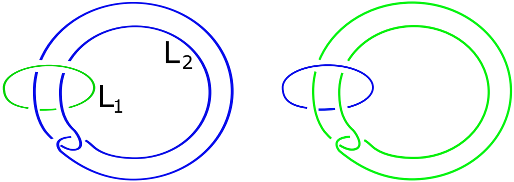
Symmetries of Knots and Links

Chuck Livingston
Indiana University

details and citations at:
Intrinsic Symmetry Groups of Links
arxiv.org/abs/2110.03502

[K-OS]
March 31, 2022

The Whitehead link: unoriented

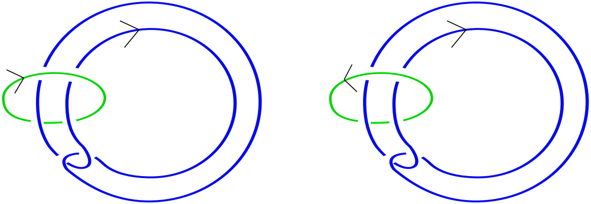


Is it possible to deform the colored link on the left to look like the link on the right?

More formally, given an ordered link (L_1, L_2), *does* $(L_1, L_2) = (L_2, L_1)$?

Problem. *Find a 2–component link* (L_1, L_2) *with unknotted components for which* $(L_1, L_2) \neq (L_2, L_1)$.

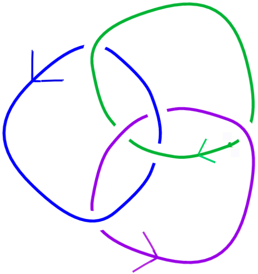
The Whitehead link: oriented



Is it possible to deform the *oriented* colored link on the left to look like the link on the right?

More formally, *given an ordered oriented link* (L_1, L_2)*, does* $(L_1^r, L_2) = (L_1, L_2)$?

The Borromean Rings



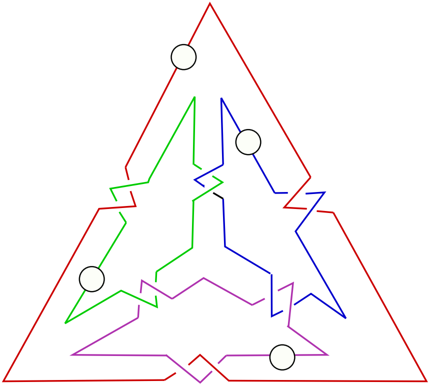
There are 6 ways to permute the colors, 8 ways to orient the components, and we can form the mirror image. Thus, we can form 96 possible "Borromean Links." **Problem.** How many are distinct?

More formally, the group $\mathbf{Z}_2 \oplus \left((\mathbf{Z}_2)^3 \rtimes S_3\right)$ *acts on the set of ordered, oriented 3–component links. What is the stabilizer of the pictured link?*

$(K, J) \to (J, K) \to (J^r, K)$ vs. $(K, J) \to (K^r, J) \to (J, K^r)$

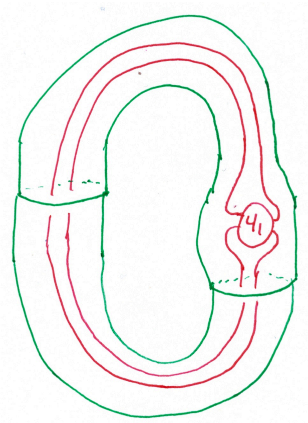
A link with symmetry group the alternating group A_4

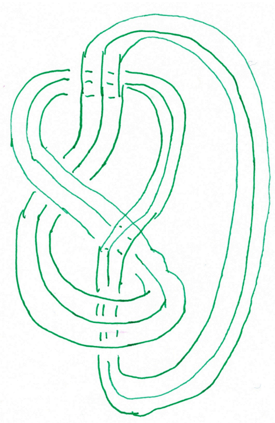
If the small circles are ignored, this (unoriented) link has symmetry group S_4. By placing appropriate knots in the circles, the symmetry group becomes A_4. Thanks to Nathan Dunfield and SnapPy for this example.



An n–component non-split link with "full" oriented symmetry group

Replace each green curve on the right with copy of the knot in solid torus on the left. (Budney found $n = 2$ case.)





Main result

1960's. Fox-Whitten defined what is now called the Whitten group:

$$\Gamma_n = \mathbf{Z}_2 \oplus \left((\mathbf{Z}_2)^n \rtimes S_n\right).$$

It acts on the set of n–component links. The intrinsic symmetry group of a link is

$$\Gamma(L) = \{g \in \Gamma_n \mid gL = L\}.$$

Question. *What subgroups of Γ_n arise as $\Gamma(L)$ for some oriented n–component link L?*

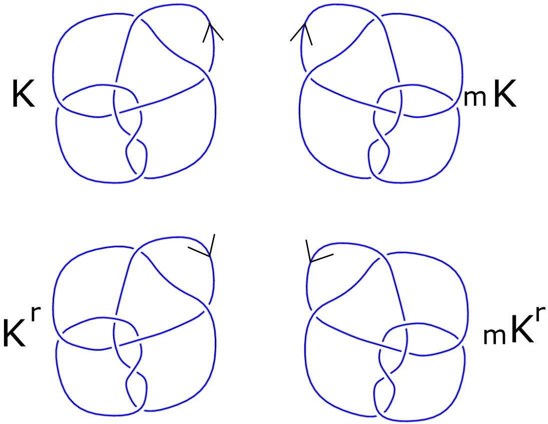
The first negative result is:

Theorem. *For $n \geq 6$, there does not exist a link with symmetry group ("projecting" onto) the alternating group A_n.*

More precisely, for all L:

$$\mathrm{Image}\left(\Gamma(L) \cap \left((\mathbf{Z}_2)^n \rtimes S_n\right) \to S_n\right) \neq A_n.$$

Basic symmetry properties of knots



$K = K^r$, Reversible

$K = mK$, Positive Amphichiral

$K = mK^r$, Negative Amphichiral

How to Prove $K \neq mK$ (Reidemeister's method)

K has a 2–fold cyclic branched cover $M_2(K)$. Suppose that $H_1(M_2) \cong Z_p$. Then there is a unique p fold cover $\widetilde{M_2}$ of M_2 in which the branch set lifts to p components. Diagrammatically,

$$\begin{array}{ccc}
(\widetilde{M_2}, \widetilde{B}) & \xrightarrow{2-fold} & (\widetilde{M'}, \widetilde{B'}) \\
\Big\downarrow {\scriptstyle p-fold\ \ cyclic} & & \Big\downarrow {\scriptstyle p-fold\ \ irregular} \\
(M_2, B) & \longrightarrow & (S^3, K)
\end{array}$$

The linking numbers (if defined) of lifts $\widetilde{B}_i$ and $\widetilde{B}_j$ can detect orientation. (*Reidemeister considered linking numbers in the p–fold irregular cover shown in red.* $\widetilde{M'}$.)

How to Prove $K \neq K^r$ (Hartley's method ~1977)

- 8_{17} has a 3–fold cyclic branched cover satisfying $H_1(M_3(K)) = (Z_{13})^2$.
- The deck transformation of M_3 acts on $(Z_{13})^2$ as a Z_{13}–vector spaces, splitting it into 3 and 9 eigenspaces, E_3 and E_9.
- Surjection $H_1(M_3) \to H_1(M_3)/E_3 \cong \mathbf{Z}_{13}$ induces a 13–fold cover.

$$\begin{array}{ccc} (\widetilde{M_3}, \widetilde{B}) & & \\ \downarrow {\scriptstyle 13-fold} & & \\ (M_3, B) & \longrightarrow & (S^3, K) \end{array}$$

- Reversing K interchanges E_3 and E_9. Thus, K and K^r can be distinguished using the homology of $\widetilde{M_3} \setminus \widetilde{B}$.

Infinite cyclic covers and twisted Alexander polynomials.

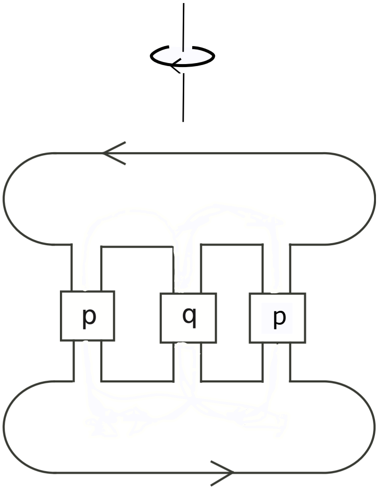
Suppose that $H_1(M_q)$ maps onto $\mathbf{Z}_p$. The surjection $H_1(S^3 \setminus K)$ induces infinite cyclic covers (deck transformation T of infinite order):

$$
\begin{array}{ccccc}
(\widetilde{M_q \setminus B})^\infty & \longrightarrow & \widetilde{M_q \setminus B} & & \\
\Big\downarrow {\scriptstyle p-fold} & & \Big\downarrow {\scriptstyle p-fold} & & \\
(M_q \setminus B)^\infty & \longrightarrow & M_q \setminus B & \longrightarrow & S^3 \setminus K
\end{array}
$$

- $H_1((M_q \setminus B)^\infty)$ is a $\mathbb{Z}[T, T^{-1}]$–module. Its order is the Alexander polynomial (after a change of variables).
- $H_1((\widetilde{M_q \setminus B})^\infty)$ is a $\mathbb{Z}[T, T^{-1}]$–module. Its order is a twisted Alexander polynomial. It provides strong constraints on reversibility, including in the setting of concordance.

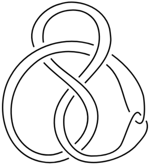
Strongly reversible knots

K is called strongly reversible if it can be reversed via an involution of S^3.



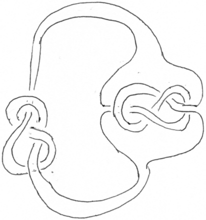
Strongly reversible knots

Theorem. (Whitten, others). A double of a knot K is reversible, but it is strongly invertible if and only if K is strongly invertible.

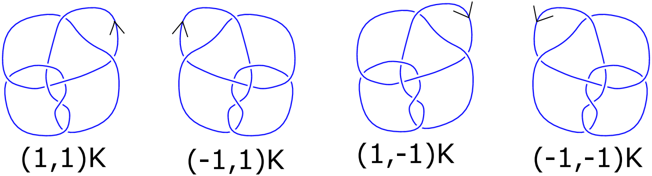


Proof. Doubled knots have unique companions (Schubert). Replace the given involution with one that preserves the separating torus.

Warning: Companions of doubled knots are (up to orientation) unique. This is not true for all companions.



Formal view of knot symmetry



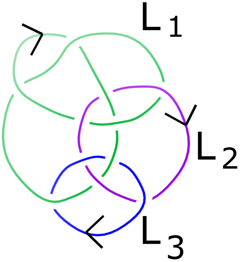
There is an action of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on the set of knots. The first factor acts by taking the mirror image. The second factor acts by reversing the string direction. For any knot K we can define the *intrinsic symmetry group* of K to be

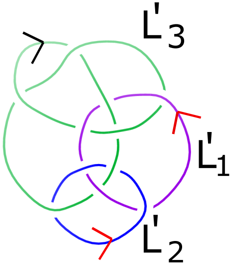
$$\Gamma(K) = \{(a,b) \in \mathbf{Z}_2 \oplus \mathbf{Z}_2 \mid (a,b)K = K\}$$

$\mathbf{Z}_2 \oplus \mathbf{Z}_2$ has three proper subgroups:
$\Gamma(K) = \langle(-1,1)\rangle$ +amphichiral
$\Gamma(K) = \langle(1,-1)\rangle$ reversible
$\Gamma(K) = \langle(-1,-1)\rangle$ −amphichiral

Link Symmetries

The Whitten group $\Gamma_n = \mathbf{Z}_2 \oplus \left((\mathbf{Z}_2)^n \rtimes S_n \right)$ acts on the set of n–component links. The first $\mathbf{Z}_2$ acts on links by taking the mirror image. S_n acts on $(\mathbf{Z}_2)^n$ by permuting the coordinates.

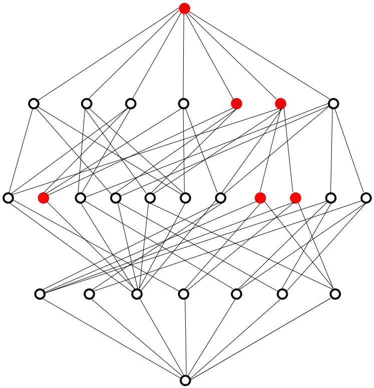




$$(1) \oplus (-1,-1,1)(123)L = (L_2^r, L_3^r, L_1).$$

$$\Gamma(K) = \{ g \in \Gamma_n \mid gK = K \}$$

Subgroups of $\Gamma_2 = \mathbf{Z}_2 \oplus \left((\mathbf{Z}_2)^2 \rtimes S_2\right)$

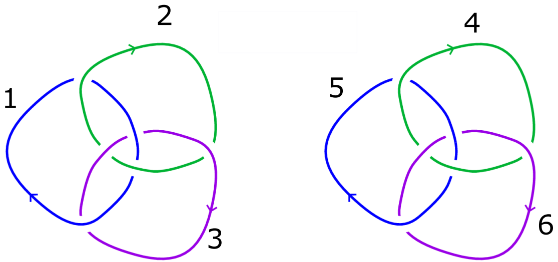


Here is the lattice of conjugacy classes of subgroups of Γ_2, taken from work of Cantarella, Cornish, Mastin, Parsley. The red dots are subgroups which do not arise for 14 crossing hyperbolic links. Some occur for general links. The subgroup $\langle(1, 1, -1)(12)\rangle$ is unknown.

Higher number of components.

Theorem. *For $n \geq 6$, there does not exist a link L with $\Gamma^*(L)$ projecting to the alternating group $A_n \subset S_n$.* ($\Gamma^*(L) \subset \Gamma(L)$ is subgroup arising from $\mathbf{Z}^n \rtimes S_n \subset \mathbf{Z}^{n+1} \rtimes S_n$ with first component 1.)

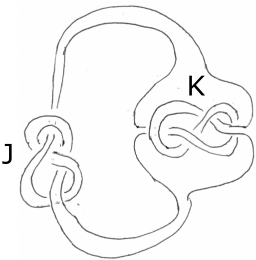
Case 1) Split links. Any symmetry preserves the non-split pieces of the link. But A_6 acts transitively on the set of pairs i, j.



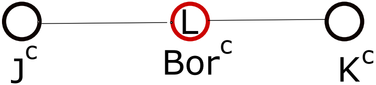
For this link there is no isotopy corresponding to $(14)(23) \in A_6$

Proof outline, ruling out A_6 in non-split case.

Case 2) A non-split link complement contains an (essentially) unique minimal set of tori $\{T_i\}$ separating it into Seifert fibered and hyperbolic pieces. (The Jaco-Shalen-Johansson decomposition. See Budney: "JSJ ... ".)

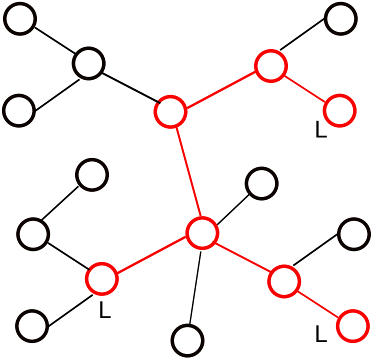


We can form a tree with vertices corresponding to components of $S^3 \setminus \{T_i\}$ and edges corresponding to tori. "Bor" denotes the Borromean link.



Proof outline: ruling out A_6 in non-split case.

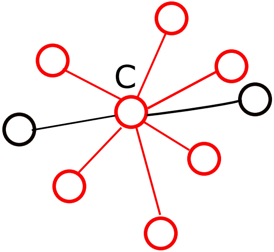
Here is a possibility for a 6 component link, with subtree spanned by the components whose closure contains elements of L. Each vertex marked "L" is a component that contains two components of L.



Ruling out A_6

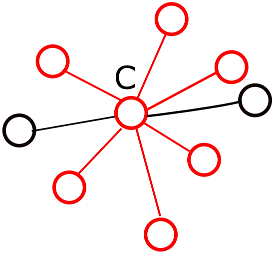
Symmetries of the link provide symmetries of the tree "spanned by the link," shown in red. In the case of A_6 one can show that some red vertex is fixed. Conclusion:

Some component C of $S^3 \setminus \{T_i\}$ has at least 6 boundary components and has symmetries that act as A_6 on 6 of those boundary components.



Ruling out A_6 Seifert fibered case

If C is Seifert fibered, it has a large set of symmetries, leading to conclusion that $\Gamma(L)$ contains a transposition, so is not A_6.



Ruling out A_6 Hyperbolic case

If C is hyperbolic, then (as observed by Budney) it reembeds into S^3 as the complement of a hyberbolic link L' (with perhaps more than 6 components).

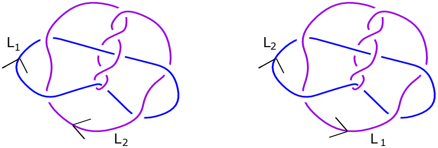
Hyperbolic geometry permits one to realize symmetries of L using a finite group of isometries. Non-free finite groups of diffeomorphisms of S^3 are isomorphic to subgroups of $SO(4)$ (Bouleau, Leeb, Porti). Such groups do not map onto A_6, as can be proved by reducing to $SO(3)$.

$$(SU(2) \times SU(2))/\langle(-1,-1)\rangle \xrightarrow{\cong} SO(4)$$

$$(SU(2) \times SU(2))/\langle(-1,-1)\rangle \xrightarrow{\text{2-fold cover}} SO(3) \times SO(3)$$

A few problems

1. Complete the determination of possible symmetry groups of 2–component links. Is there are link for which $(L_1, L_2) = (L_2, L_1^r)$ and that generates all its symmetries?



2. Do the symmetry groups that arise from links, arise from links with unknotted components? From Brunnian links? Etc.
3. Complete the analysis for hyperbolic links.
4. Is there a 5–component link with symmetry group A_5 I suspect that there is a 12–component link with symmetry group isomorphic to A_5, built from dodecahedron. A_5 acts on a set of five cubes inscribed in the dodecahedron; maybe this can be used to build a 5–component link with symmetry group A_5.