Bilinear pairings & topological theories

K-OS seminar — M. Khovanov

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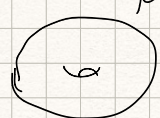
$\Bbbk$ - field (or commutative ring, even commutative semiring)

isomorphism (diffeomorphism) classes of n-dim objects $\xrightarrow{\alpha} \Bbbk$

$\alpha(M_1 \sqcup M_2) = \alpha(M_1)\,\alpha(M_2)$ multiplicative

$M \cong M'$ diffeomorphic $\Rightarrow \alpha(M) = \alpha(M')$ function on diffeomorphism classes

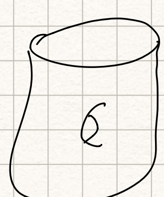


$M \longmapsto \alpha(M) \in \Bbbk$

(add involution ψ on $\Bbbk$ s.t. $\alpha(-M) = \psi(\alpha(M))$. Assume $\psi = \mathrm{id}$; easy to hide in low dimensions)

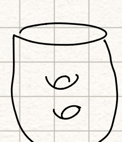
N - closed oriented $(n-1)$-manifold

$\mathrm{Fr}(N)$ - free $\Bbbk$-module, basis $[M]$ ← "symbol" of an n-manifold M, $\partial M = N$



$\Bbbk$-bilinear form on $\mathrm{Fr}(N)$

$$([M_1],[M_2])_N = \alpha\big((-M_1) \cup_N M_2\big) \in \Bbbk$$



$\xrightarrow{\alpha} \Bbbk$

$$\alpha(N) := \mathrm{Fr}(N) \big/ \ker\left((\,,\,)_N\right)$$

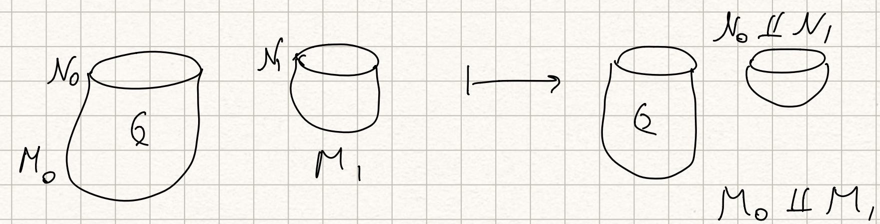
$\uparrow$

state space of N for evaluation α.

For "generic" α, $\alpha(N)$ has infinite rank/dim over $\Bbbk$.

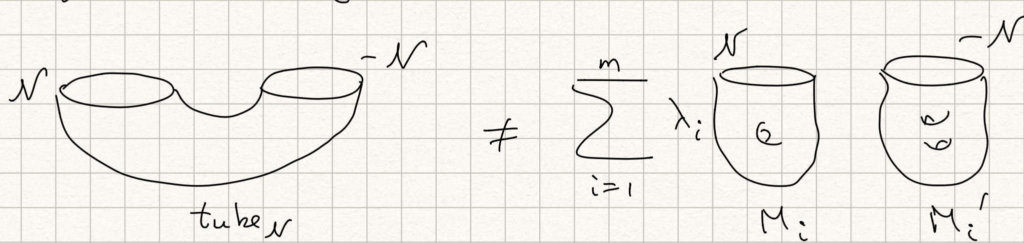
Want to study "degenerate" α with $\alpha(N)$ finite rank, $\forall N$, $\dim N = n-1$

$$\alpha(N_0) \otimes \alpha(N_1) \xrightarrow{\ \gamma\ } \alpha(N_0 \sqcup N_1)$$



Map γ is injective (if $\Bbbk$ is a field)

γ is usually <u>not</u> surjective



no neck-cutting formula, in general

$$\alpha(N_0) \otimes \alpha(N_1) \hookrightarrow \alpha(N_0 \sqcup N_1)$$

inclusion only

α is not monoidal, in general

If neck-cutting exists for all $(n-1)$-dim objects N,

$\Rightarrow \alpha$ is multiplicative in the strong sense,

$$\alpha(N_1 \sqcup N_2) \simeq \alpha(N_1) \otimes \alpha(N_2)$$

Then α is a TQFT (Atiyah's axioms, essentially).

Most of our examples are not strongly multiplicative

Origins:

1) [BHMV] Blanchet-Habegger-Masbaum-Vogel, TQFTs ... applied in the context of WRT $sl(2)$ TQFT. (1995)

2) Link homology: $sl(3)$, $sl(N)$ link homology.

Appears in the context of foam evaluation, multiplicative

L.-H. Robert, E. Wagner Foam evaluation formula 2017

3) Freedman-Kitaev-Nayak-Slingerland-Walker-Wang

Universal pairings, ... (2005)

D. Calegari-M. Freedman-K. Walker, Positivity of universal pairing in 3D (2008)

4) Recent work (2020): MK; joint papers with Radmila Sazdanovich, You Qi, Lev Rozansky, Victor Ostrik, Yakov Kononov, Robert Laugwitz (in progress)

Develop framework and work out some examples of universal construction in low dimensions $(n=2,1)$.

Relation to Deligne categories (and their generalizations)

Relation to non-commutative power series & computation $(n=1$ + defects)

Defects make these theories much richer!

Caveat: Interesting construction with lots of potential, but big applications lacking, for now.

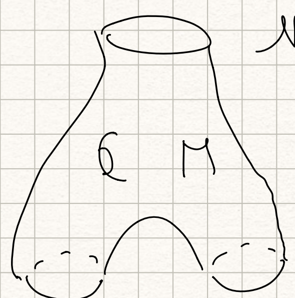
Back to universal construction.

M (n-dim) $\xrightarrow{\alpha} \alpha(M) \in \Bbbk$

N ((n-1)-dim) $\xrightarrow{\alpha} \alpha(N)$ – $\Bbbk$-module/vect. space

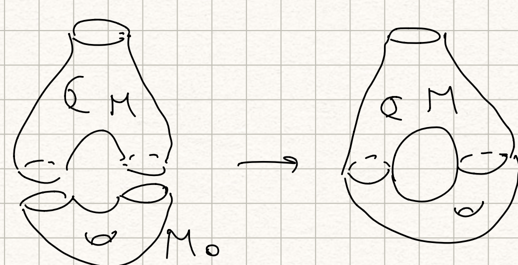
state space

Functoriality: Cobordism M induces a map



$$\alpha(N_0) \xrightarrow{[M]} \alpha(N_1)$$

$$\partial M = N_1 \sqcup (-N_0)$$

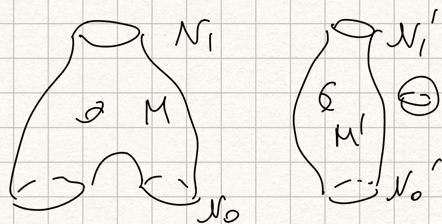


$\partial M_0 = N_0$ $\quad [M_0] \longmapsto [M M_0]$

compose cobordisms

get a functor n-cobordisms $\longrightarrow$ $\Bbbk$-vect (or $\Bbbk$-mod)

monoidal in a weak sense



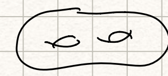
inclusions, not isomorphisms

$$\begin{tikzcd}
\alpha(N_1) \otimes \alpha(N_1') \arrow[r, hook] & \alpha(N_1 \sqcup N_1') \\
\alpha(N_0) \otimes \alpha(N_0') \arrow[r, hook] \arrow[u, "{[M] \otimes [M']}"] & \alpha(N_0 \sqcup N_0') \arrow[u, "{[M \sqcup M']}"']
\end{tikzcd}$$

$n=2$ case: Connected components determined by genus

 $S_g \xrightarrow{\alpha} \alpha(S_g) = \alpha_g \in \mathbb{k}$

$g \geq 0$

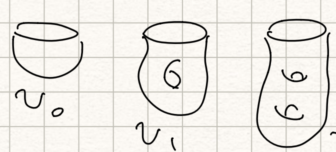
 $\longmapsto \alpha_0$  $\longmapsto \alpha_1$  $\longrightarrow \alpha_2 \quad \dots$

2-sphere 2-torus

encode in generating function

$$Z_\alpha(T) = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \dots = \sum_{n \geq 0} \alpha_n T^n$$

State space $A(1) = \alpha(S^1)$ of a circle. Spanning set

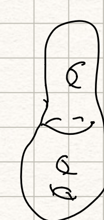


$v_0 \quad v_1 \quad v_2 \quad \dots$

$\{v_n\}_{n \geq 0}$

$$(v_n, v_m) = \alpha(\underset{\text{genus}}{S_{n+m}}) = \alpha_{n+m}$$

Gram matrix $(v_i, v_j) = \alpha_{i+j}$

 v_n, v_m $= S_{n+m}$

$$H = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \dots & j \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ i \end{matrix} & \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & & \vdots \\ \alpha_2 & \alpha_3 & & & \vdots \\ \alpha_3 & & & & \vdots \\ - & & - & - & \alpha_{i+j} \end{pmatrix} \end{array}$$

Hankel matrix H

$\mathbb{N} \times \mathbb{N}$ matrix

$$A(1) = \alpha(S^1) = \underset{\text{basis } v_0, v_1, \dots}{\mathbb{k}^{\mathbb{N}}} / \ker H$$

generic case: $\ker H = 0$.

want $\ker H$ of finite codimension $\iff \dim A(1) < \infty$

$v_0, v_1 \ldots \quad v_r$ must span $A(1)$ some r

(r+1) handles reduce to lin. comb. of $\leq r$ handles

<u>Thm</u> TFAE (essentially Kronecker)

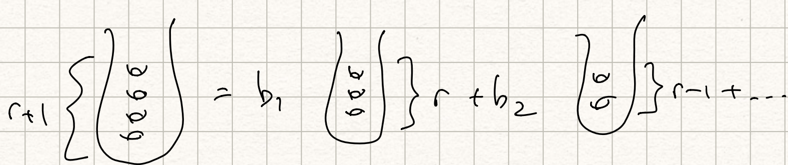
(1) $\dim A(1) < \infty$

(2) $\dim A(m) < \infty \quad \forall m = 0, 1, 2 \ldots$ state space of m circles

(3) $$Z_\alpha(T) = \frac{P(T)}{Q(T)}$$ is a rational function

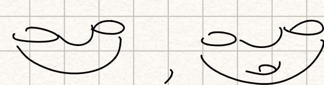
$P(T), Q(T)$ - polynomials

(1) – (3) happen when can reduce r+1 handles to fewer handles for some r



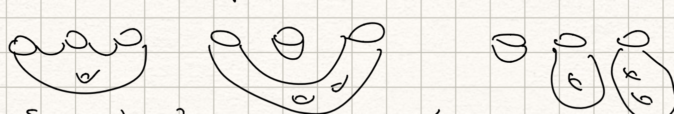
usually proper inclusion

<u>Remark:</u> Beyond $A(1)$: $A(1) \otimes A(1) \overset{i}{\subset} A(2)$



If can reduce r+1 handles, can do this reduction at each connected component

these are not always in the image

A(3) 

$\{1,2,3\}$, $\{1,3\}\{2\}$, $\{1\},\{2\},\{3\}$.

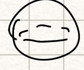
set-theoretic partition of $\underbrace{S^1 \sqcup \ldots \sqcup S^1}_{m}$ + at most r handles at each c. component

$A(m)$

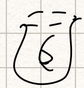
$\Downarrow$

$\dim A(m) < \infty$ if $\dim A(1) < \infty$

Example $Z(T)=\beta$ – constant function, $\beta\in\Bbbk^{*}$ invertible, char $\Bbbk=0$

 $=\beta$,  $=0$,  $=0$, ...

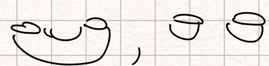
S^2

<u>Handle relation</u>  $=0$

$A(1)=\Bbbk$ 

↑ state space of S^1

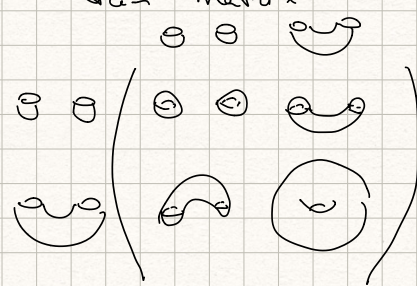
$\Rightarrow$ can assume each component has genus 0

$A(2)$: 

spanning set

actually, a basis

Gram matrix

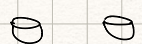


evaluation $\downarrow \alpha$

$$\begin{pmatrix}\beta^2 & \beta\\ \beta & 0\end{pmatrix}\xrightarrow{\det} -\beta^2\neq 0$$

proper inclusion

$A(1)\otimes A(1)\hookrightarrow A(2)$

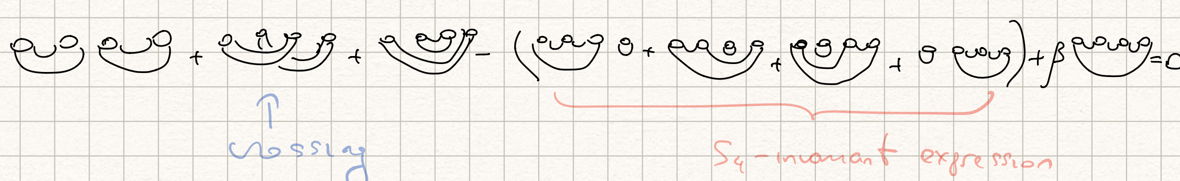
  not in the image

$A(3)$, basis: 

$\det = 2\beta^2$, basis (if char $\Bbbk\neq 2$)

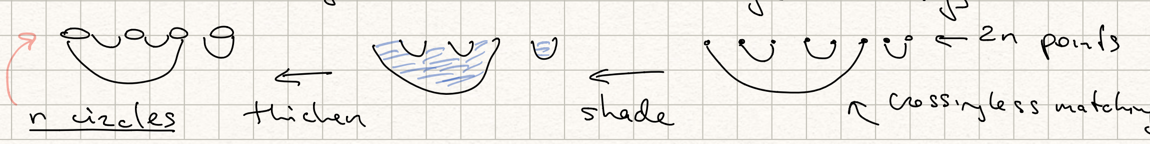
$A(4)$: spanning set of (set-theoretic) partitions of 4.

15 elements, 1 relation



$\Rightarrow$ Can inductively get rid of all crossings.

Genus 0 "crossingless" surfaces $\leftrightarrow$ crossingless matchings



Theorem (Yakov Kononov, Victor Ostrik, M.K.) If char $\Bbbk = 0$, $A(n)$ is a vector space of dim = Catalan number $\frac{1}{n+1}\binom{2n}{n}$ & basis of crossingless surfaces.

Proof uses determinant computation from Y. Kononov, N. Kitchloo, M.K. (In progress).

Alternative proof via rel-n to reps of $osp(1|2)$

General case $\quad Z_\alpha(T) = \frac{P(T)}{Q(T)}$ ← polynomials

$\deg P(T) = N$, $\deg Q(T) = M$

let $K = \max(N+1, M)$ $\qquad K = M \iff \frac{P}{Q}$ is a proper fraction

normalize $Q(0) = 1$ $\qquad$ ($\dim P < \dim Q$)

$$Q(T) = 1 + b_1 T + b_2 T^2 + \dots + b_M T^M$$

$$\frac{P(T)}{Q(T)} = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \dots$$

recurrent formula

$$(*) \quad \alpha_{n+M} + b_1 \alpha_{n+M-1} + b_2 \alpha_{n+M-2} + \dots + b_M \alpha_n = 0 \qquad n \gg 0$$

order of b_i's is reversed compared to $Q(T)$

if $\deg P < \deg Q$, $(*)$ holds for all $n = 0, 1, 2, \dots$

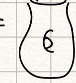
otherwise may fail for first few terms (fraction improper)

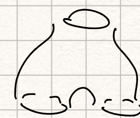
$$\frac{P(T)}{Q(T)} = \underset{\text{proper}}{\frac{\overline{P}(T)}{Q(T)}} + R(T)$$ ← polynomial, coefficients obstruct $(*)$ for small n.

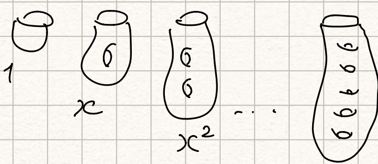
$$U(x) := x^{K-M}(x^M + b_1 x^{M-1} + \dots + b_M)$$

$K - M > 0$ $\Updownarrow$ fraction improper

"Handle polynomial" of $Z_\alpha(T)$.

Handle relation in $A(1)$: $\tau(x) = 0$ $x =$  handle

Pants multiplication in $A(1)$: 

$A(1)$: basis 

1 x x^2 $\cdots$ x^{k-1}

$A(1)$ commutative Frobenius algebra

Example $Z = \dfrac{T^2+1}{1-3T+2T^2}$ $x(x^2-3x+2)=0$ handle relation

$A(1) \otimes A(1) \hookrightarrow A(2)$ ← also a comm. Frob. alg.

$A(n) \otimes A(m) \hookrightarrow A(n+m)$

$A(n)$ · commutative Frob. alg, S_n-action, ...

LRS - linear recurrence sequences ($\Bbbk$).

well-studied, interesting NT properties.

Example: Skolem-Mahler-Lech theorem. In char 0,

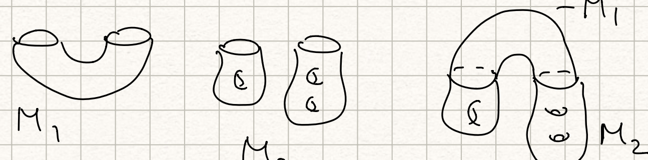
$\{n \in \mathbb{N} \mid \alpha_n = 0\}$ is a union of a fin. set & fin. many arith. progressions

Some refs: Berstel-Reutenauer, Noncomm. rational series

Lidl-Niederreiter, Finite fields

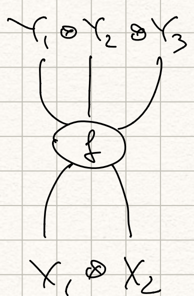
Everest et al [EPSW], Recurrence sequences

In universal construction, mod out by kernels of bilinear forms on $Fr(N)$



N M_1 M_2 $-M_1$ M_2 $\xrightarrow{\alpha} \alpha(-M_1 \cup M_2) \in \Bbbk$

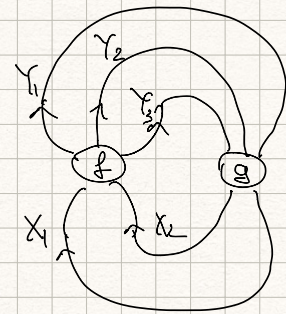
In tensor categories, mod out by negligible morphisms



f is <u>negligible</u> if

$\forall$ closure of f evaluates to 0 in ground field $\Bbbk$

$$\widehat{gf} = 0 \quad \forall g$$



C – $\otimes$ category, J ideal of negligible morphisms (2-sided ideal)

$$\boxed{\underline{C} := C/J \quad \text{quotient category by negligible morphisms}}$$

Call it "gligible quotient"

Non-standard terminology

See [EGNO], Tensor categories

In universal construction, don't have C, only <u>evaluation</u> (of closed tensor diagrams, n-manifolds, foams, etc.)

$\underline{C}$ emerges from the universal construction, sometimes can define C as well.

Such $\underline{C}$ was mostly studied when abelian $\otimes$, see [EGNO], more recently additive $\underline{C}$ are starting to be considered (K. Coulembier, arxiv, very recent)

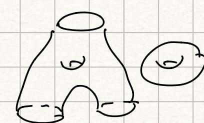
In universal constructions, C and $\underline{C}$ usually additive, rarely abelian.

Also useful to take Karoubi closure of whatever universal construction gives us.

Categories we can build from α (R. Sazdanovic, M.K.)

assume α rational, $Z_\alpha(T) = \dfrac{P(T)}{Q(T)}$

Cob_2 objects $n \geq 0$, (oriented) morphisms are 2D cobordisms



$\downarrow$

$\Bbbk Cob_2$ same objects, linearize morphisms

$\downarrow$

$VCob_2$ same objects $n \geq 0$, viewable cobordisms, evaluate closed cobordisms via α

$\to \alpha_0$, $\to \alpha_1$, ...

$\downarrow$

same objects $n \geq 0$, add handle relation $U(x) = 0$

$$\begin{tikzcd} SCob_\alpha \arrow[r, "\text{additive Karoubi closure}"'] \arrow[d, "\text{gligible quotient}"] & DCob_\alpha \arrow[d, "\text{gligible quotient}"] \\ Cob_\alpha \arrow[r, "\text{add. Karoubi closure}"] & \underline{DCob}_\alpha \end{tikzcd}$$

analogue of Deligne category $DCob_\alpha$

fin. dim hom spaces start here

objects $n \geq 0$, add all relations from $\ker(\,,\,)_n$

n circles

$A(n) = \mathrm{Hom}_{Cob_\alpha}(0, n)$

vect. spaces we get from universal construction

Square commutes in strong sense (two ways to get $\underline{DCob}_\alpha$)

Versions of this diagram:

1) 2D with boundaries & corners (You Qi, L.Rozansky, M.K.)

2) 2D with 0-dim defects (dots) (Y. Kononov, V. Ostrik, M.K.)

3) 1D with defects (decorated dots in 1-manifolds)

4) 2D with 1-dim defects (Robert Laugwitz, MK, in progress)

Eventually, this diagram of categories will get more complicated (if $D \geq 3$ and in 2D with defect webs)