a topological manifold of dimension at least 4, usually an exotic phenomena appears: it may admit more than one smooth structures. Exotic structures have been studied since Milnor's foundational work on exotic S^7.

In four dimensions, the story gets much more interesting. Unlike in higher dimensions, a smooth 4-manifold usually admits infinitely many exotic structures, and it is almost impossible to classify all possible smooth structures on a given topological 4-manifold.

Thus it is natural to ask, if two smooth 4-manifolds are homeomorphic but not diffeomorphic, can we make them diffeomorphic by possibly adding something in common?

For that purpose, it is natural to consider *stabilization*, i.e. taking a connected sum with $S^2 \times S^2$.

Theorem (Wall)

Any two simply-connected homeomorphic smooth 4-manifolds become diffeomorphic after finitely many stabilizations.

Thus it is natural to ask how many stabilizations are necessary: for any $n > 0$, is there any exotic pair of closed simply-connected 4-manifolds which remains exotic after n stabilizations?

It turns out that this is an extremely hard question; almost all known examples of exotic closed simply-connected 4-manifolds trivialize after one stabilization.

So, to start with: is there any exotic closed simply-connected 4-manifold which stays exotic after one stabilization? A natural starting point would be to construct a nice cork.

A *cork* is a triple (Y, W, f) where Y is a homology 3-sphere bounding a contractible 4-manifold W, and $f : Y \to Y$ is an orientation-preserving diffeomorphism which does not extend smoothly to W. This serves as a "building block" of exotic 4-manifolds.

Given a smooth 4-manifold with a smoothly embedded cork, one can perform a *cork twist*: one removes W and glues it back via f. This gives another smooth structure on the given space, which is potentially non-diffeomorphic to the original structure.

Cork twisting is a universal way of building exotic structures in dimension 4. Any two smooth structures (rel ∂) on an simply-connected 4-manifold are related by a single cork twist! (Curtis-Freedman-Hsiang-Stong, Matveyev)

So we should ask whether there exists a cork (Y, W, f) where f does not extend to a self-diffeomorphism of not just W, but also of $W \sharp (S^2 \times S^2)$.

Note that Wall's theorem also applies to contractible 4-manifolds with boundary, where diffeomorphism is considered rel boundary. Hence every cork gets uncorked after sufficiently many stabilizations.

So, given any cork (Y, W, f), there exists some $n > 0$ for which f extends to a self-diffomorphism of $W \sharp n(S^2 \times S^2)$. However, just as in the closed case, there has been no known example for which the minimal required n is greater than 1.

Theorem (K.)

There exists a cork which cannot be uncorked after one stabilization.

Combined with a work of Akbulut-Ruberman on absolutely exotic 4-manifolds, the above theorem implies the following corollary.

Corollary (K.)

There exists a exotic pair of contractible 4-manifolds (with boundary) W_1, W_2 such that $W_1 \sharp (S^2 \times S^2)$ and $W_2 \sharp (S^2 \times S^2)$ are not diffeomorphic.

Hence, one stabilization is not enough for corks and contractible 4-manifolds.

There are other known "one is not enough" type statements, which were usually proven using invariants defined by Seiberg-Witten theory. For example,

- There exists an exotic pair of disks in a punctured $K3$ which stays exotic after one stabilization (Lin-Mukherjee)
- There exists a self-diffeomorphism of $K3\sharp K3$ which is topologically but not smoothly isotopic to the identity, and this phenomenon persists after one stabilization (Lin)
- There exists an exotic pair of codimension 1 submanifolds of $K3\sharp(S^2 \times S^2)$ which stays exotic after arbitrarily many stabilizations (Konno-Mukherjee-Taniguchi)

However, nothing was known about exotic structures on 4-manifolds.

Constructing a cork

How do we construct our cork? We start by performing $(+1)$-surgery along a disk. Let K be a knot which bounds a smoothly embedded disk D in B^4. By removing a point in the interior of D, we get a concordance C from K to an unknot U.

Then we can perform $(+1)$-surgery along the concordance C, which gives a cobordism from $S^3_{+1}(K)$ to S^3. Then we cap off S^3 by attaching a 4-ball to get a compact 4-manifold bounding $S^3_{+1}(K)$.

This always produces a contractible 4-manifold. Its diffeomorphism class (rel ∂) depends only on the smooth isotopy class (rel ∂) of D; we write it as $B^4_{+1}(D)$, the $(+1)$-surgery of B^4 along D.

Suppose that we are given two slice disks D_1, D_2 bounding a knot K, such that they are not smoothly isotopic rel boundary in B^4, and possibly also in a punctured $S^2 \times S^2$, but there exists a self-diffeomorphism F of B^4 satisfying $F(K) = K$ and $F(D_1) = D_2$.

This induces a diffeomorphism $\tilde{F}$ between $B^4_{+1}(D_1)$ and $B^4_{+1}(D_2)$, which restricts to $F|_{\partial B^4}$ on the boundary. If we can prove that $B^4_{+1}(D_1)$ and $B^4_{+1}(D_2)$ are not diffeomorphic *rel boundary* after one stabilization, then this gives us a cork $(S^3_{+1}(K), B^4_{+1}(D_1), \tilde{F})$ which survives one stabilization.

Thus we have to find such a pair (D_1, D_2).

Now, given a knot K in S^3, suppose that we are also given a self-diffeomorphism (i.e. symmetry) f of S^3, satisfying $f|_K = \mathrm{id}_K$. Then we can consider the *deform-spun* disk $D_{K,f}$ bounding $K\sharp - K$.

Intuitively, $D_{K,f}$ is formed by rotating K half times along an axis, where we apply the diffeomorphism f as we rotate.

The key property is that $D_{K,f}$ is always diffeomorphic, but usually not even continuously isotopic, to the standard ribbon disk $D_{K,\mathrm{id}}$, so we can use them to construct a cork.

We will focus on special kinds of knots K which carries an obvious symmetry. We take

$$K = K_0 \sharp K_0,$$

where f swaps the two K_0 summands and then rotates half times along itself. Then we take

$$K_0 = (-T_{6,13} \sharp 2T_{6,7})_{3,-1} \sharp (-T_{6,13} \sharp 2T_{6,7})_{3,-1}.$$

It turns out that this choice works. The knot K_0 is carefully chosen so that an obstruction from involutive Heegaard Floer theory can be applied.

Obstruction

To a homology sphere Y, Heegaard Floer theory associates to it a $\mathbb{Z}$-graded chain complex $CF^-(Y)$ of $\mathbb{F}_2[U]$-modules, well-defined up to homotopy equivalence.

Furthermore, given a homology ball W (or in general, a Spin^c 4-manifold) bounding a 3-manifold Y, we have an associated cobordism map

$$F_W^- : CF^-(S^3) \to CF^-(Y),$$

well-defined up to homotopy. Since $CF^-(S^3) \simeq \mathbb{F}_2[U]$, the homotopy class of F_W^- is defined by the homology class of $F_W^-(1)$, denoted c_W.

For our purpose, we have to consider their involutive refinements. Hendricks-Manolescu defined an involution

$$\iota_Y : CF^-(Y) \to CF^-(Y),$$

well-defined up to homotopy, satisfying $\iota_Y^2 \sim \mathrm{id}$.

Thus, given a homology sphere Y, we take the mapping cone

$$CFI^-(Y) = \mathrm{Cone}(1 + \iota_Y),$$

which is a chain complex of $\mathbb{F}_2[U, Q]/(Q^2)$-modules. This is called *involutive Heegaard Floer homology* of Y.

Hendricks-Hom-Stoffregen-Zemke proved that involutive Heegaard Floer homology is natural and functorial up to homotopy.

This allows us to consider cobordism maps between involutive Heegaard Floer homology. Thus, given a homology ball (or in general, a Spin^c 4-manifold) W bounding Y, we have a cobordism map

$$F^I_W : CFI^-(S^3) \to CFI^-(Y),$$

from which we can define $c^I_W \in HFI^-(Y) = H_*(CFI^-(Y))$. This homology class is again a diffeomorphism (rel ∂) invariant of W.

The key property of involutive Heegaard Floer homology is that, unlike the non-involutive version, the cobordism map

$$F^I_{S^2\times S^2} : CFI^-(S^3) \to CFI^-(S^3)$$

for $S^2 \times S^2$ is nonzero: it is the multiplication map by Q. Note that $CFI^-(S^3) \simeq \mathbb{F}_2[U,Q]/(Q^2)$.

The statement that we have to prove is that

$$Q(c^I_{B^4_{+1}(D_1)} + c^I_{B^4_{+1}(D_2)}) \neq 0.$$

Note: this resembles a similar phenomenon in $\mathrm{Pin}(2)$-equivariant SWF, in which the cobordism map induced by $S^2 \times S^2$ is also Q in the base ring $\mathbb{F}_2[V,Q]/(Q^3)$.

Now we observe that this condition is equivalent to saying that

$$c_{B^4_{+1}(D_1)} + c_{B^4_{+1}(D_2)} \notin \mathbf{Im}(1 + \iota_{S^3_{+1}(K\sharp -K)}).$$

To see why, write

$$Q(c^I_{B^4_{+1}(D_1)} + c^I_{B^4_{+1}(D_2)}) = \partial(g + Qh)$$

in chain level. Then we have

$$\partial(g + Qh) = \partial g + Q(\partial h + (1 + \iota)g).$$

Hence we should have

$$c_{B^4_{+1}(D_1)} + c_{B^4_{+1}(D_2)} = \partial h + (1 + \iota)g$$

for some cycle g. This proves our observation.

Reduction to knot Floer homology

Now we reduce our problem further; this will involve knot Floer homology. Recall that Heegaard Floer homology admits cobordism maps. The same also holds for knot Floer homology and knot cobordisms, although we need extra data.

Given a slice disk D of a knot K, functoriality of CFK gives a chain map

$$F_D : \mathbb{F}_2[U, V] = CFK_{UV}(S^3, \mathrm{unknot}) \to CFK_{UV}(S^3, K).$$

We denote the homology class of $F_D(1)$ by $t_D \in CFK_{UV}(S^3, K)$; this class always generates the homology of $(U, V)^{-1}CFK_{UV}(S^3, K)$. Sometimes we will consider it as an element of $\widehat{CFK}(S^3, K)$.

Knot Floer homology determines Heegaard Floer homology of large surgery. For any knot K and integer $N \geq 2g(K)$, one can define, via holomorphic triangle counting, the large surgery isomorphism

$$\Gamma_{N,0} : CF^-(S^3_{+N}(K), [0]) \xrightarrow{\sim} A_0(K),$$

where $[0]$ denotes the zero spin structure and $A_0(K)$ denotes the Alexander grading 0 subcomplex of $CFK_{UV}(S^3, K)$.

With respect to the absolute $\mathbb{Q}$-grading on $CF^-(S^3_{+N}(K), [0])$ and the absolute $\mathbb{Z}$-grading on $A_0(K)$ induced by the bigrading on $CFK_{UV}(S^3, K)$, the map $\Gamma_{N,0}$ has degree shift $\frac{1-N}{4}$.

Since $\Gamma_{N,0}$ is defined by counting holomorphic triangles, it is easy to see that it homotopy-commutes with concordance maps.

$$\begin{tikzcd}
CF^-(S^3_{+N}(U),[0]) \arrow[r, "\Gamma_{N,0}"] \arrow[d, "F^-_{B^4_{+N}(D_i)}"'] & A_0(U) \subset CFK_{UV}(S^3,U) \arrow[d, "F_{D_i}"] \\
CF^-(S^3_{+N}(K),[0]) \arrow[r, "\Gamma_{N,0}"] & A_0(K) \subset CFK_{UV}(S^3,K)
\end{tikzcd}$$

Given a knot K, we consider the negative-definite cobordism $W_{1,N}$ from $S^3_{+1}(K)$ to $S^3_{+N}(K)$, defined by taking $N-1$ parallel copies of (-1)-framed meridians of K, which is $(+1)$-surgered. This cobordism is negative-definite, spin, and admits a unique spin structure $\mathfrak{s}$.

This cobordism commutes with the “surgery along a concordance” cobordisms. Hence we have

$$\begin{tikzcd}
CF^-(S^3_{+1}(U)) \arrow[r, "F^-_{W_{1,N},\mathfrak{s}}"] \arrow[d, "F^-_{B^4_{+1}(D_i)}"'] & CF^-(S^3_{+N}(U),[0]) \arrow[d, "F^-_{B^3_{+N}(D_i)}"'] \\
CF^-(S^3_{+1}(K)) \arrow[r, "F^-_{W_{1,N},\mathfrak{s}}"] & CF^-(S^3_{+N}(K),[0])
\end{tikzcd}$$

Given any knot K, we consider the composition

$$\Gamma_{N,0} \circ F^-_{W_{1,N},\mathfrak{s}} : HF^-(S^3_{+1}(K)) \to A_0(K).$$

The degree shifts of $\Gamma_{N,0}$ and $F^-_{W_{1,N},\mathfrak{s}}$ are $\frac{1-N}{4}$ and $\frac{N-1}{4}$, respectively, so the composed map is degree-preserving (and identity when K is unknot).

So we have

$$\begin{tikzcd}
CF^-(S^3_{+1}(U)) \arrow[r, "="] \arrow[d, "F^-_{B^4_{+1}(D_i)}"'] & A_0(U) \subset CFK_{UV}(S^3, U) \arrow[d, "F_{D_i}"] \\
CF^-(S^3_{+1}(K)) \arrow[r, "\Gamma_{N,0}\circ F^-_{W_{1,N},\mathfrak{s}}"] & A_0(K) \subset CFK_{UV}(S^3, K)
\end{tikzcd}$$

Note that $CFK_{UV}(S^3, U) \simeq \mathbb{F}_2[U, V]$. By definition, the homology class of the $F_{D_i}(1)$ is t_{D_i}. Hence we have

$$(\Gamma_{N,0} \circ F^{-}_{W_{1,N},\mathfrak{s}})(c_{B^4_{+1}(D_1)} + c_{B^4_{+1}(D_2)}) = t_{D_1} + t_{D_2}.$$

Furthermore, since $\Gamma_{N,0}$ commutes with ι (and ι_K) up to homotopy and $\mathfrak{s}$ is self-conjugate, we see that if $c_{B^4_{+1}(D_1)} + c_{B^4_{+1}(D_2)})$ is contained in the image of $1+\iota$, then $t_{D_1} + t_{D_2}$ should be contained in the image of $1+\iota_K$.

Therefore we seek to find D_1 and D_2 bounding K such that $t_{D_1} + t_{D_2} \notin \mathrm{Im}(1+\iota_K)$ in $HFK_{UV}(S^3, K)$. We will find them so that the condition actually holds on $\widehat{HFK}(S^3, K)$.

A glimpse on the proof of the main theorem

Recall that our ansatz for constructing (D_1, D_2) is by choosing a knot K_0 and taking

$$K = K_0 \sharp K_0 \sharp - K_0 \sharp - K_0,$$

where D_1 is the standard ribbon disk $D_{K_0 \sharp K_0, \mathrm{id}}$ and D_2 is the deform-spun disk $D_{K_0 \sharp K_0, f}$. Here, f denotes the "component swapping" diffeomorphism of $K_0 \sharp K_0$.

As mentioned earlier, we are taking K_0 to be the knot

$$K_0 = (-T_{6,13} \sharp 2T_{6,7})_{3,-1} \sharp (-T_{6,13} \sharp 2T_{6,7})_{3,-1}.$$

Note that the $-T_{6,13} \sharp 2T_{6,7}$ is the simplest knot in the family

$$\{-T_{2n,4n+1} \sharp 2T_{2n,2n+1} | n \text{ odd}\},$$

which was used by Hendricks-Hom-Stoffregen-Zemke to prove that $\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibered spaces.

We are making such a complicated choice because its involutive knot Floer homology looks interesting. The CFK_{UV} of the knot $-T_{6,13}\sharp 2T_{6,7}$ has a summand of the form

$$
x \qquad \oplus \qquad
\begin{tikzcd}
b \arrow[d, "V^3"'] & a \arrow[l, "U^3"'] \arrow[d, "V^3"] \\
d & c \arrow[l, "U^3"]
\end{tikzcd}
$$

This component is the "essential component" - other components can be ignored up to ι_K-local equivalence.

The involution ι_K is given by $a \mapsto a + U^2V^2x$, $x \mapsto x + d$, $b \leftrightarrow c$, $d \mapsto d$.

This is very similar to the involutive knot Floer homology of the figure-eight knot 4_1. The complex $CFK_{UV}(S^3, 4_1)$ is given by

$$x \qquad \oplus \qquad \begin{tikzcd} b \arrow[d, "V"'] & a \arrow[l, "U"'] \arrow[d, "V"] \\ d & c \arrow[l, "U"] \end{tikzcd}$$

The involution ι_K in this case is given by $a \mapsto a + x$, $x \mapsto x + d$, $b \leftrightarrow c$, $d \mapsto d$.

We are now taking (3,-1)-cable of the knot $-T_{6,13}\sharp 2T_{6,7}$; then we have to figure out how the involution ι_K acts on its $\widehat{HFK}$. However, there is no "involutive cabling formula" right now. We will have to use a very technical argument via involutive bordered Floer homology to carry out a partial computation.

We will not explain the bordered techniques used in the proof. Instead, we will focus on the prediction: based on the ι_K action on $(4_1)_{3,-1}$, we will predict the ι_K action on $(-T_{6,13}\sharp 2T_{6,7})_{3,-1}$.

The action of ι_K on $CFK_{UV}(S^3, (4_1)_{3,-1})$ was computed (for $(2n+1,-1)$-cables, $n \geq 1$) by Hom-K.-Park-Stoffregen. It consists of four irreducible summands:

$$
\begin{tikzcd}
\bullet \arrow[d, "V"'] & a_1 \arrow[l, "U^3"'] \arrow[d, "V"] \\
d_1 & \bullet \arrow[l, "U^3"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "V"'] & & a_2 \arrow[ll, "U^2"'] \arrow[dl, "UV"'] \arrow[dd, "V^2"] \\
d_2 & \bullet \arrow[l, "U"] \arrow[d, "V"'] & \\
 & \bullet & \bullet \arrow[l, "U"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "V^3"'] & a_3 \arrow[l, "U"'] \arrow[d, "V^3"] \\
d_3 & \bullet \arrow[l, "U"']
\end{tikzcd}
\qquad
x
$$

The action of ι_K maps $a_i \mapsto a_i + x$ and $x \mapsto x + U^2 d_1 + U d_2 + V^2 d_3$. So in $\widehat{HFK}$, x is ι_K-invariant, but a_i is still mapped to $a_i + x$.

This is because we had both $a \mapsto a + x$ and $x \mapsto x + d$ in $CFK_{UV}(S^3, 4_1)$. Intuitively, taking a (3,-1)-cable killed the arrow $x \mapsto x + d$, but could not kill $a \mapsto a + x$.

On the other hand, if we take (3,1)-cable instead of (3,-1), we would have killed $a \mapsto a + x$ instead, while $x \mapsto x + d$ would still remain in the ι_K action of CFK of the cabled knot.

So what should we do? First possible idea would be to take the (3,1;3,-1) iterated cable of 4_1. But one can check that this also doesn't work!

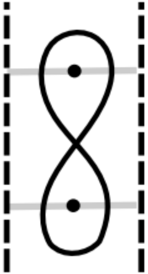
This leads us to consider the (3,-1)-cable of $-T_{6,13}\sharp 2T_{6,7}$. The difference between this case and the case of 4_1 is that, while we still have $x \mapsto x + d$, instead of $a \mapsto a + x$ in the case of 4_1, we have $a \mapsto a + U^2V^2x$.

As in the 4_1 case, we should first compute the *CFK* of the (3,-1)-cable. This is possible over $\mathcal{R}$, i.e. modulo diagonal arrows, via Hanselman's cabling formula in terms of immersed curves.

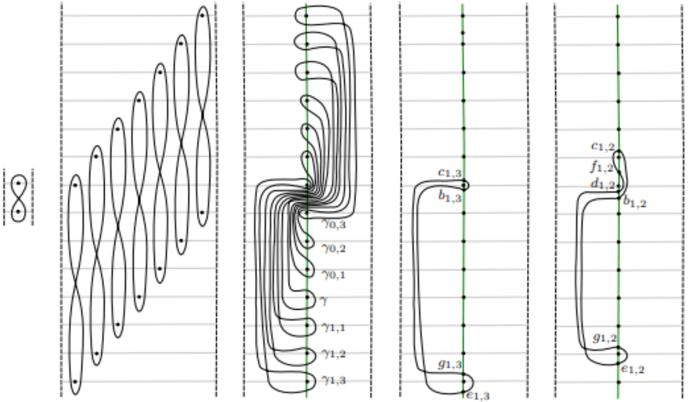
The formula goes: CFK $\rightarrow$ immersed curve $\rightarrow$ immersed curve of cable $\rightarrow$ CFK (modulo $UV = 0$) of cabled knot.

We have an algorithm for drawing an immersed curve from $CFK_{UV}(S^3, K)$, which becomes very simple when CFK admits a horizontally and vertically simplified basis.

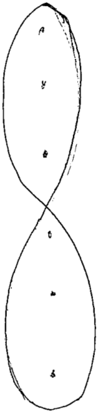
For example, the 1-by-1 box summand in the CFK of 4_1 corresponds to the figure-eight curve.



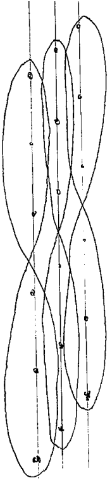
Applying Hanselman's cabling formula allows us to easily compute $CFK_{\mathcal{R}}(S^3, (4_1)_{2n+1,-1})$.



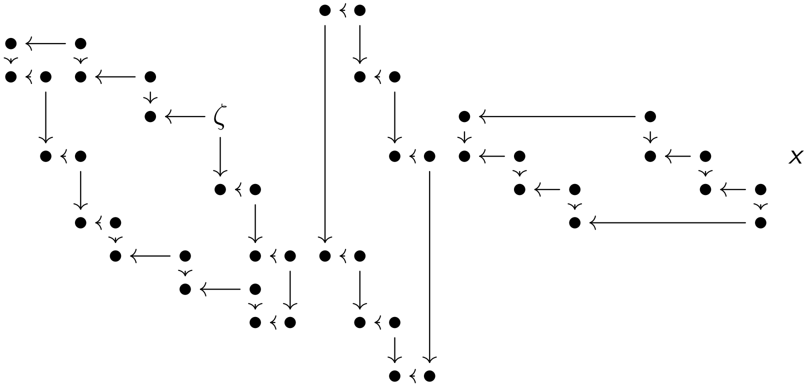
The n-by-n box summand of the CFK of $-T_{6,13}\sharp 2T_{6,7}$ corresponds to the n-times stretched figure-eight curve.



Applying Hanselman's cabling formula gives the immersed (multi)curve invariant for the cabled knot, $(-T_{6,13}\sharp 2T_{6,7})_{3,-1}$.



Components of immersed curves correspond to irreducible summands of CFD of knot complement, which then corresponds to those of $\mathcal{R}$-coefficient CFK complex. This allows us to compute (a part of) the $\mathcal{R}$-coefficient CFK of $(-T_{6,13}\sharp 2T_{6,7})_{3,-1}$. (Note: there are other components as well)



Then one proves that the free summand x is "isolated" under the action of ι_K on $\widehat{HFK}$.

However it turns out that the element ζ is also isolated. First of all, we have $\iota_K(\zeta) = \zeta$ in $\widehat{HFK}$. Furthermore, for any other basis element ξ, its image $\iota_K(\xi)$ in $\widehat{HFK}$ does not contain ζ when we write it as a linear combination of basis elements.

The same also holds for the "basepoint actions" Φ and Ψ. This is because x generates the free summand and all "arrows" connected to ζ have coefficients U^2 or V^2, i.e. no length 1 arrows.

These "isolated elements" (and the fact that we have more than one) are crucial in our computation.

Juhasz and Zemke proved that, under the identification

$$\widehat{HFK}(S^3, K\sharp -K) \simeq \mathrm{End}_{\mathbb{F}_2}(\widehat{HFK}(S^3, K)),$$

the element $t_{D_{K,f}}$, associated to the deform-spun disk $D_{K,f}$ defined by a symmetry f of (S^3, K), corresponds to the induced action f_* of f on $\widehat{HFK}(S^3, K)$.

Furthermore, if f denotes the "swapping symmetry" when $K = K_0 \sharp K_0$, then the action f_* is given by

$$f_* = \mathrm{Sw} \circ (1 \otimes (1 + \Phi\Psi) + \Phi \otimes \Psi),$$

where Sw denotes the "swapping isomorphism", i.e. $x \otimes y \mapsto y \otimes x$.

Clearly, the invariant $t_{D_{K,\mathrm{id}}}$, which corresponds to the identity endomorphism, can be written as the cotrace class:

$$t_{D_{K,\mathrm{id}}} = \mathrm{cotr}(1) = \sum_{\text{basis element } b} b^* \otimes b.$$

Hence, in our case when $K = 4K_0$ where $K_0 = (-T_{6,13} \sharp 2T_{6,7})_{3,-1}$, the linear combination expression of $t_{D_{K,\mathrm{id}}}$ contains the term s:

$$s = x^* \otimes \zeta^* \otimes \zeta^* \otimes x^* \otimes x \otimes \zeta \otimes \zeta \otimes x.$$

Note that we are identifying $\widehat{HFK}(S^3, -K_0)$ with the dual space of $\widehat{HFK}(S^3, K_0)$.

Since both x and ζ are "isolated" under the (hat-flavored) actions of ι_K, Φ, and Ψ, we know that f_* acts by

$$f_*(x \otimes \zeta \otimes \zeta \otimes x) = \zeta \otimes x \otimes x \otimes \zeta.$$

Thus the term s does not appear in $t_{D_{K,f}}$; instead it contains the term

$$x^* \otimes \zeta^* \otimes \zeta^* \otimes x^* \otimes \zeta \otimes x \otimes x \otimes \zeta.$$

Hence s still appears in $t_{D_{K,\mathrm{id}}} + t_{D_{K,f}}$.

Now assume that $t_{D_{K,\mathrm{id}}} + t_{D_{K,f}}$ is contained in the image of $1 + \iota_K$; write it as $y + \iota_K(y)$. Then s should appear in either y or $\iota_K(y)$, but not both.

Suppose first that s appears only in y. Then we know that it also appears in $\iota_K(y)$, because s is ι_K-invariant and cannot be cancelled by terms in ι_K(other things). A contradiction.

Now suppose that s appears only in $\iota_K(y)$. By applying the same logic three times, we can say that s still appears in $\iota_K^3(\iota_K(y)) = \iota_K^4(y)$. But ι_K^4 is homotopic to the identity map! Thus s should also appear in y, a contradiction again.

Therefore $t_{D_{K,\mathrm{id}}} + t_{D_{K,f}}$ cannot be contained in the image of $1 + \iota_K$, and hence our main theorem is proven.

Further remarks

One may ask whether using such a complicated knot is necessary. Our lives are already so complicated, so why add more?

It turns out that simplest possible choices, like linear combinations of torus knots, cannot work, due to an argument of Zemke. First of all, for my argument to work, there should exist a homology class of $HFK_{UV}(S^3, K\sharp -K)$ which is not contained in the image of $1+\iota_K$.

This is impossible if the knot K "dualizes perfectly", i.e. we have a splitting

$$CFK_{UV}(S^3, K\sharp -K) \simeq \mathbb{F}_2[U,V] \oplus C \oplus C',$$

where the free summand is ι_K-invariant and ι_K maps C to C', and also C' to C, isomorphically.

Clearly, if K dualizes perfectly, then $-K$ also does. Also, if K_1, K_2 dualize perfectly, then $K_1 \sharp K_2$ also dualizes perfectly.

Furthermore, Floer-thin knots and torus knots dualize perfectly. (I think (odd,1)-cables of the figure-eight knot also dualizes perfectly, but not sure) Hence linear combinations of those knots cannot be used in my arguments.

However, those knots are the almost all cases in which the action of ι_K is completely understood. Actually, the knot $(-T_{6,13} \sharp T_{6,7})_{3,-1}$ is the first example of a knot which does not dualize perfectly. So my argument was destined to be complicated!

One can also try to use "twisted" stabilizations. Instead of $S^2 \times S^2$, we may use $S^2 \tilde{\times} S^2$ instead. (This is necessary for nonorientable 4-manifolds)

What happens if we attach $S^2 \tilde{\times} S^2$ to my cork? Observe:

$$S^2 \tilde{\times} S^2 \simeq \mathbb{C}P^2 \sharp -\mathbb{C}P^2.$$

When we attach $\mathbb{C}P^2$ to my cork, the boundary diffeomorphism extends.

This is because, for any knot K bounding a smooth slice disk D, the diffeomorphism class (rel ∂) of $B^4_{+1}(D)\sharp\mathbb{C}P^2$ does not depend on the isotopy class of D. (informed to me by Hayden)

It might also be very interesting to provide an upper bound on the number of stabilizations needed to trivialize my cork. We know that one stabilization is not enough and finitely many stabilization is enough, but don't know how many is needed.

In general, given a knot K and two smooth slice disks D_1, D_2 bounding K, how can we find an upper bound on the number n satisfying

$$B^4_{+1}(D_1)\sharp n(S^2 \times S^2) \simeq B^4_{+1}(D_2)\sharp n(S^2 \times S^2) \text{ rel } \partial?$$

Thing might become easier if $K = K_0 \sharp - K_0$, $D_1 = D_{K_0,\mathrm{id}}$, and $D_2 = D_{K_0,f}$ for some symmetry f of K_0.

One can ask other questions as well. For example,

- Can we perform a cork twist using the cork that we just described to construct an exotic pair of closed simply-connected 4-manifolds?
- Is there a cork which survives two stabilizations? How about n stabilizations for general $n > 1$?
- How about surfaces? Is there an exotic pair of two properly embedded smooth disks in B^4 which stays exotic after one stabilization?

Thank you!