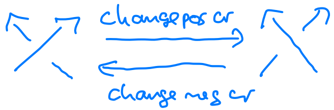
Unknotting & cobordism distances (joint w/ P. Feller)

Def: K, J knots (up to isotopy)

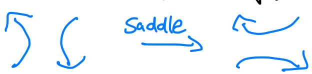
metric $d_{bu}(K,J) := \min\{n \mid K \rightsquigarrow J$ by changing at most n pos & at most n neg cr.$\}$



metric $d_{3d}(K,J) := \min\{g(\Sigma) \mid \Sigma$ 3d-cobordism $K \to J\}$

ie $\Sigma^2 \subset S^3$, $\partial\Sigma = K^{rev} \cup J$

metric $d_{1e}(K,J) := \min\{n \mid K \rightsquigarrow J$ by $2n$ saddle moves$\}$



pseudometric $d_{ee}(K,J) := \min\{g(F) \mid$ F epiepi cobordism $K \to J\}$

cobordism:
$F^2 \subset S^3 \times [0,1]$
oriented, smooth
$\partial F = F \cap S^3 \times \{0,1\}$
$= K^{rev} \times \{0\} \cup J \times \{1\}$

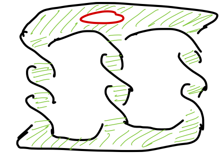
epiepi:
surjective induced maps
$\pi_1(X_{K^{rev}} = S^3 \setminus K^{rev}) \twoheadrightarrow \pi_1(X_F)$
$\pi_1(X_J) \twoheadrightarrow$

Prop.: $d_{bu} \geq d_{3d} = d_{1e} \geq d_{ee}$ (in any 3-mfd)

Thm: $\mathcal{A} := \{$knots with Alex pol $1\}$ (in any $\mathbb{Z}HS^3$). $\forall K$:

$d_{bu}(K,\mathcal{A}) = d_{3d}(K,\mathcal{A}) = d_{1e}(K,\mathcal{A}) = d_{ee}(K,\mathcal{A})$

Example for $d_{bu} > d_{3d}$:



$= K$

$u(K) = 3$ (Owens '08, using Heegaard-Floer & Jones)

$\Rightarrow d_{bu}(K, \text{unknot}) \in \{2,3\}$

But $d_{3d}(K, \text{unknot}) \leq g(K) = 1$ for all knots K

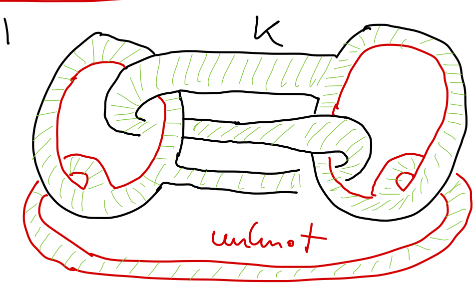
Conjecture: d_{bu}, d_{3d} not quasi-isometric

Example for $g(K) > d_{3d}(K, \text{unknot})$:

$K =$ Trefoil # -Trefoil

$g(K) = 2$

$d_{3d}(K, \text{unknot}) = 1$



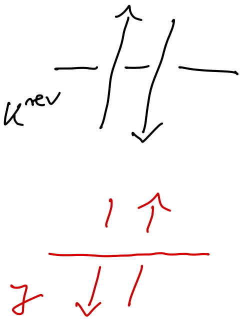
Plan of talk:

1. Proof prop
2. dee
3. Sketch Proof of Thm
4. Further results

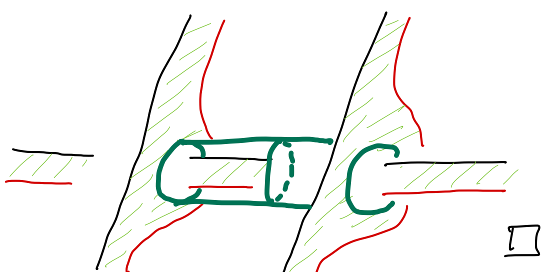
Pf $d_{bu} \geq d_{3d}$: $d_{bu}(K,J)=n \Rightarrow K \rightsquigarrow J$ by changing n pos & n neg cr.

May arrange all change to happen simultaneously.

May arrange change in pairs of one neg & one pos change each that are next to each other:



3D-cobordism $K \rightsquigarrow J$ of genus n:



$\square$

Pf of $d_{3d} = d_{1h}$: "$\geq$" Σ 3d-cobordism $K \to J$

Σ = twisted annulus with $\partial = K^{rev} \cup K$

$\cup$ $2g(\Sigma)$ 1-handles $\rightsquigarrow$ saddle

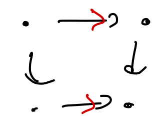
$K =$  $\xrightarrow{\text{saddle}}$ J $\square$

"$\leq$": May arrange saddles to happen simultaneously

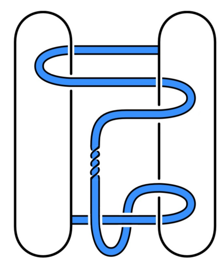
$\Sigma :=$ untwisted annulus $\cup$ 1-handle for each saddle move $\square$

Pf $d_{1h} \geq d_{ee}$: $2n$ saddle moves $K \rightsquigarrow J$ give cobordisms $K \to J$ and $J \to K$ with only 1-handles

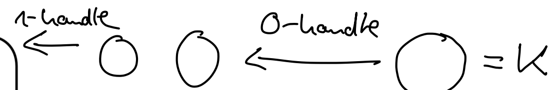
Handle attached to F	Handle attached to X_F	Effect on $\pi_1(X_F)$
0	1	new generator
1	2	new relation
2	3	no effect $\square$

Pf of triangle inequality for dee: use Seifert van Kampen,  $\square$

Example for $K \neq J$, $dee(K,J)=0$ $(< d_{1h}(K,J))$



(Meier-Livingston) 11_{n42}



genus-0-cobordism F $K \to J$

$\pi_1(X_J) \twoheadrightarrow \pi_1(X_F)$

(no 0-handles)

$\pi_1(X_K) \twoheadrightarrow \pi_1(X_F)$

(choice of 1-handle)

(in fact $\mathbb{Z} \cong \pi_1(X_K) \xrightarrow{\cong} \pi_1(X_F)$)

Proof Sketch of Thm: To show $d_{ee}(K,\mathcal{A}) \geq d_{bu}(K,\mathcal{A})$.

1. $d_{ee}(K,\mathcal{A}) = g_{\mathbb{Z}}(K) :=$ "$\mathbb{Z}$-slice genus"
$\min\{g(F) \mid F^2 \subset B^4$ $\mathbb{Z}$-slice surface, ie oriented, locally flat, $\partial F = F \cap S^3 = K$, $\pi_1(X_F) \cong \mathbb{Z}\}$

Given epiep: C from K to J, J Alex pol 1, take D $\mathbb{Z}$-slice disk for J (Freedman-Quinn), $F = C \cup D$.

2. $\exists\, W^4$, $\partial W = 0$-framed surgery of K and further properties.

3. W gives Presentation matrix for Blanchfield pairing of K.

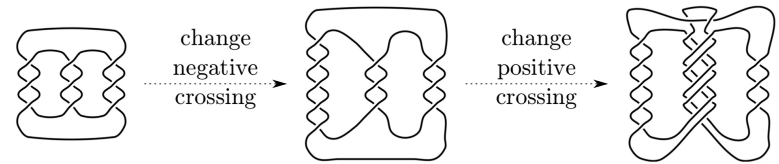
4. $\rightsquigarrow$ generalized crossing change from K to an Alex-pol-1 knot

5. generalized cr change may be replaced by usual ones without changing the S-eq. class of the target. $\square$

(in 3-5, using Borodzik-Friedl).

Corollary $g(K) = 1$

$\Rightarrow$ $K \rightsquigarrow$ knot with Alex pol 1 by two crossing change.



$$\begin{tikzcd}
\pi_1(X_K) \arrow[d, two heads] \arrow[dr, two heads, "ab"] & \\
\pi_1(X_F) \arrow[r, two heads, "ab"] & \mathbb{Z} \\
\pi_1(X_J) \arrow[u] \arrow[ur, two heads, "ab"'] &
\end{tikzcd}$$