Automorphisms of the character variety

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April 2020

Notation and main statement

Let Σ be a closed, connected, oriented surface of genus $g \geq 1$. We define the mapping class group $\mathrm{Mod}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\,\mathrm{Diff}_0^+(\Sigma)$ and the character variety

$$X(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$$

which as a topological space is the space of equivalence classes of representations $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ where $\rho_1 \sim \rho_2 \iff \mathrm{Tr}\,\rho_1(\gamma) = \mathrm{Tr}\,\rho_2(\gamma)$ for all $\gamma \in \pi_1(\Sigma)$.

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The group $\mathrm{Mod}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma)$ acts on $X(\Sigma)$ via $\phi.[\rho] = [\rho \circ \phi^{-1}]$. This action is still not completely understood (for instance we don't know its ergodic components). The main purpose of this talk is to explain the proof of the following

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Theorem [M-S]: If $g \geq 3$, the automorphism group of the character variety is a finite extensions of $\mathsf{Mod}(\Sigma)$.

Algebraic structure

The character variety is an affine algebraic variety and by automorphism we mean automorphisms of this structure. This structure is completely described by the algebra of regular functions on $X(\Sigma)$ whose typical element is the trace function t_γ for $\gamma \in \pi_1(\Sigma)$ mapping $[\rho]$ to $\operatorname{Tr} \rho(\gamma)$.

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Theorem [Procesi '70]: The algebra of regular functions on $X(\Sigma)$ is

$$\mathbb{C}[X(\Sigma)] = \mathbb{C}[t_\gamma, \gamma \in \pi_1(\Sigma)]/(t_1 - 2, t_\gamma t_\delta - t_{\gamma\delta} - t_{\gamma\delta^{-1}}).$$

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For people wo know, this is Kauffman skein algebra at $A = -1$. So that we reformulate our statement as follows:

Theorem V2: There exists an exact sequence

$$0 \to H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{alg}}(\mathbb{C}[X(\Sigma)]) \to \mathrm{Mod}(\Sigma) \to 0.$$

where $\lambda : \pi_1(\Sigma) \to \{\pm 1\}$ maps t_γ to $\lambda(\gamma) t_\gamma$.

Context: rigidity of mapping class group actions

There is a long list of similar statements: $\mathrm{Mod}(\Sigma)$ is the group of

1. biholomorphisms of the Teichmüller space.
2. isometries of the Teichmüller space w.r.t the Weil-Petersen metric.
3. homeomorphisms of the space of measured laminations preserving the intersection pairing.
4. automorphisms of the curve complex.

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Our proof is indeed a reduction to the last statement, recalled below.

Let $\mathcal{C}$ be the graph whose vertices are isotopy classes of non-trivial simple curves and edges correspond to disjoint non-parallel curves.

Theorem [Ivanov 1990]: The automorphism group of the graph $\mathcal{C}$ is $\mathrm{Mod}(\Sigma)$.

Context: compactification of character variety

The proof consists in studying Morgan-Shalen like compactification of $X(\Sigma)$. Recall that the Teichmüller space is a component of the real part of $X(\Sigma)$ and that its most famous compactification is Thuston boundary sphere of projective measured laminations (homeomorphic to S^{6g-7}).

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The main step of the proof is to provide a natural compactification extending Thurston's such that automorphisms preserve Thurston's boundary.
My initial motivation was to compute the dual boundary complex of $X(\Sigma)$ defined as follows. Take a 'smooth' projective compactification $\overline{X}$ of $X(\Sigma)$ such that $\overline{X} \setminus X(\Sigma) = \bigcup_{i\in I} D_i$ where D_i are normal crossing divisors.
The dual boundary complex is a simplicial complex whose points are indexed by I and simplices correspond to $J \subset I$ such that $\bigcap_{j\in J} D_j \neq \emptyset$.
Conjecture: This simplicial complex is homotopic to S^{6g-7}.

Valuations

We define a set $\mathcal{V}$ of all functions $v : \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup [0, +\infty)$ satisfying for all $f, g \in \mathbb{C}[X(\Sigma)]$:

1. $v(f) = -\infty \iff f = 0$.
2. $v(fg) = v(f) + v(g)$.
3. $v(f + g) \leq \max(v(f), v(g))$.

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Replacing $\mathbb{C}[X(\Sigma)]$ with the simpler algebra $\mathbb{C}[x_1, \ldots, x_n]$ we define monomial valuations by setting $v(x_i) = v_i \in \mathbb{R}$ and

$$v\left(\sum_{\alpha} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) = \max\left\{\sum v_i \alpha_i \text{ s.t. } c_\alpha \neq 0\right\}.$$

To define an analogous notion for character varieties, we replace 'monomial' by 'multicurves'.

Simple valuations

First observe that $t_{\alpha\gamma\alpha^{-1}} = t_\gamma$ hence t_γ only depends on the free homotopy class of γ.

A *multicurve* Γ is a submanifold of Σ without components bounding a disc. For such $\Gamma = \bigcup_{i \in I} \gamma_i$ we set $t_\Gamma = \prod_{i \in I} t_{\gamma_i}$ (and $t_\emptyset = 1$).

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Idea of proof

1. Show by induction on the number of crossing points that t_Γ is a spanning set by using trace relations.
2. Use Reidemeister moves to show that the expansion of t_γ as a linear combination of t_Γ is well-defined.

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A valuation $v \in \mathcal{V}$ is said *simple* if it satisfies for all $f = \sum_\Gamma c_\Gamma t_\Gamma \in \mathbb{C}[X(\Sigma)] \setminus \{0\}$:

$$v(f) = \max\{v(t_\Gamma) \text{ s.t. } c_\Gamma \neq 0\}.$$

Measured laminations as simple valuations

Given a simple curve δ, define $v_\delta(t_\gamma) = i(\delta, \gamma)$ and extend it as

$$v(\sum_\Gamma c_\Gamma t_\Gamma) = \max\{i(\delta, \Gamma) \text{ s.t. } c_\Gamma \neq 0\}.$$

It is a non-obvious fact that this actually defines an element of $\mathcal{V}$. The proof (not the statement) appears in the preprint *Geometric intersection of curves on surfaces* by D. Thurston.

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As the condition of being simple is closed, we deduce that ML is included in the set of simple valuations. I believe that our most interesting result is the following equivalence:

Theorem M-S: The simple valuations are the one associated to measured laminations.

Untameable valuations

Now that we have seen that $\mathrm{ML} \subset \mathcal{V}$, the main step in the proof of Theorem 1 is to show that $\mathrm{Aut}(\mathbb{C}[X(\Sigma)])$ preserves ML.
This is not clear: what is clear is that it preserves the set of untameable valuations v which by definition satisfy

$$v \leq w \iff w = Cv \text{ for some } C \geq 1$$

where $v \leq w$ means $v(f) \leq w(f)$ for all $f \in \mathbb{C}[X(\Sigma)]$.

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We will prove that

1. Any untameable valuation is in ML (uses Morgan-Otal-Skora theorem, see next proposition).
2. Almost all valuations in ML are untameable.

The second statement is equivalent to a theorem of Masur stating that almost all measured laminations are uniquely ergodic.

Domination of valuations by measured laminations

Theorem : For any $v \in \mathcal{V}$ there exists a unique measured lamination $\lambda \in \mathrm{ML}$ such that $v \leq v_\lambda$ and $v(t_\gamma) = v_\lambda(t_\gamma) = i(\lambda, \gamma)$ for all $\gamma \in \pi_1(\Sigma)$.

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Sketch of proof

1. Let $K = \mathbb{C}(X(\Sigma))$ and $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(K)$ be the tautological representation satisfying $\mathrm{Tr}\,\rho(\gamma) = t_\gamma$ (Cheat) and $v : K^* \to \mathbb{R}$ defined by $v(\frac{f}{g}) = v(f) - v(g)$. The Bass-Serre tree associated to (K, v) is a real tree T_v on which $\pi_1(\Sigma)$ acts via ρ.

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2. Morgan-Otal theorem (a real version of Stallings theorem associating dual curves to action on trees) says that there exists a measured lamination λ with an equivariant contracting map $\Phi : T_\lambda \to T_v$.
   This shows that for any $\gamma \in \pi_1(\Sigma)$, the translation length of the action of γ on T_v is less than the corresponding one on T_λ. In formula

$$\max(0, 2v(t_\gamma)) \leq v_\lambda(t_\gamma) = 2i(\lambda, \gamma).$$

Preuve de la domination

3. For any $f = \sum_\Gamma c_\Gamma t_\Gamma \in \mathbb{C}[X(\Sigma)]$ we have

$$v(f) \underset{\text{valuation}}{\leq} \max\{v(t_\Gamma)\} \underset{Point2.}{\leq} \frac{1}{2}\max\{v_\lambda(t_\Gamma)\} \underset{\text{simple}}{=} \frac{1}{2} v_\lambda(f).$$

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4. If the equality $v(t_\gamma) = \frac{1}{2}v_\lambda(t_\gamma)$ does not hold, $\Phi : T_\lambda \to T_v$ is not an isometry on its image. By Skora's theorem, there exists an edge of T_v whose stabiliser contains a free subgroup $\langle \alpha, \beta \rangle$. One shows that $v(t_{[\alpha,\beta]} - 2) < 0$: a contradiction.

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Corollary:

1. If v is simple and λ is such that v coincide with v_λ on t_γ then $v = v_\lambda$ i.e. $v \in \mathrm{ML}$.
2. If v is untameable, $v \leq v_\lambda$ implies $v = Cv_\lambda$ i.e. $v \in \mathrm{ML}$.

Reduction to Ivanov's theorem

Once we know that $\mathrm{Aut}(X(\Sigma))$ preserves ML, it also preserves

$$\mathrm{ML}_{\mathbb{Z}} = \{\lambda \in \mathrm{ML}, v_\lambda(t_\gamma) = i(\lambda, \gamma) \in \mathbb{N} \quad \forall \gamma \in \pi_1(\Sigma)\}$$

which correspond to (half-weighted) multi curves. This shows that $\mathrm{Aut}\, X(\Sigma)$ acts on $\mathcal{C}$ where we identify $\gamma \in \mathcal{C}$ with v_γ.

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To apply Ivanov's theorem, it remains to prove that $\mathrm{Aut}(X(\Sigma))$ maps disjoint curves to disjoint curves. To that aim we introduce the positive valuation ring

$$\begin{aligned} \mathcal{O}_\gamma^+ &= \{f \in \mathbb{C}[X(\Sigma)] \text{ s.t. } v_\gamma(f) = 0\} \\ &= \mathrm{Im}(\mathbb{C}[X(\Sigma \setminus \gamma)] \to \mathbb{C}[X(\Sigma)]). \end{aligned}$$

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Proposition: For non-parallel simple curves γ, δ one has

$$\dim \mathcal{O}_\gamma^+ \cap \mathcal{O}_\delta^+ \leq \dim X(\Sigma) - 2$$

with equality if and only if γ and δ are disjoint.

End of the proof

Applying Ivanov's theorem, any automorphism ϕ of $X(\Sigma)$ acts on $\mathcal{C}$ as an element $\Phi \in \mathrm{Mod}(\Sigma)$. The surjection $\phi \mapsto \Phi$ gives the map on the right in the sequence

$$0 \to H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Aut}(X(\Sigma)) \to \mathrm{Mod}(\Sigma) \to 0.$$

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We are reduced to considering automorphisms ϕ such that $v_\gamma \circ \phi = v_\gamma$ for all γ simple. This implies that $\phi(t_\gamma) \in \mathbb{C}[t_\gamma]$. As ϕ is an automorphism, one has $\phi(t_\gamma) = a_\gamma t_\gamma + b_\gamma$. Checking some trace relations gives $b_\gamma = 0$ and $a_\gamma = \lambda(\gamma)$ for some morphism $\lambda : \pi_1(\Sigma) \to \{\pm 1\}$, proving the result.

Connection to the dual boundary complex

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$$x^2 + y^2 + z^2 - xyz - 2 = \lambda.$$

Compactifying in $\mathbb{P}^3$ gives as boundary the divisors $xyz = 0$ in $\mathbb{P}^2$. The dual complex is a triangle, homotopic to a circle, as the boundary of the Teichmüller space of the punctured torus.

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In the general case, the dual boundary complex is homotopic to the projectivization of the space of all valuations $v : \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup \mathbb{R}$. The conjecture would follow if one shows that this space retracts on PML.