Exotic Brunnian Surface links

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Theme: exotic surfaces

Two surfaces S_1, S_2 in a 4-manifold X are an <u>exotic pair</u> if they are topologically isotopic but there is <u>no</u> diffeomorphism

$(X, S_1) \to (X, S_2)$ "S_1, S_2 not smoothly equivalent"

Open question:

There are non-orientable ones
Finashin-Kreck-Viro

Do there exist exotic orientable surfaces in S^4?

Next closest thing

B^4 ?

Thm (Juhasz-M-Zemke; Hayden 2020)

There exist exotic orientable surfaces in B^4.

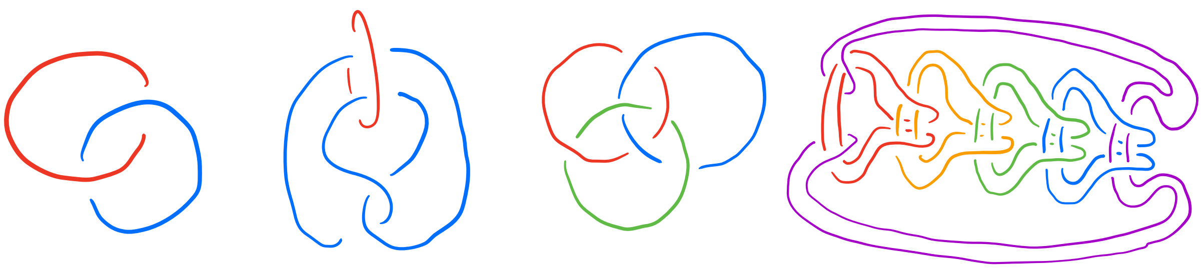
Hayden: there exist pairs of exotic disks

JMZ: there exist infinite families of pairwise exotic positive genus surfaces

Q Does there exist an infinite family of pairwise exotic disks in B^4?

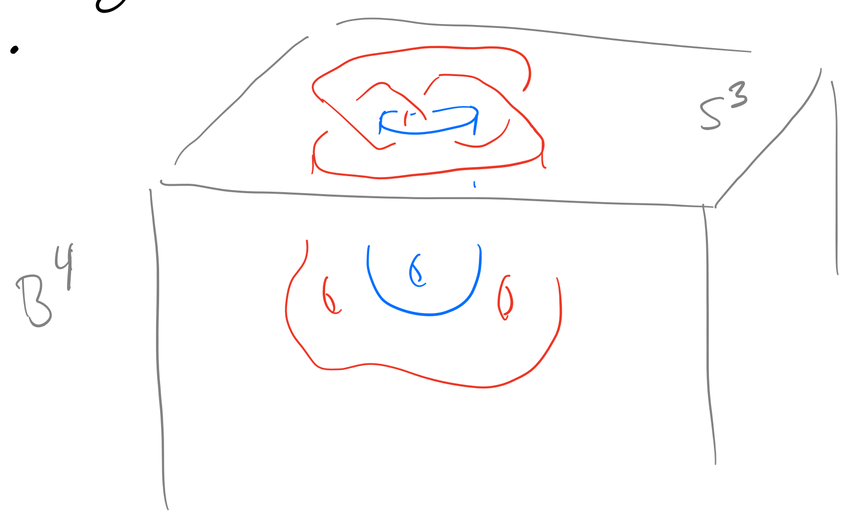
New direction: focus on multiple component surfaces and exhibit increasingly subtle forms of exotica.

<u>Def</u> A multi-component link is <u>Brunnian</u> if removing any one component yields an unlink.

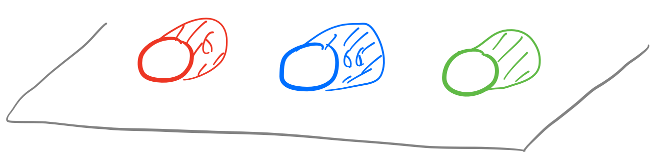


Def

A surface link is a disjoint union of connected, orientable surfaces, each with one boundary component, properly embedded in B^4.



A surface link is an unlink if it is isotopic to a Seifert surface for an unlink in S^3



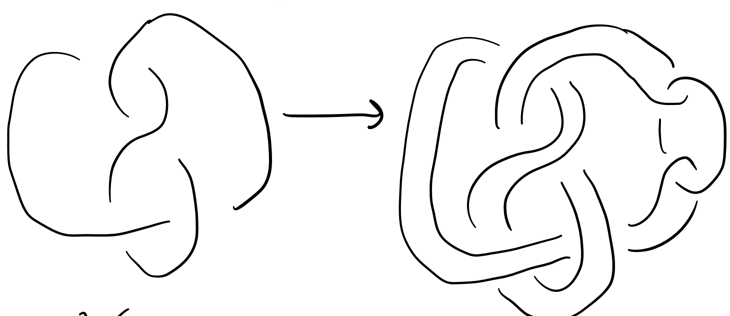
Def A multi-component surface link is **Brunnian** if removing any one component yields an unlink.

How do we produce examples?

Example:

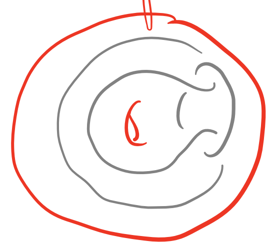
Bing double a slice disk

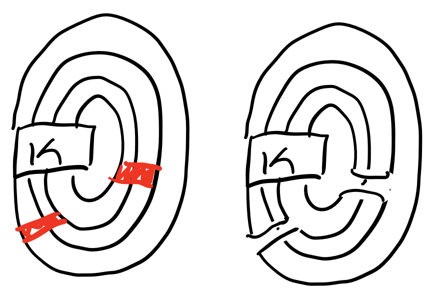
Recall Bing doubling a knot:



K BD(K)

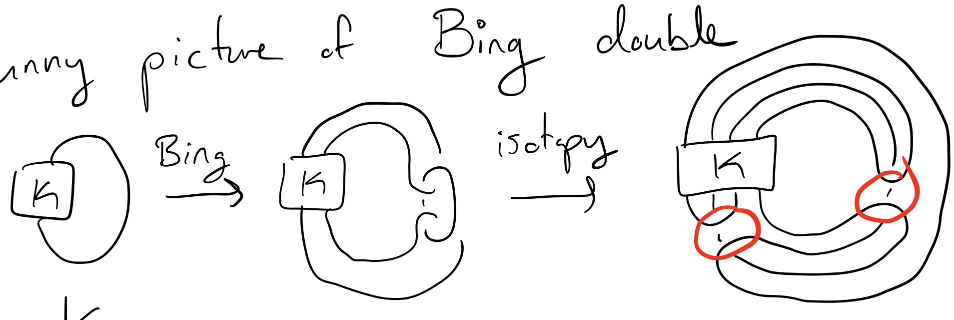
Satellite with pattern





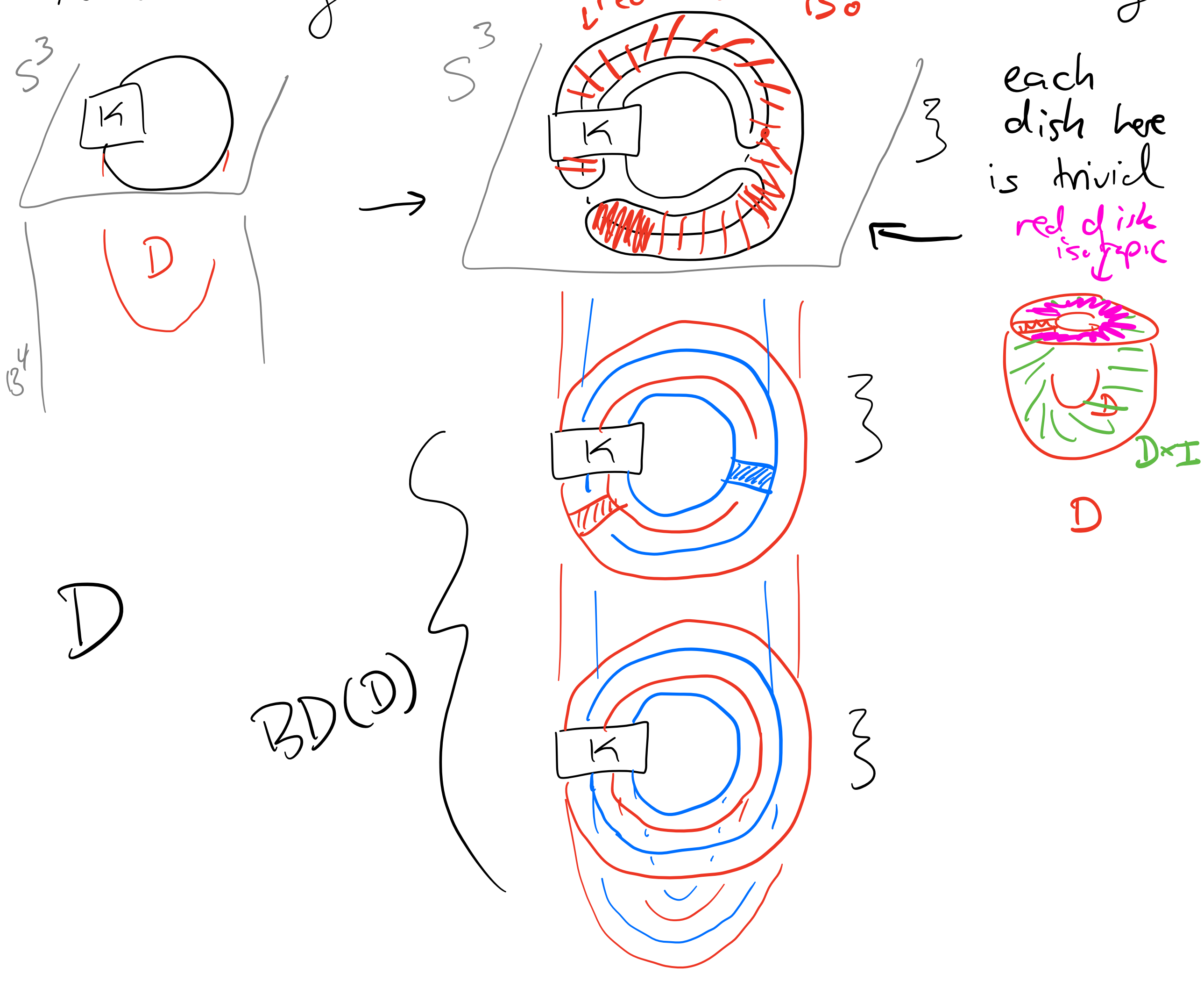
i.e. take 4 parallel K's and surger in pairs

funny picture of Bing double

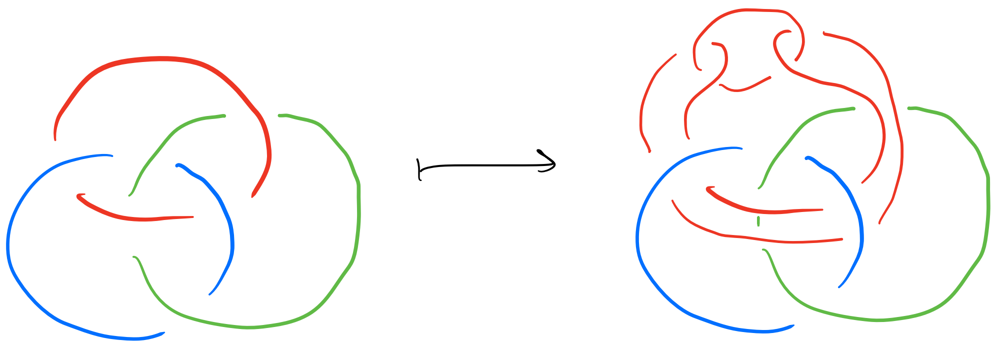


K BD(K) BD(K)

Now Bing double a disk D bounded by K:



Bing doubling one component of an n-component Brunnian link yields an (n+1)-component Brunnian link.



Thm (HKKMPS)

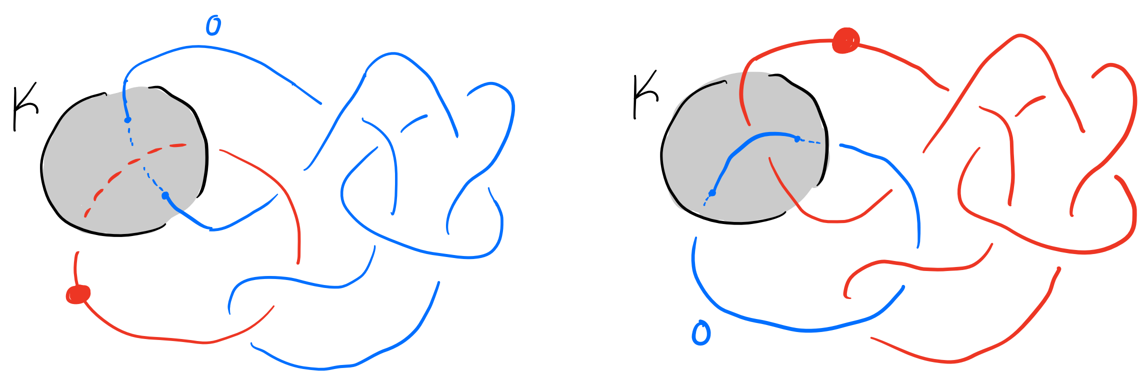
For any $n > 1$, there exist exotic orientable n-component Brunnian surface links.

1. There exist exotic pairs of n-component Brunnian disk links.
2. There exist infinite families of pairwise exotic Brunnian surface links with one positive-genus component.

<u>Q</u> Does there exist an infinite family of pairwise exotic n-component Brunnian disk links?

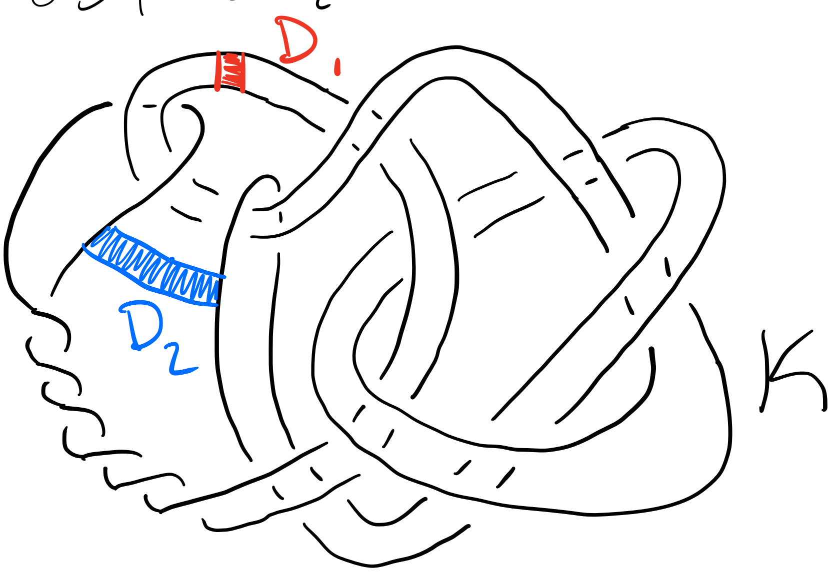
Constructing Brunnian exotic disk links

Following Hayden's construction of exotic disks:



D_1 $\qquad\qquad$ D_2

$$\partial D_1 = \partial D_2 = K$$

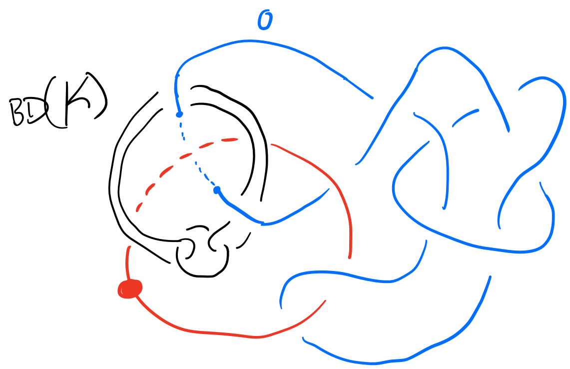


Easy computation:

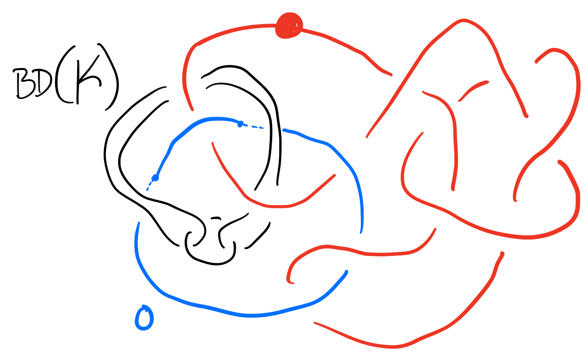
$$\pi_1(B^4 \setminus D_1) \cong \pi_1(B^4 \setminus D_2) \cong \mathbb{Z}$$

$\overset{\text{Conway-Powell}}{\Longrightarrow}$ D_1 and D_2 are topologically isotopic rel boundary.

Now Bing double D_1 and D_2.



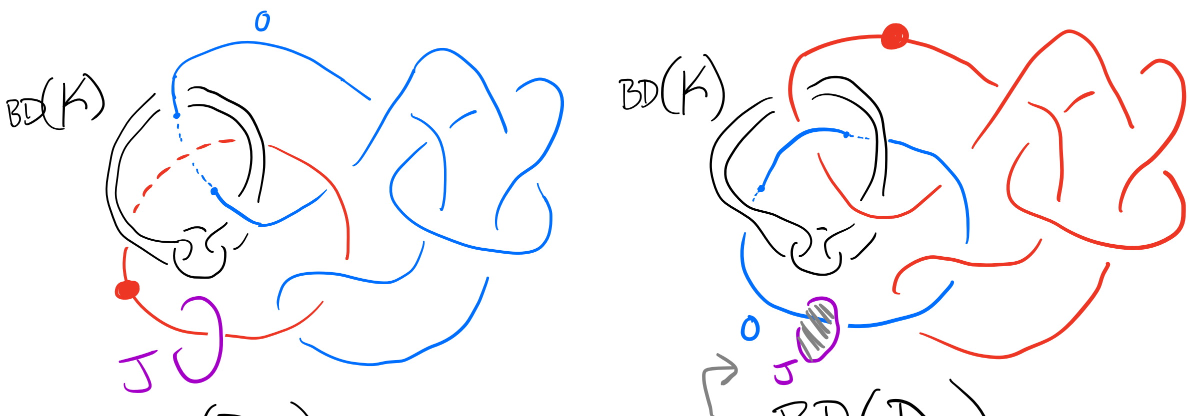
$BD(D_1)$



$BD(D_2)$

- $BD(D_i)$ is Brunnian
- Since D_1, D_2 are top iso. rel ∂, so are $BD(D_1)$ & $BD(D_2)$.

To distinguish $BD(D_i)$ $(i=1,2)$ smoothly, we use another knot $J \subset S^3$.



$BD(D_1)$ $BD(D_2)$

Observe: J is smoothly slice into $B^4 \setminus BD(D_2)$.

Fact: J is <u>not</u> smoothly slice into $B^4 \setminus BD(D_1)$

(Redraw diagram as Legendrian and use tb.)

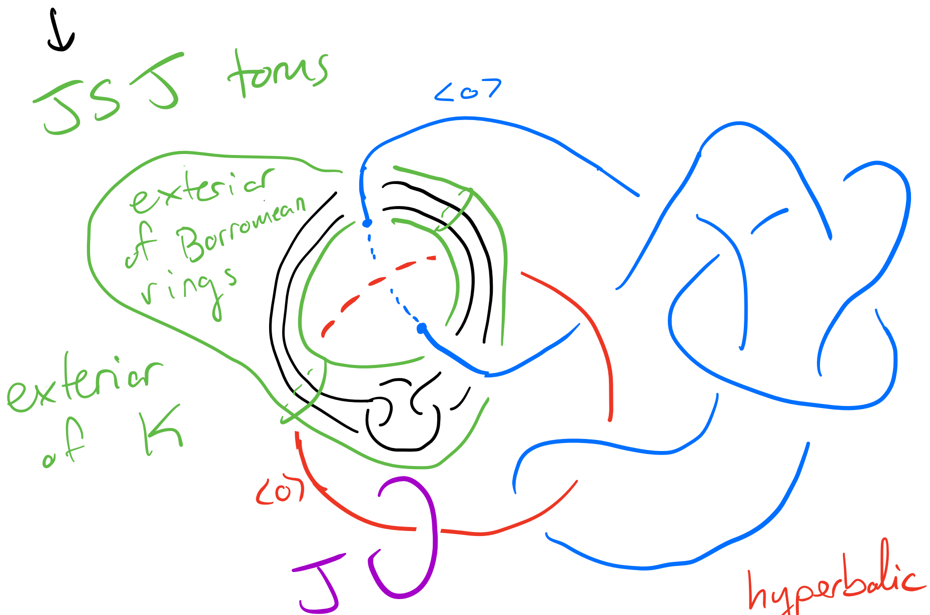
$\Rightarrow BD(D_1)$ and $BD(D_2)$ are not smoothly isotopic rel boundary.

Problem: we want to show

$\varphi : (B^4, BD(D_1)) \cong (B^4, BD(D_2))$

but to conclude this, need to know $\varphi(J) = J$ up to iso.

To obstruct smooth equivalence, sufficient to show any homeomorphism $f : (S^3, BD(K)) \circlearrowleft$ preserves J up to isotopy.



$$S^3 \setminus \nu(BD(K)) = \underbrace{\overbrace{\left(S^3 \setminus \nu(K)\right)}^{\text{hyperbolic}} \cup \left(S^3 \setminus \nu\left(\text{Borromean rings}\right)\right)}_{\text{JSJ decomposition}}$$

Uniqueness of JSJ decompositions
$\Rightarrow f$ restricts to an automorphism of $S^3 \setminus \nu(K)$

SnapPea says $\mathrm{Diff}_+(S^3 \setminus \nu(K)) = 1$
$\Rightarrow f(J) = J$ up to isotopy.

Conclude $BD(D_1)$, $BD(D_2)$ an exotic pair of 2-component Brunnian disk links.

(end of $n=2$, disk case.)

More components

Bing double the first component n times

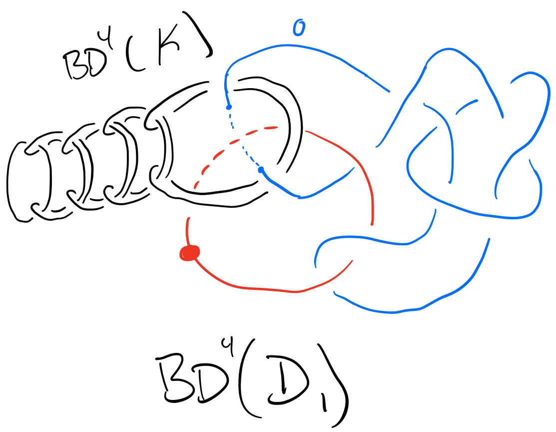
D  $\to BD$ 

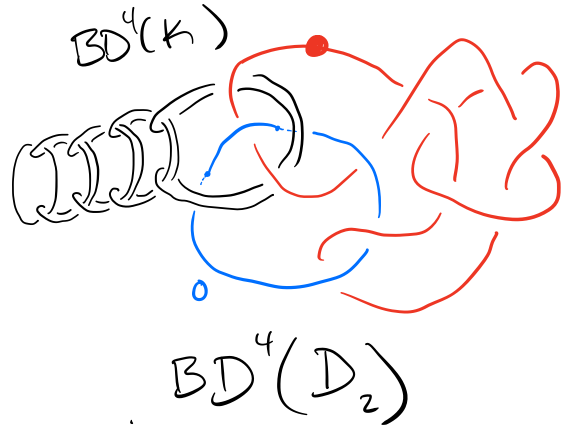
$\to BD^2(D)$ 

Thm

Let D_1, D_2 be as above.

Then $BD^n(D_1)$, $BD^n(D_2)$ are an exotic pair of Brunnian $(n+1)$-component disk links $\forall\ n > 0$.


BD^4(K)
0
BD^4(D$_1$)


BD^4(K)
0
BD^4(D$_2$)

Pf

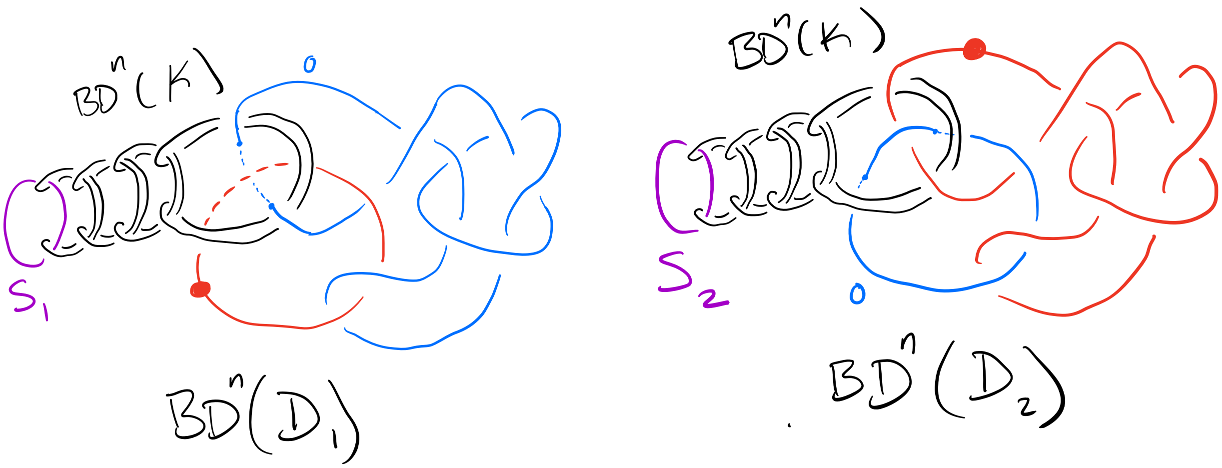
- $BD^n(D_i)$ is Brunnian
- Since D_1, D_2 are top iso. rel ∂, so are $BD^n(D_1), BD^n(D_2)$.

Obstruction to smooth equivalence via induction:

Base case: $n=1$, just showed $BD(D_1)$, $BD(D_2)$ not smoothly equivalent ✓

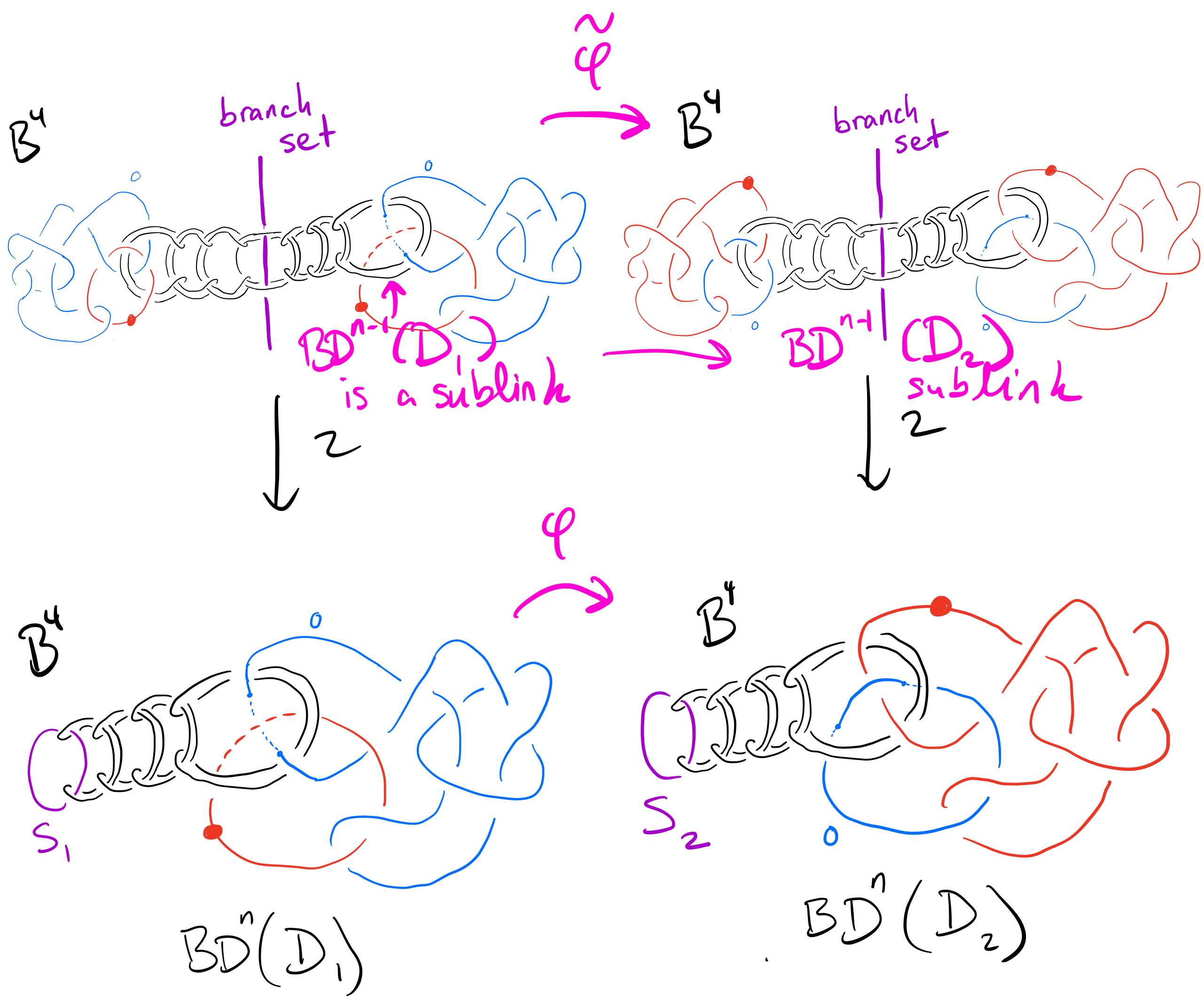
Induct: Assume $BD^{n-1}(D_1)$, $BD^{n-1}(D_2)$ are not smoothly equivalent $(n>1)$.

Let S_i be an "innermost" component of $BD^n(D_i)$.



Since S_1, S_2 are trivial disks, the double branched cover of B^4 branched over S_1 or S_2 is B^4.

Branch over S_i and consider the preimage of the rest of $BD^n(D_i)$ in the double cover.



Note $BD^{n-1}(D_i)$ is a sublink of the covering link!

So if $f : (B^4, BD^n(D_1)) \to (B^4, BD^n(D_2))$ a diffeomorphism:

WLOG $f(S_1) = S_2$ (JSJ)

$\Rightarrow f$ lifts to a diffeomorphism of the covering surface links

More JSJ stuff $\Rightarrow$ f in the case restricts to a diffeo

$$(B^4, BD^{n-1}(D_1)) \to (B^4, BD^{n-1}(D_2)),$$

contradicting inductive hypothesis. $\blacksquare$

Conclusion:

$BD^n(D_1)$, $BD^n(D_2)$ an exotic pair of $(n+1)$-component Brunnian disk links $\forall\, n > 0$.

Now let's construct an infinite family.

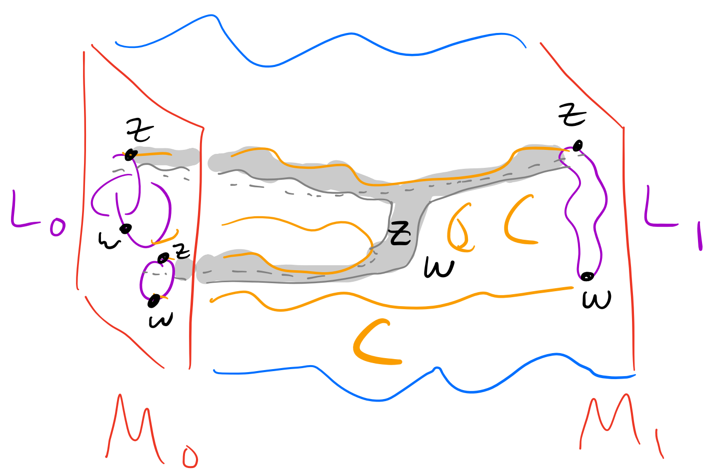
Recall:

we want to construct an infinite family $\{\Sigma_n\}_{n \in \mathbb{Z}^{\geq 0}}$ of Brunnian surface links so Σ_n, Σ_m are topologically isotopic rel ∂ but not smoothly equivalent $\forall\, n \neq m$.

So how do we obstruct <u>smooth</u> equivalence of surfaces?

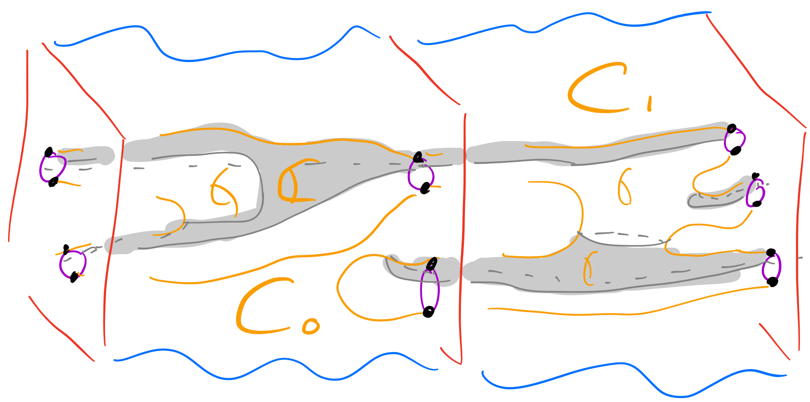
Juhász:

A (suitably decorated) cobordism C from L_0 to L_1 induces a map $F_C : \widehat{HFL}(L_0) \to \widehat{HFL}(L_1)$



These maps are functorial

Juhász: $F_{C_1 C_0} = F_{C_1} \circ F_{C_0}$



short: surface induces element of $\widehat{HFL}$ (boundary)

We can view a properly embedded surface as a cobordism C from $\emptyset$ to L.

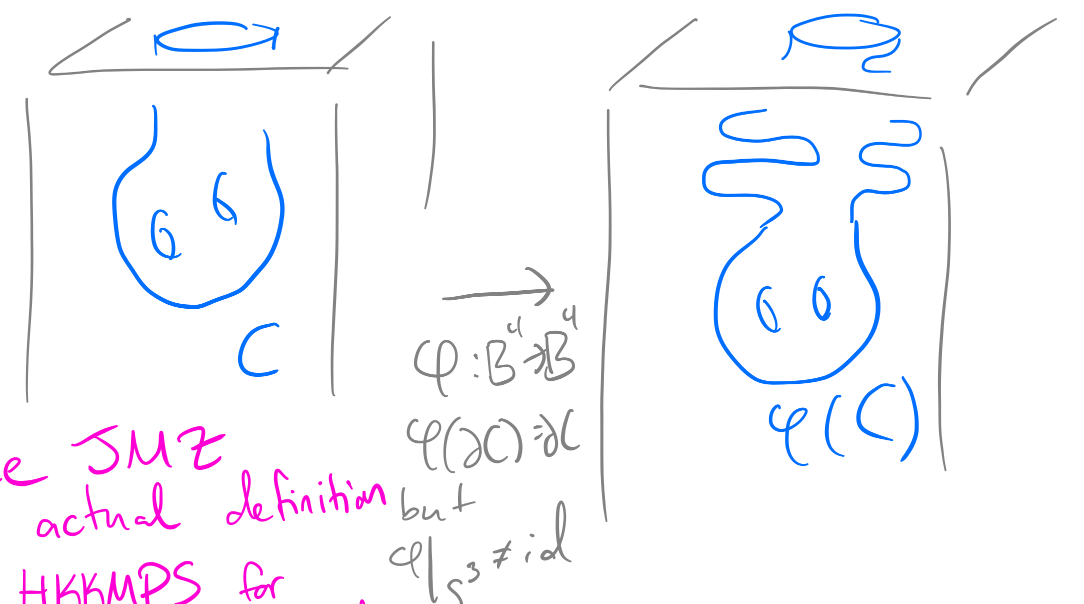
then $F_C : \mathbb{F}_2 \to \widehat{HFL}(L)$ determined by $F_C(1)$.

$F_C(1)$ is an invariant of C up to smooth isotopy rel boundary!

JMZ: Up to smooth equivalence, there is an invariant $\Omega(C)$ $\in \mathbb{Z}^{\geq 0} \cup \{-\infty\}$ well-defined.

Juhász-M-Zemke:

A diffeomorphism of ~~bandary~~ induces an automorphism of HFL, so $F_C(1)$ might not be preserved. But we can extract numerical invariant $\Omega(C)$ that is.



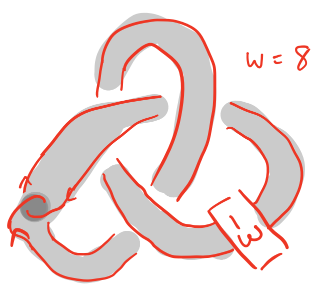
See JMZ for actual definition or HKKMPS for some shorter (but less detailed) explanation

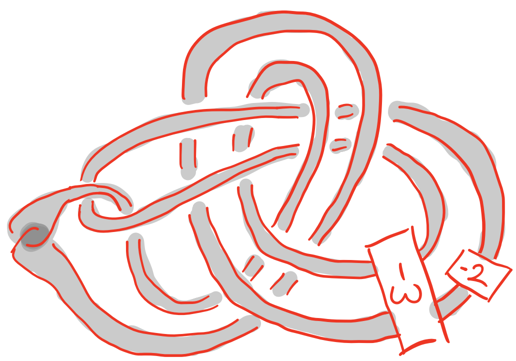
$$\Omega(C) \in \mathbb{Z}^{\geq 0} \cup \{-\infty\}$$

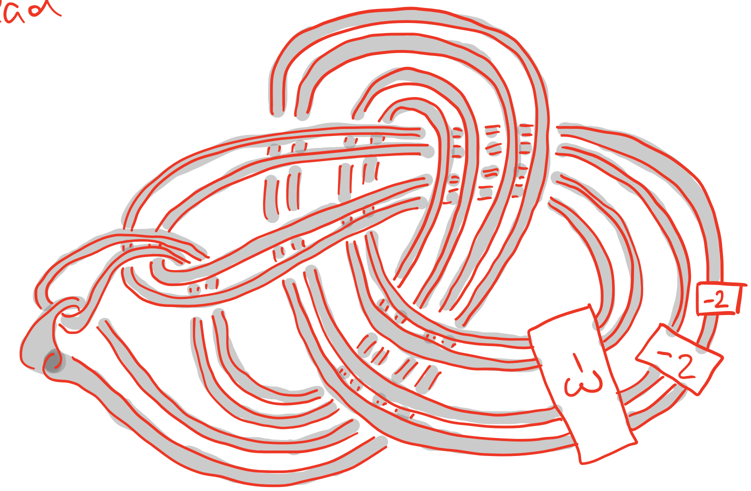
F_C and $\Omega(C)$ are hard to compute. But fact (JMZ): if C is smoothly isotopic to a strongly quasipositive Seifert surface, then $F_C(1) \neq 0$

$\Omega(C) = 0$.

e.g. genus-1 surfaces for iterated positive Whitehead doubles of a positive knot

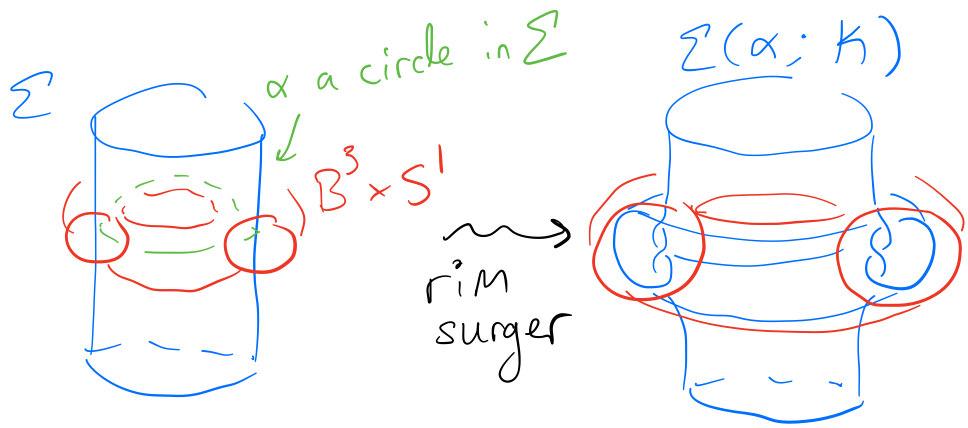






Knot Floer maps play well with rim surgery!

Rim surgery replaces $(B^3 \times S^1, I \times S^1)$
with $(B^3 \times S^1, \mathring{K} \times S^1)$
for $\mathring{K}$ a 1-stranded tangle.



Note this involves choice of not only $\alpha \subset \Sigma$ and K a knot, but also a framing of α.

Juhász-Zemke (α a non-separating curve)

$$F_{\Sigma(\alpha;K)} = \underbrace{\Delta_K(z)}_{\text{Alexander polynomial}} F_\Sigma$$

$$\Rightarrow \quad \Omega(\Sigma(\alpha;K)) = \Omega(\Sigma) + \#\text{ irreducible factors of } \Delta_K \text{ w/ multiplicity}$$

Conclude:

If $F_{B^4,\Sigma} \not\equiv 0$ and K_1, K_2 knots:

$F \xrightarrow[\text{surgery}]{\text{rim}} \Delta_K F$

- If $\Delta_{K_1} \not\doteq \Delta_{K_2}$, then $\Sigma(\alpha; K_1)$ and $\Sigma(\alpha; K_2)$ not smoothly isotopic rel boundary.

- If $\Delta_{K_1}, \Delta_{K_2}$ have different numbers of irreducible factors then $\Sigma(\alpha; K_1)$ and $\Sigma(\alpha; K_2)$ not smoothly equivalent!

$\Omega \mapsto \Omega^{+}$

factors of Δ_K

However, in certain situations rim surgery can preserve isotopy type.

Thm (JMZ, very similar to Zeeman)
If α bounds a framed (locally flat/smooth) disk into the complement of Σ, then we can arrange for $\Sigma(\alpha;K)$ to be top/smooth isotopic rel ∂ to Σ.

Here I mean that if we frame α correctly.

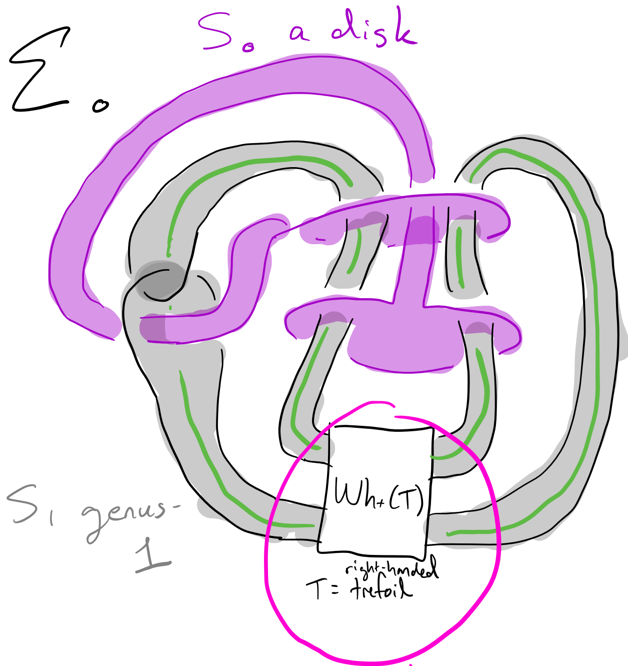
This is very similar to Zeeman's proof that 1-twist spun 2-knots are unknotted. See JMZ (or HKKMPS) for details.

Now to construct infinite family of Brunnian pairwise exotic surfaces $\{\Sigma_n\}_{n\geq 0}$ need the following:

- $\Sigma_0 = S_0 \cup S_1$ a Brunnian surface with $F_{B^4, \Sigma_0} \not\equiv 0$.

- $\alpha \subset S_1$ a nonseparating curve
  α bounds a framed
  - smooth disk into complement of S_1.
  - locally flat disk into complement of $S_0 \cup S_1$.

Then set $\Sigma_n = \Sigma_0(\alpha; \#_n(\text{Trefoil}))$.

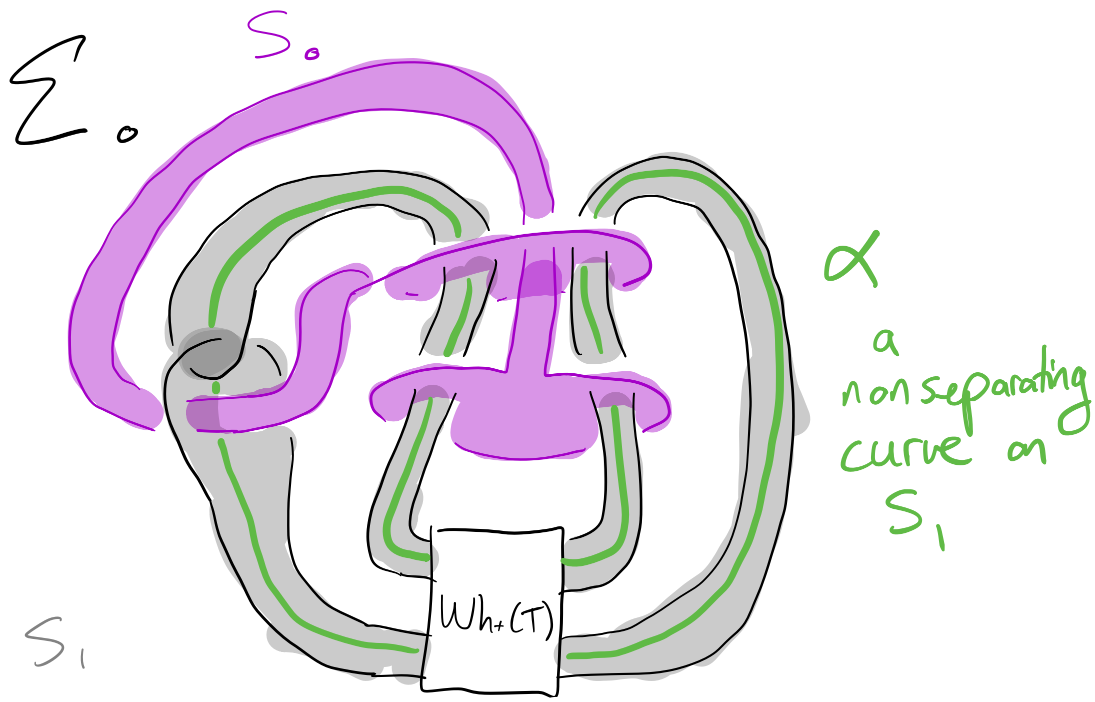


The proposed surface

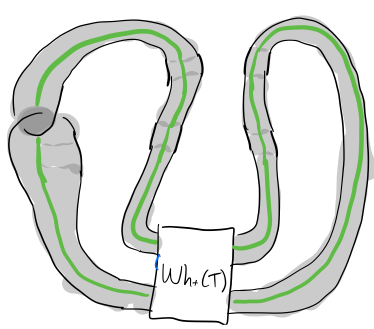
Important to use a knot here that is SQP and topologically slice, e.g. $Wh_+(T)$.

Will use this to prove $F_{B^4, \Sigma_0} \neq 0$

will use this to show α bounds a framed locally flat disk into complement of Σ_0



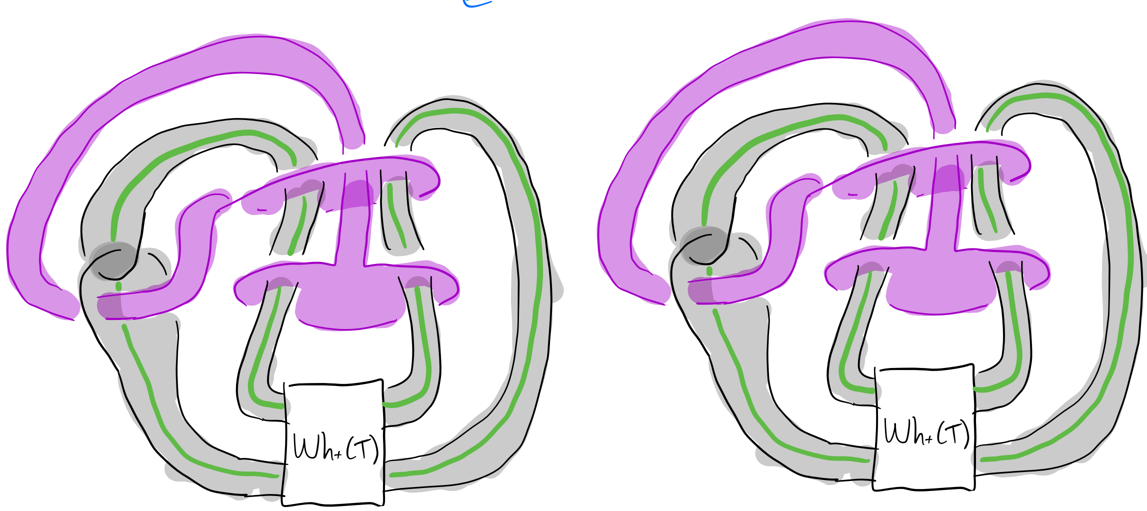
Observation: α bounds a smooth framed disk into complement of S_1



Another observation:

Since $Wh_+(T)$ is top slice, α bounds a locally flat framed disk into complement of $\Sigma = S_0 \cup S$ . .

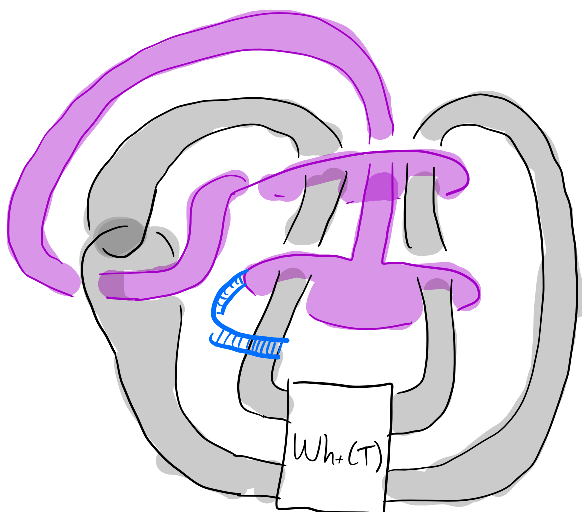
Glue these two locally flat disks together



Our Σ_0 can't be SQP since S_1 compressible, but since $\widehat{HFL} : DLink \to Vect$ factorial, if we show Σ_0 is a factor of an SQP surface we'll know $F_{B^4, \Sigma_0} \neq 0$.

Final observation:

If we glue S_0, S_1 along this band, we get SQP surface for $Wh_+(Wh_+(Wh_+(\text{Right-handed trefoil})))$.

$S :=$ 

so $F_{B^4, \Sigma_0} \neq 0$

Conclude: $\{\Sigma_n\}_{n\in\mathbb{Z}^{\geq 0}}$ are pairwise exotic 2-component Brunnian surface links.

(And previous branched covering argument shows $\{BD^k(\Sigma_n)\}_{n\in\mathbb{Z}^{\geq 0}}$ are pairwise exotic $(2+k)$-component Brunnian surface links.)

Open problems

1. Find infinite family of pairwise exotic disks in B^4

2. Find infinite family of pairwise exotic Brunnian 2-component disk links in B^4

   (Almost surely you can then extend this to $n>2$-component via Bing doubling)

3. Find an exotic pair of orientable surfaces in S^4