4-sided pegs fitting round holes fit all smooth holes.

Joint work with Josh Greene.

Andrew Lobb

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The square peg problem.

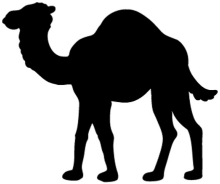
Suppose that $\gamma \subset \mathbb{R}^2 = \mathbb{C}$ is a Jordan curve (the image of an injective continuous map $S^1 \to \mathbb{R}^2 = \mathbb{C}$).

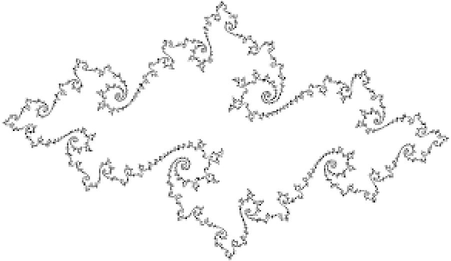
Conjecture (Toeplitz 1912)

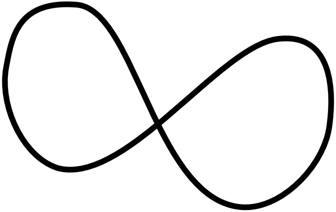
There exist four points on γ forming the vertices of a square.

Although unsolved and over a century old, this conjecture still seems rather left-field and under-motivated. But it posits a surprising connection between the topology and geometry of the Euclidean plane.

Jordan curves

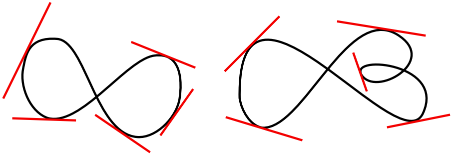






Can we deform an immersed curve to be embedded?

Suppose that we have an immersed curve $C \subset \mathbb{R}^2$. When can we deform C through immersed curves to become an embedded curve? Let us parametrize C by a smooth map $f\colon S^1 \to \mathbb{R}^2$.

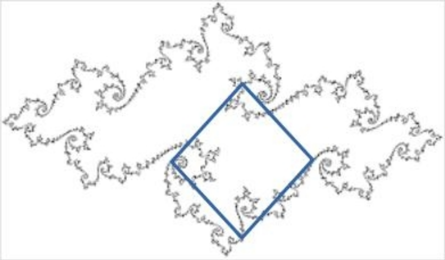


We consider the induced map $S^1 \to \mathbb{RP}^1 \colon p \mapsto f'(p)$.
After some choices this gives a map

$$\phi\colon \pi_1(S^1) = \mathbb{Z} \longrightarrow \mathbb{Z} = \pi_1(\mathbb{RP}^1).$$

This determines a non-negative integer n by $\mathrm{im}(\phi) = n\mathbb{Z}$. If C is deformable to an embedded curve then $n = 2$.

Inscribed squares.



Why ask this question?

- Each Jordan curve contains an inscribed triangle of each similarity class.
- On the other hand, dissimilar ellipses intersect in at most four points, so two dissimilar ellipses can never inscribe similar pentagons.
- Quadrilaterals are more interesting. The square is the first quadrilateral that one thinks of.

Early history.

- Emch (1913) solved the problem for smooth convex curves (a proof involving the moduli space of rhombi).
- Schnirelmann (1929) solved it for smooth Jordan curves (a bordism argument).

Since continuous Jordan curves can be approximated by smooth Jordan curves, why not attack the original problem via a limiting argument? The difficulty is that a sequence of squares may limit to a single point rather than to a square.

Vaughan's idea.

In about 1977 (see Meyerson, Balancing Acts, 1981) Vaughan considered inscribed rectangles in a continuous Jordan curve. The way he conceptualized rectangles quickly leads to an existence proof.

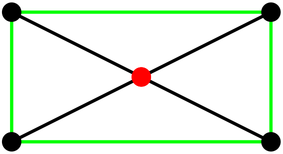
Theorem (Vaughan 1977)

All Jordan curves γ admit an inscribed rectangle.

Vaughan's first insight was to think about the diagonals of a rectangle rather than the edges.

What is a rectangle?

How do we know when the distinct points $a, b, c, d \in \mathbb{R}^2$ are the cyclically ordered vertices of a rectangle? This is exactly when the line segments *ac* and *bd* have the same length and the same midpoint.



Let us begin then by thinking about the space of *inscribed arcs* in a Jordan curve, and then try to write down what it would mean for two such arcs to give the diagonals of a rectangle.

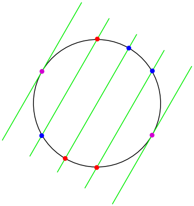
Inscribed arcs.

We define the symmetric product

$$\mathrm{Sym}^2(\gamma) := \{\{x, y\} : x, y \in \gamma\}.$$

In other words, this is pairs of unordered points on the curve γ, and so parametrizes the inscribed arcs of γ. In fact $\mathrm{Sym}^2(\gamma)$ is a Möbius band. To see this, we may as well think about the case when γ is the unit circle.

- send $\{z, w\} \in \mathrm{Sym}^2(S^1)$ to the parallelism class of [tangent] line $\overleftrightarrow{zw}$
- obtain $\mathrm{Sym}^2(S^1) \to \mathbb{RP}^1$ as an interval bundle over $\mathbb{RP}^1$



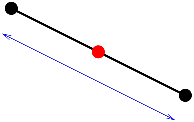
Boundary $\partial\mathrm{Sym}^2(S^1) = \{\{z, z\} : z \in S^1\} = S^1$.

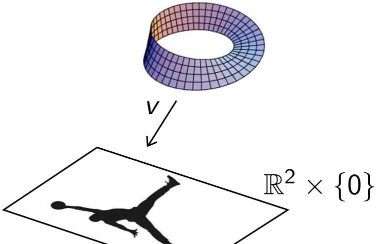
Vaughan's map of the Möbius band into 3-space.

Define a continuous map $v\colon \mathrm{Sym}^2(\gamma) \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$:

$$v\colon \{z, w\} \longmapsto \left(\frac{z + w}{2}, |z - w| \right).$$

The "midpoint, distance" map.

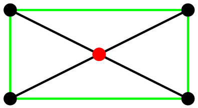




Note that $\mathrm{im}(v)$ hits $\mathbb{R}^2 \times \{0\}$ in $v(\partial \mathrm{Sym}^2(\gamma)) = \gamma \times \{0\}$.

Self-intersections correspond to inscribed rectangles.

$$v(\{z, w\}) = v(\{x, y\}) \iff$$



$$\iff \{z, w\} \text{ and } \{x, y\} \text{ span diagonals of an inscribed rectangle}$$

Principle:

$$\boxed{\{\text{inscribed rectangles in } \gamma\} \longleftrightarrow \{\text{self-intersections of } v\}}$$

A Klein bottle in $\mathbb{R}^3$.

Now reflect $\mathrm{im}(v)$ across $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$:

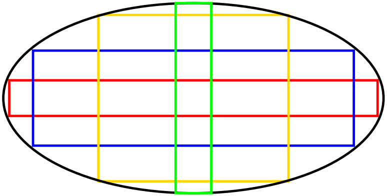
This gives a continuous map of the Klein bottle to $\mathbb{R}^3$, 1-to-1 at the join of the Möbius bands $\gamma \times \{0\}$.
Any map of the Klein bottle to $\mathbb{R}^3$ must have self-intersection, so there must exist an inscribed rectangle in γ.

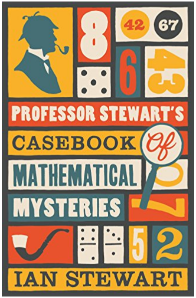
Can we do any better than just one rectangle?

Rectangular peg problem.

The rectangular peg problem

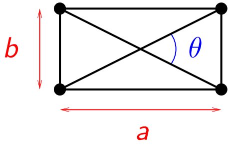
For every smooth Jordan curve and every rectangle in the Euclidean plane, do there exist four points on the curve at the vertices of a rectangle similar to the one given?





Hugelmeyer 2018-2019.

In 2018-2019, Cole Hugelmeyer recovered a new case of the rectangular peg problem.



Aspect ratio: $a/b \geq 1$.

Aspect angle: $0 < \theta \leq \pi/2$.

Theorem (Hugelmeyer 2018-2019)

Every smooth Jordan curve inscribes a rectangle of aspect ratio $\sqrt{3}$, (equivalently aspect angle $\pi/3$).

Hugelmeyer's topological proof strategy.

Consider $h\colon \mathrm{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}$,

$$h\colon \{z, w\} \longmapsto \left(\frac{z+w}{2}, (z-w)^2 \right)$$

It is a smooth embedding. The image $M = \mathrm{im}(h)$ is a Möbius band with boundary $\partial M = \gamma \times \{0\} \subset \mathbb{C} \times \mathbb{C}$. For $\phi \in \mathbb{R}$, let $R_\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ denote rotation by ϕ in the second coordinate:

$$R_\phi\colon (z, w) \mapsto (z, e^{i\phi} \cdot w).$$

$$\boxed{\left\{ \begin{array}{c} \text{inscribed rectangles in } \gamma \\ \text{with aspect angle } \phi \end{array} \right\} \longleftrightarrow \mathring{M} \cap R_{2\phi}(\mathring{M})}$$

Theorem (Greene-L 2020)

For every smooth Jordan curve and rectangle in the Euclidean plane, there exist four points on the curve that form the vertices of a rectangle similar to the one given.

Proof without details.

Define $f : \mathrm{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}$,

$$f\colon \{z, w\} \longmapsto \left(\frac{z+w}{2}, \frac{(z-w)^2}{2\sqrt{2}|z-w|} \right) \quad (z \neq w)$$

Möbius band $M = \mathrm{im}(f)$. M meets $\mathbb{C} \times \{0\}$ in $\partial M = \gamma \times \{0\}$.

$$\left\{ \begin{matrix} \text{inscribed rectangles in } \gamma \\ \text{with aspect angle } \phi \end{matrix} \right\} \longleftrightarrow \mathring{M} \cap R_{2\phi}(\mathring{M})$$

We can smooth $M \cup R_{2\phi}(M)$ nearby $\gamma \times \{0\}$ to get a smoothly mapped *Lagrangian* Klein bottle. But Shevchishin and Nemirovski showed (2007) that there does not exist a smooth, Lagrangian embedding of the Klein bottle. Hence $\mathring{M} \cap R_{2\phi}(\mathring{M})$ is non-empty so there exists an inscribed rectangle in γ of aspect angle ϕ.

The most general smooth peg problem.

The cyclic quadrilateral peg problem

For every smooth Jordan curve and every quadrilateral inscribed in the unit circle, do there exist four points on the Jordan curve at the vertices of a quadrilateral similar to the one given?

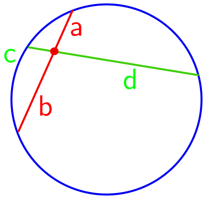
This is a question that has a negative answer in the non-smooth case. The only cyclic quadrilaterals inscribable in all triangles are the isosceles trapezoids.
It is also the most general question that one can ask in the smooth case, since the unit circle is itself a smooth Jordan curve.

Theorem (Greene-L 2020)

For every smooth Jordan curve and every quadrilateral inscribed in the unit circle, there do exist four points on the Jordan curve at the vertices of a quadrilateral similar to the one given.

A result due to Euclid.

Euclid proved a result which tells us how to recognize when quadrilaterals are cyclic.

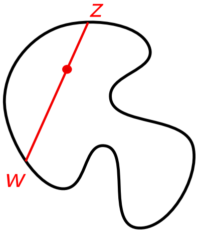


A convex quadrilateral is cyclic if and only if the lengths of the diagonal sections as above satisfy $ab = cd$.
It follows that to specific a similarity class of cyclic quadrilateral we need to give two ratios a/b and c/d and an angle ϕ that the chords make with each other.

Tori and cyclic quadrilaterals

We define a map (for $0 < r \leq 1/2$)

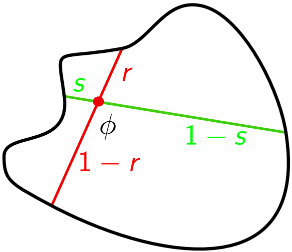
$$F_r \colon \gamma \times \gamma \hookrightarrow \mathbb{C} \times \mathbb{C} \colon (z, w) \mapsto (rz + (1-r)w, (z-w)\sqrt{r(1-r)}).$$



Note that the first coordinate gives the marked point which divides the chord in the ratio $r/(1-r)$. The second coordinate gives us the length and the direction of the chord.
We write $T_r = F_r(\gamma \times \gamma)$ for the smoothly embedded torus. Note that the degenerate chords get taken to $\gamma \times \{0\} \subset T_r \subset \mathbb{C} \times \mathbb{C}$.

Tori and cyclic quadrilaterals.

If $R_\phi\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}\times\mathbb{C}\colon (z,w)\mapsto(z,e^{i\phi}w)$, which pairs of chords correspond to intersection points between T_r and $R_\phi T_s$?



Recall that the first coordinates correspond to the points dividing up the chords in the ratios $r/(1-r)$ and $s/(1-s)$. The second coordinate gives the direction, meaning that the chords must make an angle of ϕ with each other. The second coordinate also depends on the length of the chords (and r and s) – it turns out that the pair of chords must then satisfy Euclid's criterion.

Tori and cyclic quadrilaterals.

In conclusion, intersection points between T_r and $R_\phi T_s$ correspond to inscribed cyclic quadrilaterals in γ of similarity class given by the ratios $r/(1-r)$ and $s/(1-s)$ and angle ϕ.

$$\left\{ \begin{array}{c} \text{Inscribed cyclic quadrilaterals} \\ \text{(possibly degenerate) in } \gamma \\ \text{of given ratios and angle} \end{array} \right\} \longleftrightarrow T_r \cap R_\phi T_s.$$

Enter symplectic geometry.

It turns out that T_r and $R_\phi T_s$ are Lagrangian with respect to the standard symplectic structure on $\mathbb{C}^2$,

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

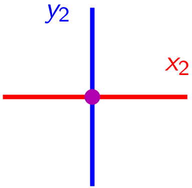
Being Lagrangian means exactly that restricting the 2-form ω to each tangent plane of T_r or $R_\phi T_s$ gives the vanishing 2-form.

Lagrangian intersection questions are often amenable to geometric techniques, including Floer homology. We will be able to lean on a result due to Polterovich and Viterbo independently.

Smoothing the circle of degenerate quadrilaterals.

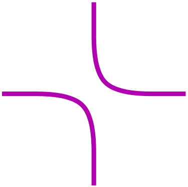
The tori T_r and $R_\phi T_s$ always have one circle of intersection $\gamma \times \{0\} \subset \mathbb{C} \times \mathbb{C}$ which corresponds to the degenerate quadrilaterals whose four vertices coincide. This is a *clean* intersection, meaning that we can smooth it away to result in a single immersed Lagrangian torus T.

Local model:



$\times S^1(x_1)$

Smoothing:



$\times S^1(x_1)$

- $\omega(\partial/\partial x_1, \partial/\partial x_2) = 0$
- $\omega(\partial/\partial x_1, \partial/\partial y_2) = 0$

The self-intersections of T correspond to non-degenerate inscribed quadrilaterals of the given similarity class.

Winding of Lagrangian 2-planes

We shall argue that T is not embedded. In fact, T is not even isotopic through Lagrangian immersions to a Lagrangian embedding.
Given a circle $S^1 \subset T$, we can consider the tangent planes to T as we travel around this S^1. We get a map (after a few choices)

$$\phi_T : \mathbb{Z} \oplus \mathbb{Z} = \pi_1(T) \longrightarrow \pi_1(\text{Lagrangian } 2-\text{planes in } \mathbb{C}^2) = \mathbb{Z}.$$

We obtain a non-negative integer n_T, an invariant of T up to isotopy through Lagrangian immersions, by

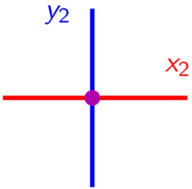
$$\text{im}(\phi_T) = n_T\mathbb{Z}.$$

Polterovich and Viterbo showed independently in around 1991 that if T were embedded then $n_T = 2$.

Winding of Lagrangian 2-planes.

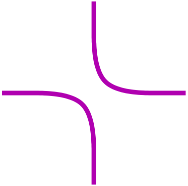
However, it is fairly straightforward to see that, in our case, we have $n_T = 4$ from how the smoothing was performed.

Local model:



$\times S^1(x_1)$

Smoothing:



$\times S^1(x_1)$

In fact T is Lagrangian isotopic to a double cover of the embedded Lagrangian torus T_r, non-trivially double covering all curves in T_r which map to $2 \in \pi_1(\text{Lagrangian } 2-\text{planes in } \mathbb{C}^2) = \mathbb{Z}$.

Conclusion.

It follows that T cannot be embedded, hence it has self-intersection. But these self-intersections must arise from intersection points of T_r and $R_\phi T_s$ corresponding to non-degenerate cyclic quadrilaterals inscribed in γ. Hence we are done.

For a detailed account of the history of the square peg problem and relatives, see the AMS Notices article by Matschke.