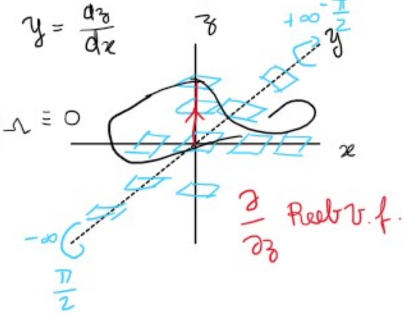
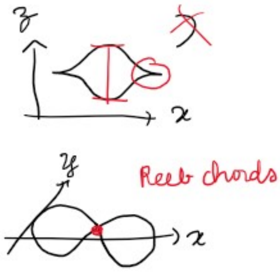
ABOUT REVERSING SURGERY IN LAGRANGIAN FILLINGS OF LEGENDRIAN KNOTS

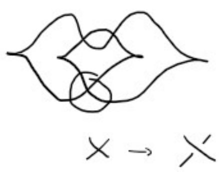
j.w/ Capovilla-Searle, Legout, Murphy, Pan & Traynor

$\Lambda \subset (\mathbb{R}^3, \xi = \ker(dz - y\,dx))$
leg.
$\Leftarrow T\Lambda \subset \xi$
$\Leftrightarrow (dz - y\,dx)|_\Lambda \equiv 0$

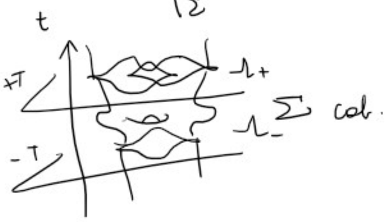
$y = \frac{dz}{dx}$

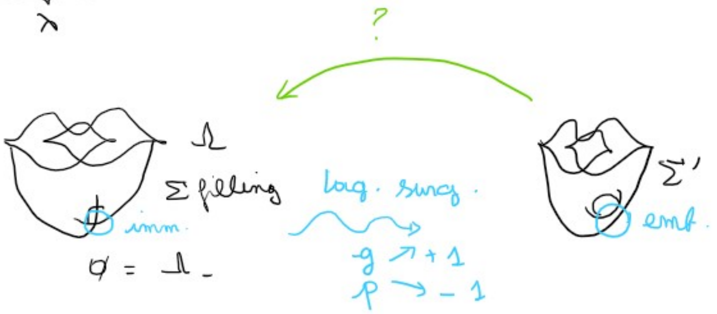






$\Sigma \subset (\mathbb{R}_t \times \mathbb{R}^3 ; \omega = d(\underbrace{e^t(dz - y\,dx)}_{\lambda}))$
Lag.
$\Leftarrow \omega|_\Sigma \equiv 0$





question: Does any Lag. filling of Λ w/ $g > 0$ come from a Lag. filling of genus $g-1$, w/ 1 double pt and Lag. surgery?

Approach by counting augmentations of the Chekanov-Eliashberg dga of Λ.

all our Lag. are exact w/ Maslov 0

Thm 1: $\forall g > 0$, $\exists \Lambda_g$ leg. knot which has a "ViA" filling Σ w/ genus g and 0 double pt, but no ViA filling Σ' w/ $g(\Sigma') = g-1$ and 1 double pt.

Thm 2: (Pan '17 + extension for links '20):
If Σ is a Lag. cob. from Λ_- to Λ_+ emb, then the map $f_\Sigma : \mathcal{A}ug_+(\Lambda_-) \longrightarrow \mathcal{A}ug_+(\Lambda_+)$ is injective on equi. classes.
i.e. $\#\{\text{augm. } \Lambda_+\}/_\sim \geq \#\{\text{augm. } \Lambda_-\}/_\sim$.

Plan: (1) Leg. knots and Lag. fillings
(2) The A_∞ - category of augmentations

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(2) The A_∞ - category of augmentations
(3) The proof of Thm 1.

(1) Leg. knots and Lag. fillings:

$\{$smooth knots$\}$ $\xleftarrow{1:\infty}$ $\{$Leg. knots$\}$ classification | classical invariants
- smooth type
- Thurston-Bennequin inv
- Rotational number.

Those are not enough to classify Leg. knots.

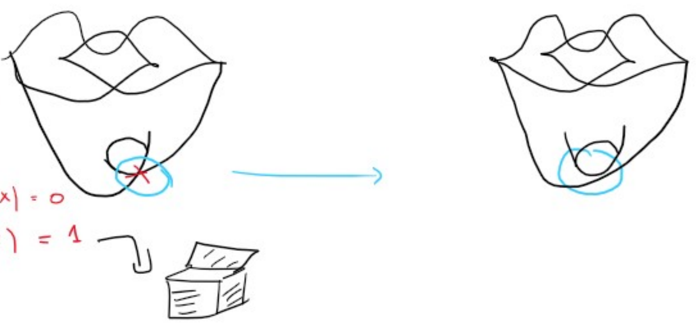
Σ lag. filling of Λ $\Rightarrow$ $g(\Sigma) = g_4(\Lambda)$.

Lagrangian surgery : ref : Polterovich '91

$\omega = d\lambda$
$\lambda_{|\Sigma} = df$

exact
Maslov 0
+
Action$(x) = 0$
$CZ(x) = 1$

. Action $(x) = f(x_1) - f(x_2)$



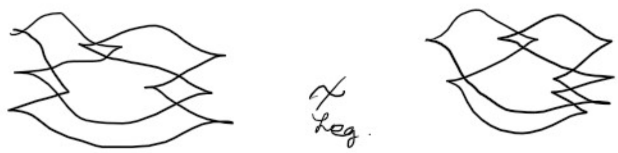
exact
Maslov 0

Def: A vanishing index and action filling / cob. is a lag. filling / cob., immersed, s.t. all imm. pts are $\pitchfork$ double pts w/ CZ index 1 and Action 0.

(2) The A_∞-category of augmentations:

Λ leg. $\rightsquigarrow$ Chekanov-Eliashberg dga : $(\mathcal{A}(\Lambda) = \mathbb{Z}_2\langle$Reeb chords$\rangle ; \partial)$

$\hookrightarrow$ distinguish :



$\not\simeq$ Leg.

Chekanov knots

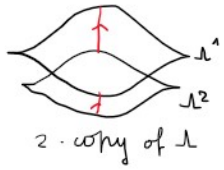
An augmentation of Λ is a dga map: $\varepsilon : (\mathcal{A}(\Lambda) ; \partial) \longrightarrow (\mathbb{Z}_2 ; 0)$

The A_∞-augmentation category of Λ: $\nearrow$ $\mathcal{A}ug_-(\Lambda)$ [Bourgeois-Chantraine '14]
$\searrow$ $\mathcal{A}ug_+(\Lambda)$ [Ng-Rutherford-Shende-Sivek-Zaslow '15]

The unital A_∞-augmentation category of Λ: $\to Aug_-(\Lambda)$ [Bourgeois-Chantraine ...]
$\to \mathcal{A}ug_+(\Lambda)$ [Ng-Rutherford-Shende-Sivek-Zaslow '15]

Objects: augm.

Morphisms: $\mathcal{H}om_+(\varepsilon^1, \varepsilon^2)$
$=$ v.s. generated by Reeb chords from Λ^2 to Λ^1



} Morse f ct w/ 1 min and 1 max per component of Λ

A_∞-operations: $m_1 : \mathcal{H}om_+(\varepsilon^1, \varepsilon^2) \to \mathcal{H}om_+(\varepsilon^1, \varepsilon^2)$
has degree $+1$ and $m_1 \circ m_1 = 0$ $\rightsquigarrow H^*\mathcal{H}om_+(\varepsilon^1, \varepsilon^2)$

$m_2 : \mathcal{H}om_+(\varepsilon^2, \varepsilon^3) \otimes \mathcal{H}om_+(\varepsilon^1, \varepsilon^2) \longrightarrow \mathcal{H}om_+(\varepsilon^1, \varepsilon^3)$
$\rightsquigarrow$ product structures on $H^*\mathcal{H}om_+$.

unit: $e_\varepsilon \in \mathcal{H}om_+^0(\varepsilon, \varepsilon)$
$=$ sum of the "minima"

Def: $\varepsilon^1, \varepsilon^2$ are equivalent if $\exists$ $[\alpha] \in H^0\mathcal{H}om_+(\varepsilon^1, \varepsilon^2)$ and $[\beta] \in H^0\mathcal{H}om_+(\varepsilon^2, \varepsilon^1)$ s.t. $m_2([\alpha][\beta]) = [e_{\varepsilon^2}]$ and $m_2([\beta],[\alpha]) = [e_{\varepsilon^1}]$.

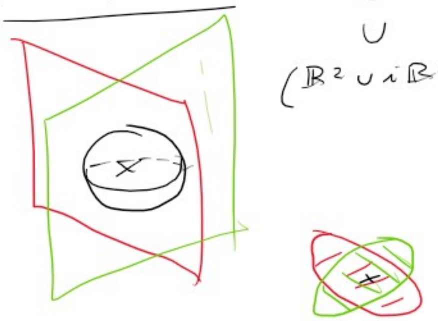
The augm. category map f_Σ:

Σ emb. Lag cob. from Λ_- to Λ_+ $\xrightarrow{\text{[Ekholm-Honda-Kálmán '12]}}$ $\mathcal{A}(\Lambda_+) \xrightarrow{\Phi_\Sigma} \mathcal{A}(\Lambda_-)$

$f_\Sigma : \mathcal{A}ug_+(\Lambda_-) \longrightarrow \mathcal{A}ug_+(\Lambda_+)$ $\xleftarrow{\text{[NRSSZ]}}$ $f_\Sigma^0 : \{\text{augm. of } \Lambda_-\} \longrightarrow \{\text{augm. of } \Lambda_+\}$
$\varepsilon_- \longmapsto \varepsilon_+ = \varepsilon_- \circ \Phi_\Sigma$

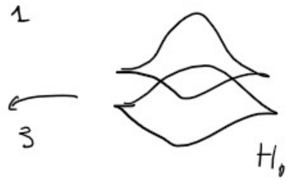
Thm 2: If $\varepsilon_+^1 = f_\Sigma(\varepsilon_-^1)$ and $\varepsilon_+^2 = f_\Sigma(\varepsilon_-^2)$ and $\varepsilon_+^1 \sim \varepsilon_+^2$, then $\varepsilon_-^1 \sim \varepsilon_-^2$.

3 Proof thm 1:



$$\begin{array}{c} \mathbb{C}^2 \\ \cup \\ (\mathbb{R}^2 \cup i\mathbb{R}^2) \cap B^4 \end{array} = \text{a VIA Lag. filling of the Hopf link}$$
w/ $g = 0$ and $p = 1$

$\#\{\text{augm.}\}/_\sim = 3$

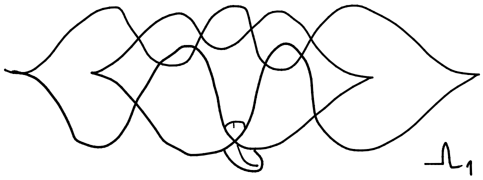


$\nexists$ a VIA filling of Λ w/ genus g and 1 double pt

$\Leftrightarrow \nexists$ a VIA filling of Λ w/ genus g and 1 double pt

$\Leftrightarrow \nexists$ a VIA cob. from H_0 to Λ w/ genus g and 0 double pt

Thm 2 $\Leftarrow$ $\#\{\text{augm. of } \Lambda\}/_\sim < 3$



Λ_1

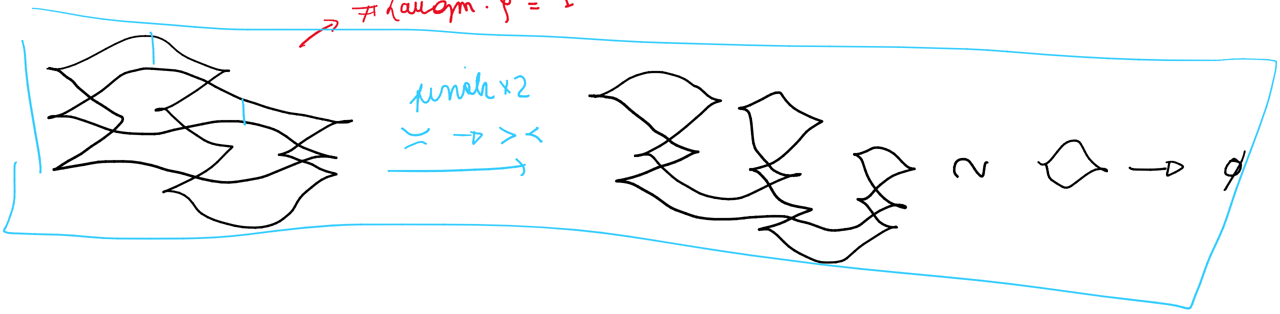
smooth type 7_4, $tb = +1$, $rot = 0$

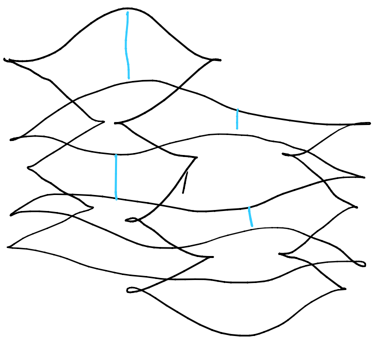
$\boxed{\#\{\text{augm}\}/_\sim = 1}$

$\boxed{\exists \text{ a VIA filling w/ } g = 1 \text{ and } p = 0}$

also: $\sim$

$\#\{\text{augm.}\} = 1$





Λ_2

etc.

Thm 1 ✓.

Rmk : Smooth world : . Paper from Owens-Strle $\longrightarrow$ The min. # double pts for a smooth disk filling of 7_4 is 2.

. talk from Peter Kronheimer : webinar of Regensburg.