The meridional rank conjecture

(an attack with crayons)

Joint with:
- Ryan Blair (2015 –)
- Roman Velazquez & Paul Villanueva (2016–2017)
- Sebastian Baader (2017 –)
- Nathaniel Morrison (2019 –)
- Filip Misev (2019 –)

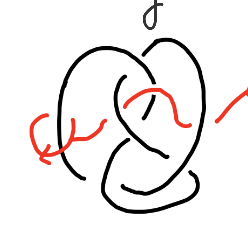
I. Introduction

MRC (Kirby List # 1.11, due to Cappell–Shaneson)

Given a link $L \subset S^3$, does $\beta(L) = \mu(L)$?
(β: bridge number; μ: meridional rank)

Today: sketch a proof of MRC for some new infinite classes of links via $a \geq \beta \geq \mu \geq a$
(first a: via the "Wirtinger number" of L; second a: via Coxeter quotients of $\pi_1(S^3 \setminus L)$)

Recall: a meridian of L is a (based) loop in $S^3 \setminus L$ which is freely homotopic to the boundary of a small disk intersecting L transversally once:



$\pi_1(S^3 \setminus L)$ is generated by meridians. How many are needed?

Def The meridional rank of L, $\mu(L)$, is the minimal number of meridians of L which suffice to generate $\pi_1(S^3 \setminus L)$.

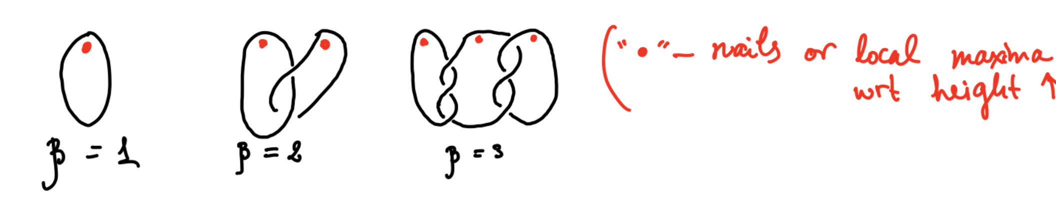
⚠ $\mu(L) \neq \text{rank}(\pi_1(S^3 \setminus L))$.

Ex. $T_{p,q}$, the (p,q)-torus link. $\pi_1(S^3 \setminus T_{p,q}) = \langle x, y \mid x^p = y^q \rangle$.

Thm (Rost–Zieschang '87) $\mu(T_{p,q}) = \min\{p,q\} = \beta(T_{p,q})$, so MRC holds for torus links.

Recall: the bridge number of a link L, $\beta(L)$, is the smallest number of nails you need to display L in a museum.

ex:



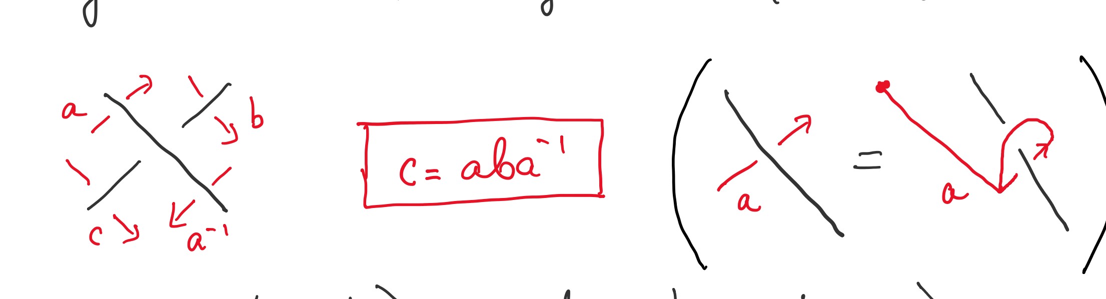
MRC: does $\beta(L) = \mu(L)$?

Known cases:
1. Torus links (Rost–Zieschang '87)
2. $\mu = 2 \Rightarrow \beta = 2$ (Boileau–Zimmermann '89)
3. Pretzel links, Montesinos links
4. Other special cases (Boileau, Zieschang, Weidmann, Lustig-Moriah...)

(Our methods recover / generalize 3.)

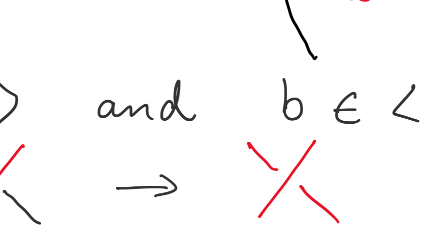
"Obvious" inequality: $\beta(L) \geq \mu(L)$.

For a link L in bridge position, meridians of the arcs containing the local maxima generate $\pi_1(S^3 \setminus L)$:



$\Rightarrow c \in \langle a, b \rangle$ and $b \in \langle a, c \rangle$

Shorthand:



(1)

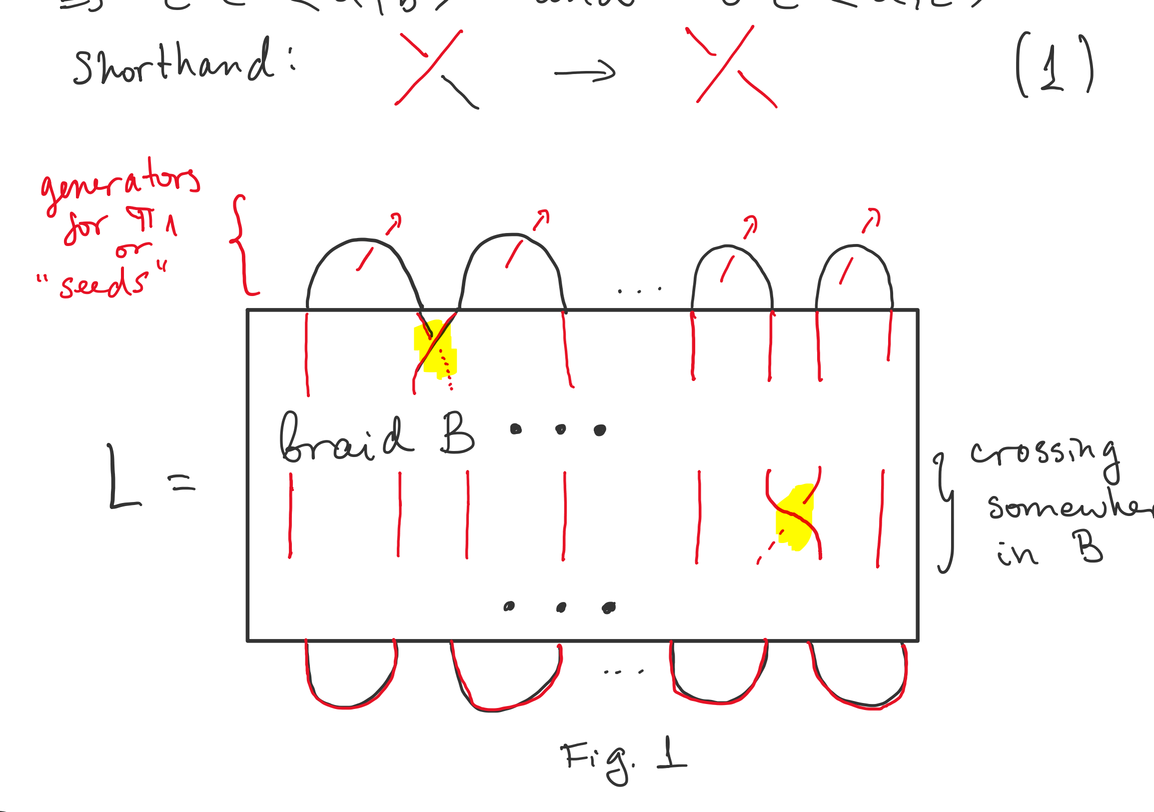


Fig. 1

II. Upper bounds on β

Def Given a link diagram D, its Wirtinger number, $w(D)$ is the minimal number of seed strands $\{s_1, s_2, \ldots, s_{w(D)}\}$ in D, s.t. coloring the s_i allows us to exend the partial coloring to all of D, using (1).

$$w(L) = \min_{D\text{ - diagram of } L} \{ w(D) \}$$

Idea: meridians of the seed strands generate $\pi_1(S^3 \setminus L)$.

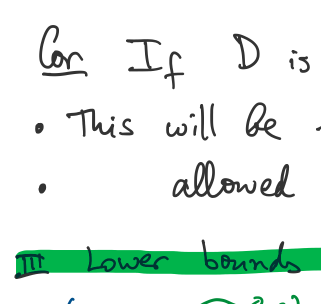
Remark $\beta(L) \geq w(L) \geq \mu(L)$
(first $\geq$: Fig. 1; second $\geq$: def of w)

Thm 1. (Blair–K.–Velazquez–Villanueva '17)
For every link $L \subset S^3$, $w(L) = \beta(L)$.

Cor If D is a diagram of L, then $w(D) \geq \beta(L)$.
- This will be the upper bound used in our proof of MRC.
- allowed us to compute the bridge #s of over 500,000 knots.

III. Lower bounds on μ

ex.



3-coloring of the trefoil K is a map $\pi_1(S^3 \setminus K) \twoheadrightarrow S_3$ sending meridians of K to transpositions.

$\Rightarrow$
- $\pi_1(S^3 \setminus K) \not\cong \mathbb{Z}$
- $K \neq \bigcirc$
- $\mu(K) = 2$

More generally, if $\exists\, \varphi : \pi_1(S^3 \setminus K) \twoheadrightarrow S_n$, meridians $\longrightarrow$ transpositions (ij)

$\Rightarrow \mu(K) \geq n-1$

(It takes $n-1$ transpositions to generate S_n.)

Baader–K '17 used S_n quotients to prove MRC for a (very specialized) class of knots.

Only works for about 14% of tabulated knots (Blair–K–Morrison)

i.e. maps $\left\{ \begin{array}{l} \pi_1(S^3 \setminus K) \twoheadrightarrow S_{\mu(K)+1} \\ \text{meridians} \longrightarrow (ij) \end{array} \right\}$ are rare

Solution: consider maps onto more general Coxeter groups. Roughly 30% of tabulated knots admit μ-detecting quotients to finite Coxeter groups (Blair–K–Morrison). But, many admit μ-detecting quotients to infinite Coxeter groups, as we see next.

Want: $\pi_1(S^3 \setminus L) \twoheadrightarrow G$, s.t. G needs many generators (in fixed conjugacy classes)

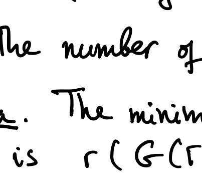
Key idea: do not fix G ahead of time. Define G in terms of L.

III.1. Coxeter groups for children

Def G is a Coxeter group if it admits a presentation of the form $\langle a_1, \ldots, a_n \mid a_i^2 = 1, (a_i a_j)^{k_{ij}} = 1 \rangle$

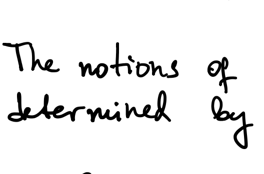
Γ - graph with weighted edges $\rightsquigarrow G(\Gamma)$ - Coxeter group

ex



$\rightsquigarrow S_3 \quad (a = (12),\ b = (23),\ (ab)^3 = 1)$

ex



$\rightsquigarrow S_4 \quad (a = (12),\ b = (23),\ c = (34),\ (ab)^3 = 1, \ldots)$

(Not the convention used in Dynkin diagrams.)

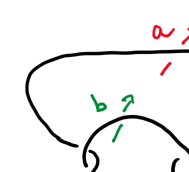
Def An element $g \in G(\Gamma)$ is a reflection if g is conjugate to one of the generators (identified with vertices of Γ).

Def The number of vertices of Γ is the (reflection) rank of $G(\Gamma)$, denoted $r(G(\Gamma))$.

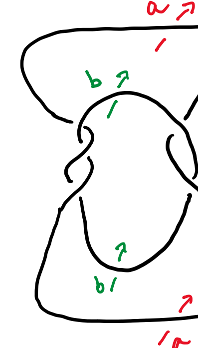
Lemma. The minimal number of reflections in $G(\Gamma)$ needed to generate $G(\Gamma)$ is $r(G(\Gamma))$.

⚠ The notions of reflection and rank depend on Γ, i.e. they are not determined by the group $G(\Gamma)$.

ex.



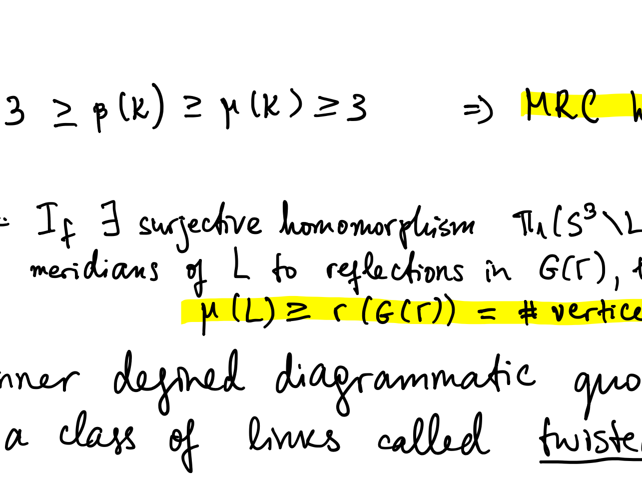
$\rightsquigarrow D_6 = \langle x, y \mid x^2 = y^6 = 1,\ xyx^{-1} = y^{-1} \rangle$
reflection rank $= 2$



$\rightsquigarrow D_6$ again
reflection rank $= 3$

III.2. Diagrammatic quotients of link groups (Brunner '92)

ex. $K = P(3,-2,3)$



$\Rightarrow babbbab = babab = a$

$\Rightarrow (ba)^3 = 1$

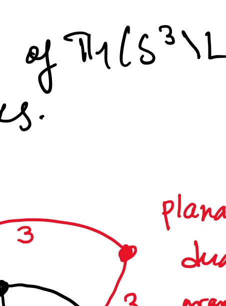
Let Γ =

$\pi_1(S^3 \setminus K) \twoheadrightarrow G(\Gamma)$

$G(\Gamma) = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ (ab)^3 = 1,\ (bc)^2 = 1,\ (ac)^3 = 1 \rangle$

$\Rightarrow \mu(K) \geq 3$

But also, $w(D) = 3$ since the partial coloring extends to all of D.



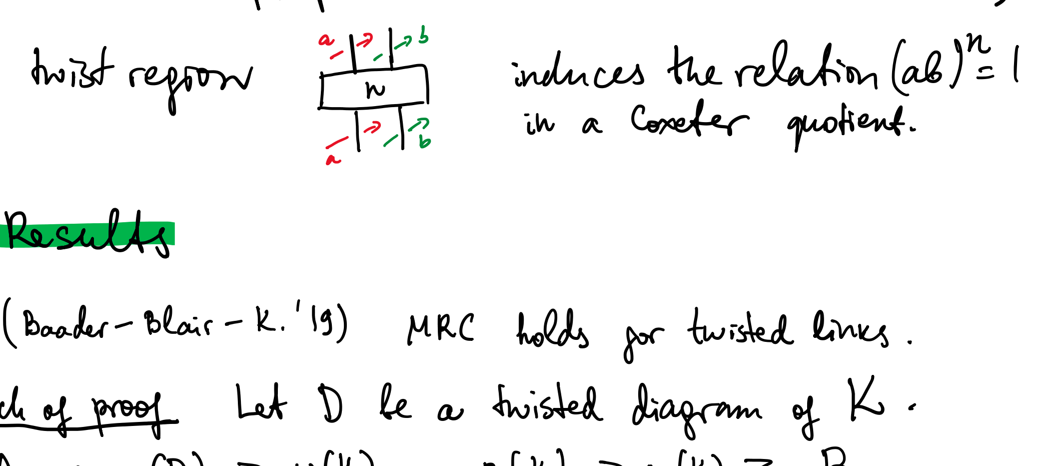
$3 \geq \beta(K) \geq \mu(K) \geq 3 \quad \Rightarrow$ MRC holds for K.

Prop 1. If $\exists$ surjective homomorphism $\pi_1(S^3 \setminus L) \twoheadrightarrow G(\Gamma)$ sending meridians of L to reflections in $G(\Gamma)$, then:

$$\mu(L) \geq r(G(\Gamma)) = \#\text{vertices in } \Gamma.$$

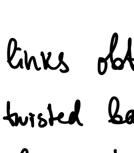
Brunner defined diagrammatic quotients of $\pi_1(S^3 \setminus L)$ for a class of links called twisted links.

ex $P(3,-2,3)$ again:



(checkerboard coloring produces a "twisted" surface)

A twisted link is the boundary of such a surface, reduced in the appropriate sense. In Brunner's construction, each twist region



induces the relation $(ab)^n = 1$ in a Coxeter quotient.

IV. Results

Thm (Baader–Blair–K. '19) MRC holds for twisted links.

Sketch of proof Let D be a twisted diagram of K.

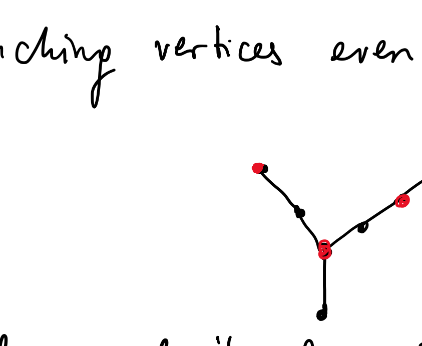
$A \geq w(D) \geq w(K) = \beta(K) \geq \mu(K) \geq B$
(def) (Thm BKVV) (Fig 1) (Brunner + Prop 1)

We show we can choose $A = B \Rightarrow$ MRC holds. ∎

This strategy works for other families of links as well.

Arborescent links: links obtained from plumbing together twisted bands "in treelike patterns", i.e. planar tree with weighted vertices $\rightsquigarrow$ link

ex. T_1



$\rightsquigarrow$ 2-bridge link $L(T_1)$

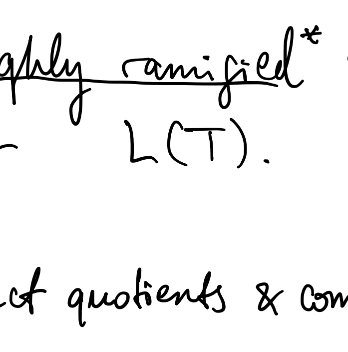
ex. T_2

$\rightsquigarrow$ Montesinos link $L(T_2)$

Thm (Baader–Blair–K. '19)

Let T be a bipartite* tree with non-zero even weights. MRC holds for $L(T)$ and $\beta(L(T)) = |v(T)|$.

* branching vertices even distance apart, e.g.:



New idea: we don't rely on Brunner. Construct Coxeter quotients explicitly (inductively by the complexity of the underlying tree).

Thm (Baader–Blair–K–Misev '21)

Let T be a highly ramified* tree with weights $\neq 0, \pm 1$. MRC holds for $L(T)$.

* TBDef.

(New ways to construct quotients & compute Wirtinger numbers.)

Thank you!