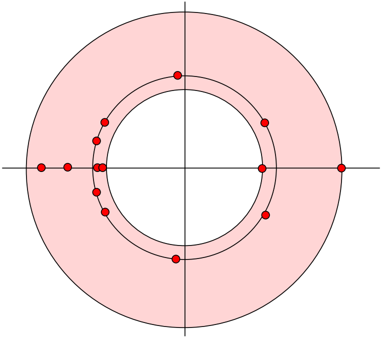
Bi-Perron numbers and the Alexander polynomial

K-OS, October 8, 2020



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I. Bi-Perron numbers

Particular geometric situations often give rise to particular algebraic numbers.

Problem

Relate the numbers to their geometry.

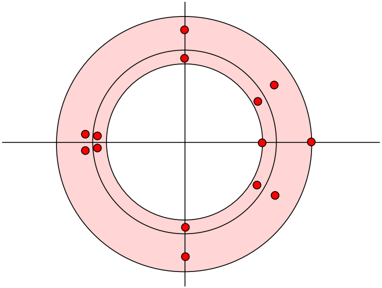
Example: coordinates of points constructed with ruler and compass are algebraic of degree a power of two.

"Cannot double the cube"

In this talk: bi-Perron numbers.

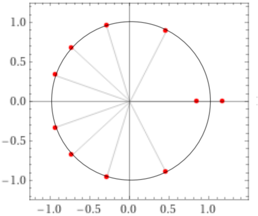
Definition

A bi-Perron number λ is a real algebraic unit $\lambda > 1$ all of whose Galois conjugates have modulus in the open interval $(1/\lambda, \lambda)$, except for λ itself and possibly one of $\pm 1/\lambda$.



Examples

1. *The golden ratio* $\varphi = \frac{1+\sqrt{5}}{2}$,
2. *the square of the golden ratio,*
3. *Salem numbers, eg. Lehmer's number with minimal polynomial* $t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$,



4. *growth rates of surface homeomorphisms (Fried '85).*

Main result

Theorem (L. – Pankau '20)

For λ bi-Perron, the following are equivalent

a) *all Galois conjugates of λ are contained in $S^1 \cup \mathbb{R}$,*

b) *for some positive integer k, λ^k is the stretch factor of a pseudo-Anosov map arising from Thurston's construction,*

c) *for some positive integer k, λ^k is the spectral radius of a bipartite Coxeter transformation of a bipartite and simple Coxeter diagram.*

Main result

Theorem (Knot-theoretic variation)

For λ bi-Perron, the following are equivalent

a) *all Galois conjugates of λ are contained in $S^1 \cup \mathbb{R}$,*

b) *for some positive integer k, the number $-\lambda^k$ is the maximal root (in modulus) of the Alexander polynomial of a link with upper triangular block 2×2 Seifert matrix with identity blocks on the diagonal.*

Problem

Find a more geometric class of links for b).

Properties of bi-Perron numbers

Lemma (Strenner '17)

Let λ be bi-Perron with minimal polynomial $(t-\lambda_1)\cdots(t-\lambda_n)$. Then λ^k is bi-Perron with minimal polynomial $(t-\lambda_1^k)\cdots(t-\lambda_n^k)$.

Lemma

Let λ be bi-Perron number. Let $\mu \notin \mathbb{R}$ be a Galois conjugate of λ. Then the argument of μ is an irrational multiple of π.

Proof.

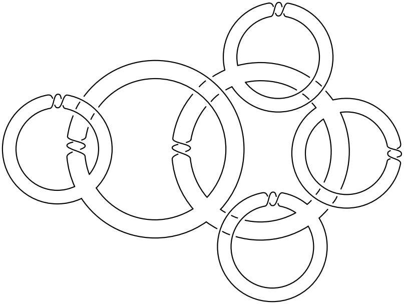
Assume $\mu \notin \mathbb{R}$ has argument $\frac{2\pi p}{q}$. Both $\mu \neq \bar{\mu}$ are Galois conjugates of λ. By the lemma, μ^q and $\bar{\mu}^q$ are different Galois conjugates of λ^q. But $\mu^q = \bar{\mu}^q$, a contradiction. □

II. Links and the Alexander polynomial

Let L be a link with upper triangular block 2×2 Seifert matrix with identity blocks on the diagonal.

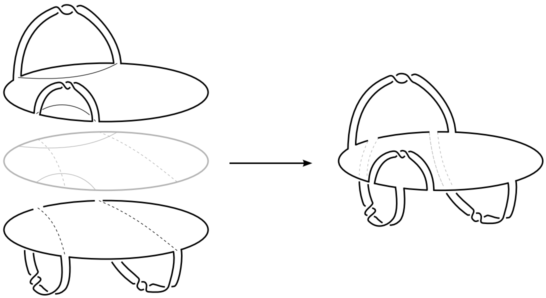
Example

Positive arborescent Hopf plumbings



Example

a Murasugi sum of two arbitrary connected sums of positive Hopf bands



The Alexander polynomial

Suppose $A = \begin{pmatrix} I_n & X \\ 0 & I_m \end{pmatrix}$ is a Seifert matrix of L.

Then $\Delta_L(t) = \det(tA^\top - A) = \det(tI - AA^{-\top}) = \chi_M(t)$, where $M = AA^{-\top}$.

Lemma

The eigenvalues λ_i of the matrix M are related to the eigenvalues μ_i of $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$ by

$$\mu_i^2 = 2 - \lambda_i - \lambda_i^{-1}.$$

Proof.

$\Omega^2 = 2I - M - M^{-1}$. □

Proposition

Let L be a link with upper triangular block 2×2 Seifert matrix with identity blocks on the diagonal. Then all zeros of $\Delta_L(t)$ are contained in $S^1 \cup \mathbb{R}_{<0}$.

Proof. (cf. A'Campo '76, McMullen '02, etc.)

The matrix $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$ is symmetric, so every μ_i is real and every μ_i^2 is positive.

In particular, every solution λ_i of $\mu_i^2 = 2 - \lambda_i - \lambda_i^{-1}$ must be contained in $S^1 \cup \mathbb{R}_{<0}$. □

Theorem

For λ bi-Perron, the following are equivalent

a) *all Galois conjugates of λ are contained in $S^1 \cup \mathbb{R}$,*

b) *for some positive integer k, the number $-\lambda^k$ is the maximal root (in modulus) of the Alexander polynomial of a link with upper triangular block 2×2 Seifert matrix with identity blocks on the diagonal.*

Proof of $b) \implies a)$:

Let L be a link with upper triangular block 2×2 Seifert matrix with identity blocks on the diagonal.

Let λ be bi-Perron such that $-\lambda^k$ is the maximal root of $\Delta_L(t)$ in modulus.

All roots of $\Delta_L(t)$ are contained in $S^1 \cup \mathbb{R}_{<0}$. So all Galois conjugates of λ^k are contained in $S^1 \cup \mathbb{R}_{>0}$.

Assume μ is a Galois conjugate of λ and $\mu \notin S^1 \cup \mathbb{R}$.
The argument of μ is an irrational multiple of π.
But μ^k is a Galois conjugate of λ^k and so in $S^1 \cup \mathbb{R}_{>0}$, a contradiction. □

III. Spectral radii of integer symmetric matrices

For $a) \implies b)$, we need the following constructive result.

Proposition (Pankau '17 for Salem numbers)

Let λ be a bi-Perron number all of whose Galois conjugates are contained in $S^1 \cup \mathbb{R}$. Then there exists a positive integer k such that $\lambda^k + \lambda^{-k}$ equals the spectral radius of a positive symmetric integer matrix.

Proof of $a) \implies b)$:

Let λ be a bi-Perron number all of whose Galois conjugates are contained in $S^1 \cup \mathbb{R}$.

Let X be a symmetric integer matrix with spectral radius $\lambda^k + \lambda^{-k}$, for some positive integer k.

Take a link with Seifert matrix $A = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$.

The eigenvalues α_i of the matrix $M = AA^{-\top}$ are related to the eigenvalues μ_i of $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$ by $\mu_i^2 = 2 - \alpha_i - \alpha_i^{-1}$.

The maximal eigenvalue μ_i^2 is $(\lambda^k + \lambda^{-k})^2 = \lambda^{2k} + \lambda^{-2k} + 2$.

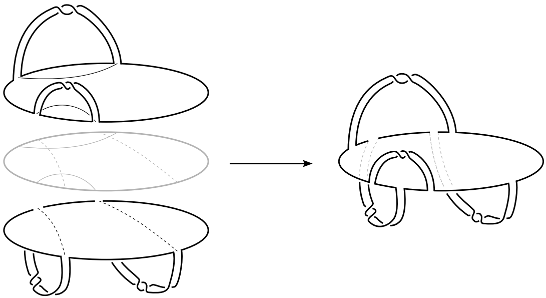
So the maximal (in modulus) α_i equals $-\lambda^{2k}$. □

A more geometric class of links?

Remark

Can take the matrix X to only have entries 0 and 1.

Fairly close to this kind of construction:



Problem

Find a more geometric class of links for b).

Merci!