Quantum trace for SL_n skein algebra

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Overview

• Two quantizations of SL_n-character varieties of surfaces.

• Algebra homomorphism between the two quantized algebras, quantizing the classical homomorphism of Fock-Goncharov.

• $n = 2$: quantum trace map of Bonahon-Wong.

Character variety

$\Sigma = \Sigma_g \setminus \mathcal{P}$, finite set $\mathcal{P}$. $\qquad SL_n = SL_n(\mathbb{C})$.

$$C_n(\Sigma) = \{\rho : \pi_1(\Sigma) \to SL_n\}//SL_n$$
$$\rho_1 \equiv \rho_2 \Leftrightarrow \mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)$$

$C_n(\Sigma)$ affine variety.
• A closed curve c on Σ $\longrightarrow$ function $c : C_n(\Sigma) \to \mathbb{C}$

$$c(\rho) = \mathrm{tr}(\rho(c))$$

Curve functions generate $\mathbb{C}[C_n(\Sigma)]$.
• Surface structure $\to \mathbb{C}[C_n(\Sigma)]$ Poisson algebra
(Atiyah-Bott, Goldman, Fock-Rosly...)
$\to$ quantization $\mathbb{C}_q[C_n(\Sigma)]$, skein algebra.

Fock-Goncharov framed PSL_n character variety

$$X_n = \{\text{twisted } \rho : \pi_1(\Sigma) \to PSL_n,\ \text{boundary data}\}/PSL_n.$$

Not as intrinsic as $C_n(\Sigma)$. Algebra and Geometry are simpler.
Birationally $X_n \cong (\mathbb{C}^*)^r$, a torus.
Ideal triangulation $\lambda \to$ coordinates $x_1, \dots, x_r \in \mathbb{C}^*$ for generic points of X_n. Quantization $\mathscr{X}_n(\Sigma, \lambda)$.

$$\mathrm{tr} : \mathbb{C}[C_n] \to \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$$

• $n = 2$: X_n is more or less the "enhanced" Teichmuller space, with Thurston's shear coordinates. tr expresses trace of a curve as a Laurent polynomial in shear coordinates.
Want: quantization of tr.

Kauffman bracket, Jones polynomial

$D \subset \mathbb{R}^2$: link diagram $\longrightarrow \langle D \rangle \in \mathbb{Z}[q^{\pm 1/2}]$

 $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$  (1)

 $= (-q - q^{-1})$ (2)

$\emptyset = 1.$ (3)

$\langle D \rangle$ invariant of unoriented framed links in $\mathbb{R}^3$.
Framing = continuous vector field, nowhere tangent.

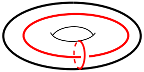
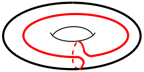
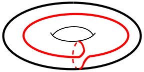
- Jones polynomial.

Reshetikhin-Turaev SL_2-quantum invariant associated to the fundamental representation.

Kauffman bracket skein algebra

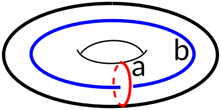
(Przytycki, Turaev, 1987)
$\Sigma = \Sigma_g \setminus \mathcal{P}$, finite set $\mathcal{P}$. Thickening $\tilde{\Sigma} = \Sigma \times (-1, 1)$.
$\mathcal{R} = \mathbb{Z}[q^{\pm 1/2}]$, or $\mathcal{R} = \mathbb{C}$, $q^{1/2} \in \mathbb{C} \setminus \{0\}$.

$\mathcal{S}(\Sigma) =$ $\mathcal{R}$-span of unoriented framed link on $\tilde{\Sigma}$ / ( $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$  ,  $= (-q - q^{-1})$, $\emptyset = 1$)

Example:  $= q^{1/2}$  $+ q^{-1/2}$ 

- **Algebra** (Turaev):

$\alpha_1 \alpha_2 =$ $\begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \end{array}$

example: $a.b =$ 

Quantization of Character variety

• $\mathcal{S}(\Sigma)$ non-comutative in general.
• Turaev, Bullock, Przytycki-Sikora,
Bullock-Frohman-Kania-Bartoszynska, Charles-Marché:
$\mathcal{S}(\Sigma)$ is the quantization of SL_2-character variety

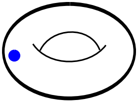
$$\mathcal{S}(\Sigma; q = \pm 1) \equiv \mathbb{C}[C_2(\Sigma)]$$

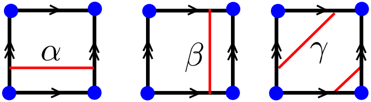
Atiyah-Bott-Weil-Petersson-Goldman bracket.

• Relations/applications:
Witten-Reshetikhin-Turaev TQFT (BHMV)
AJ conjecture. (Frohman-Gelca-Lofaro, Garoufalidis, L.)
(quantum) Teichmüller spaces, (quantum) cluster algebras
(Bonahon-Wong, Muller....), ...

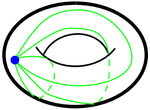
• $\mathcal{S}(\Sigma)$: Defined by geometric terms.
Algebraic structure? (Re)presentations?

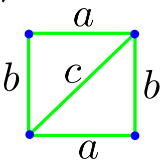
Bonahon-Wong quantum trace, punctured torus

$\Sigma = \Sigma_{1,1}$ 



• $\mathcal{S}(\Sigma)$ is generated by α, β, γ (Bullock-Przytycki).

triangulation, edges *a*, *b*, *c* 



$$\mathscr{X}(\Sigma; \lambda) := \mathcal{R}\langle a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\rangle / (ba = qab, ac = qca, cb = qbc).$$

$\mathscr{X}(\Sigma; \lambda)$: Laurent polynomials in 3 q-commuting variables.
Chekhov-Fock algebra, quantum Teichmüller space.

• BW q-trace: algebra embedding $\mathrm{tr}_q^{\lambda} : \mathcal{S}(\Sigma) \hookrightarrow \mathscr{X}(\Sigma; \lambda)$

$$\mathrm{tr}_q(\alpha) = q^{1/2}(bc + b^{-1}c^{-1} + cb^{-1}).$$

rotations: $\alpha \to \beta \to \gamma \to \alpha$ and $a \to b \to c \to a$.

Bonahon-Wong quantum trace (2010)

- Ideal triang. $\lambda = \{x_1, \ldots, x_r\}$ of $\Sigma \to$ CF algebra (Bonahon-Liu)

$$\mathscr{X}(\Sigma; \lambda) = \mathcal{R}\langle x_i^{\pm 1}, i = 1, \ldots, r\rangle / (x_i x_j = q^{\frac{1}{2} Q_{ij}} x_j x_i),$$

where $Q = Q(\Sigma, \lambda)$: $r \times r$ anti-symmetric integer matrix.

- $\mathscr{X}(\Sigma; \lambda)$ ring of Laurent polynomials in r variables, q-commuting $r = 3|e(\Sigma)| = \#$ edges. (quantum Teichmüller space).
- $\mathscr{X}(\Sigma; \lambda)$: Noetherian domain, GK dim r. Reps are known.

Theorem (Bonahon-Wong)

$\exists$ *algebra embedding, natural with respect to triangulation change*

$$\mathrm{tr}_q^{\lambda} : \mathcal{S}(\Sigma) \hookrightarrow \mathscr{X}(\Sigma; \lambda).$$

- $q = 1$, trace of curves in shear coordinates of Teichmüller space.
- tight (same GK dimension). Study $\mathscr{X}(\Sigma; \lambda)$ via $\mathscr{X}(\Sigma; \lambda)$.

Quantum trace SL_n case

(1) Sikora (2005): SL_n-skein algebra $\mathcal{S}_n(\Sigma)$. Quantization of SL_n-character variety $C_n(\Sigma)$, Atiyah-Bott Goldman bracket.

$$\mathcal{S}_n(\Sigma; q = 1) = \mathbb{C}[C_n(\Sigma)].$$

(2) Ideal triangulation λ of $\Sigma \to \mathscr{X}_n(\Sigma; \lambda)$: Fock-Gontcharov quantum framed PSL_n-character variety (2007).
a quantum torus (Laurent polynomials in q-commuting variables.)

Theorem (Main Theorem, L. & Yu)

$\exists$ *algebra homomorphism, natural with respect to trangulation change*

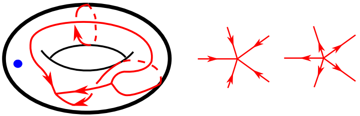
$$\mathrm{tr}_q^\lambda : \mathcal{S}_n(\Sigma) \to \mathscr{X}_n(\Sigma; \lambda).$$

(i) $q = 1$ recovers classical map of Fock-Goncharov.
(ii) $n = 2$ Bonahon-Wong map.
(iii) $n = 3$ injective.

$n = 3$ independent work of H. Kim, partial result of D. Douglas.

SL_n-skein algebra (Sikora)

n-web in $\tilde{\Sigma} = \Sigma \times (-1,1)$
n-valent sink or source, cylic order
framing (normal vector field)

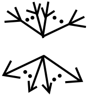
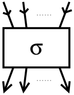


$\mathcal{S}_n(\Sigma) = \mathcal{R}$ – isotopy classes of n-webs/Rel

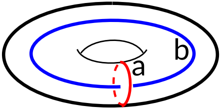
$q^{1/n}$  $- q^{-1/n}$  $= (q - q^{-1})$ 

 $= (-1)^{n-1} q^{n-1/n}$ 

 $= (-1)^{n-1}[n], \ [n] := \dfrac{q^n - q^{-n}}{q - q^{-1}}$

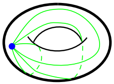
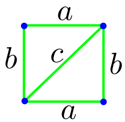
 $= (-q)^{\frac{n^2-n}{2}} \cdot \sum_{\sigma \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma)}$ 

SL_n-skein algebra

$\alpha_1\alpha_2 = \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline\end{array}$ example: $a.b =$ 

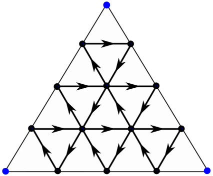
• local n-webs: generating intertwiners of SL_n fundamental representation.
Defining Relations of skein algebra: (all) relations among elementary intertwiners.
• $n = 2$, Kauffman bracket skein algebra.
• $n = 3$, Kuperberg skein algebra.
• sinks and sources are “necessary”: basic intertwiners of $U_q(sl_n)$ fundamental module.

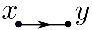
Fock Goncharov algebra (quantum higher Teichmüller theory, 2007)

λ: ideal triangulation of $\Sigma = \Sigma_g \setminus \mathcal{P}$,  

- Subdivide each triangle into n^2 small triangles. Arrows on edges.

$n = 4$





$$xy = q^{2/n^2} yx$$

- X = set of vertices (except $\mathcal{P}$) = set of q-commuting variables.

Poisson matrix $\quad Q(x, y) = \#(x \to y) - \#(y \to x)$

$$\mathscr{X}_n(\Sigma; \lambda) := \mathcal{R}\langle x^{\pm 1}, x \in X\rangle / (xy = q^{\frac{2}{n^2} Q(x,y)} yx)$$

- Quantization of X-variety (q instead of q^{1/n^2}.)

Construction of $\mathrm{tr}_q^\lambda : \mathcal{S}_n(\Sigma) \to \mathscr{X}_n(\Sigma; \lambda)$

- localize the problem, reducing to triangle case (for $n = 2$ L. 2017)
- Step 1: (Stated skein algebra, L. & Sikora 2020)

Extend $\mathcal{S}_n(\Sigma)$ to Σ having boundary. $\mathcal{S}_n(\Sigma)$ behaves well under splitting. $\exists$ algebra homomorphism

$$\Psi : \mathcal{S}_n(\Sigma) \to \bigotimes_{\tau\text{:faces of triangulation}} \mathcal{S}_n(\tau).$$

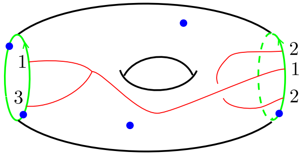
- Step 2 (L. & Yu 2020): Construct quantum trace for triangle

$$T : \mathcal{S}_n(\tau) \to \mathscr{X}_n(\tau)$$

tr_q^λ is the composition:

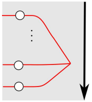
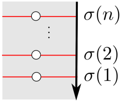
$$\mathcal{S}_n \xrightarrow{\Psi} \bigotimes \mathcal{S}_n(\tau) \xrightarrow{\otimes T} \bigotimes \mathscr{X}_n(\tau) \supset \mathscr{X}_n(\Sigma; \lambda)$$

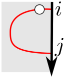
Stated skein (L.-Sikora, L. $n=2$, Higgins $n=3$)



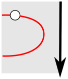
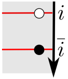
- $\partial\Sigma = \sqcup$boundary edge, each $\cong (0,1)$
- n-web α in $\tilde{\Sigma} = \Sigma \times (-1,1)$ has endpoints; distinct heights over each boundary edge.
- state of a boundary point $\in \{1,2,\dots,n\}$.

$$\mathcal{S}_n(\Sigma) = \frac{\mathcal{R}\langle \text{stated } n\text{-webs}\rangle}{\text{old, new boundary relations}}$$

 $= q^{\frac{(1-n)(2n+1)}{4}} \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}$ 



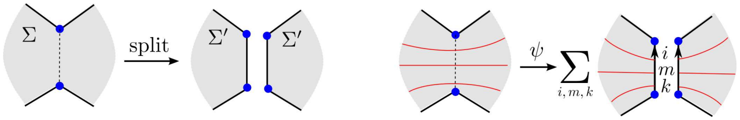
$= \delta_{\bar{j},i}\, c_i$, where $c_i = (-1)^{n-i} q^{\frac{2n^2+n-1}{2n} - i}$

 $= \sum_{i=1}^{n} (c_{\bar{i}})^{-1}$ 

where $\bar{i} = n+1-i$

Here  stands for  or .

Splitting



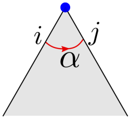
Theorem ($n = 2$, L. (2017), $n = 3$ Higgins (2020), $n \geq 2$, L.-Sikora)

ψ is algebra homomorphism $\psi : \mathcal{S}_q(\Sigma) \to \mathcal{S}_q(\Sigma')$.
injective if $\partial\Sigma \neq \emptyset$ (or $n \leq 3$).

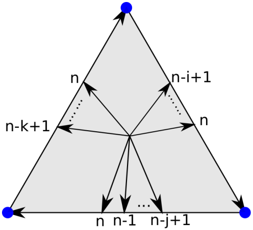
Completion of Step 1 in proof of existence of SL_n quantum trace:

$$\Psi : \mathcal{S}_n(\Sigma) \to \bigotimes_{\tau:\text{faces}} \mathcal{S}_n(\tau)$$

Reduced skein algebra; Triangle



$\overline{\mathcal{S}}_n(\Sigma) = \mathcal{S}_n(\Sigma)/(\alpha_{ij} = 0, i < j)$.
($n = 2$ Costantino &L., L.& Yu:
$\overline{\mathcal{S}}_2(\Sigma) =$ quantum cluster algebra of Muller.)
$\overline{\mathcal{S}}_n(\tau)$ contains quantum torus frame
quantum space $\subset \overline{\mathcal{S}}_n(\tau) \subset$ quantum torus



$a_{ijk}, i + j + k = n, (ijk) \neq (00n), (0n0), (n00)$

$a_K a_J = q^{P_{KJ}} a_J a_K$

$\mathcal{A}_+(\tau) = R\langle a_J \rangle / (a_K a_J = q^{P_{KJ}} a_J a_K)$

$\mathcal{A}(\tau) = R\langle a_J^{\pm 1} \rangle / (a_K a_J = q^{P_{KJ}} a_J a_K)$

$\mathcal{A}_+(\tau) \hookrightarrow \overline{\mathcal{S}}_n(\tau) \hookrightarrow \mathcal{A}(\tau)$

step 2: $\quad \mathrm{tr}_q = T : \overline{\mathcal{S}}_n(\tau) \hookrightarrow \mathcal{A}(\tau) \xrightarrow{\cong (\text{ linear })} \mathscr{X}^{\mathrm{bl}}(\tau) \subset \mathscr{X}(\tau)$

A-version, embedding

Σ: connected, no interior punctures, $\partial\Sigma \neq \emptyset$.
λ: triangulation.
$\mathrm{tr}_q : \mathcal{S}_n(\Sigma) \to \mathscr{X}_n(\Sigma, \lambda)$ is not injective.
L. & Yu introduced

- $\widetilde{\mathscr{X}_n}(\Sigma, \lambda)$: extension of the Fock-Goncharov algebra.
- $\widetilde{\mathcal{A}}_n(\Sigma, \lambda)$: a quantization of A-version character variety of Fock-Goncharov.

$$\begin{array}{ccc} & \overset{\mathrm{tr}^A}{\nearrow} & \widetilde{\mathcal{A}}_n(\Sigma, \lambda) \\ \mathcal{S}_n(\Sigma) & & \uparrow \cong \\ & \underset{\mathrm{tr}^X}{\searrow} & \widetilde{\mathscr{X}}_n^{\mathrm{bl}}(\Sigma, \lambda) \end{array}$$

$$\widetilde{\mathcal{A}}_+(\Sigma, \lambda) \subset \mathcal{S}(\Sigma) \subset \widetilde{\mathcal{A}}(\Sigma, \lambda)$$

THANK YOU!