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de TOULOUSE
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UNIVERSITÉ
TOULOUSE III
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A homological model for quantum Verma modules and braid groups representations

Jules Martel

Thursday april 30th

[K-OS] seminar

Notations for braid groups

Definition (Braid groups)

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- $$\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \middle| \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \leq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \ldots, n-2 \end{array} \right\rangle$$

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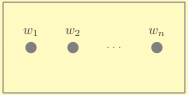

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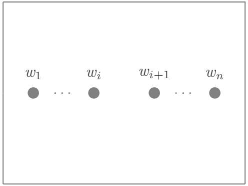

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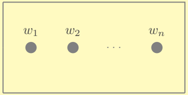

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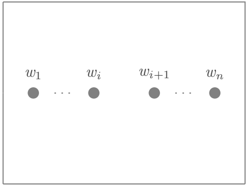
- Let D_n be the disk with n punctures, the group $\mathcal{B}_n$ is the *mapping class group* of D_n:

$$\mathcal{B}_n = Mod(D_n).$$



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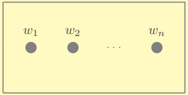

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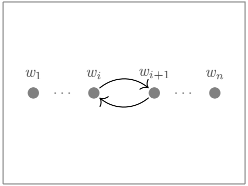
$\sigma_i \quad \longleftrightarrow \quad$ half Dehn-twist:



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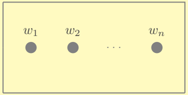

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Let $U_q\mathfrak{sl}(2)$ be the quantized enveloping algebra of $\mathfrak{sl}(2)$.

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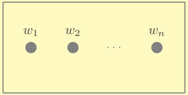

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Quantum rep. of braids ($\sim$ 1985)

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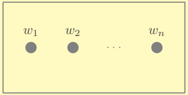

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$$\text{Quantum rep. of braids } (\sim 1985) \xrightarrow[(\sim 1990)]{\text{Reshetikhin -- Turaev}} \text{Knot invariants}$$

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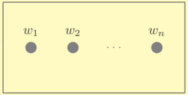

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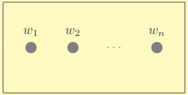

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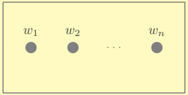

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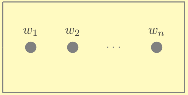

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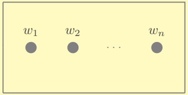

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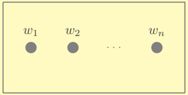

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The present work extends these relations.

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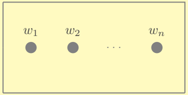
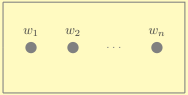


Plan:

1. Prerequisite of quantum algebra
2. Homology of configuration spaces of points
3. Structure of the homology
4. Homological representations and results

D_n :



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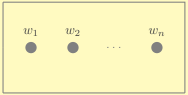

$U_q\mathfrak{sl}(2)$

Definition

$U_q\mathfrak{sl}(2)$ is the $\mathbb{Q}(q)$-algebra generated by elements E, F and $K^{\pm 1}$, satisfying the following relations:

$$KEK^{-1} = q^2E, \ \ KFK^{-1} = q^{-2}F$$

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \ \ KK^{-1} = K^{-1}K = 1.$$

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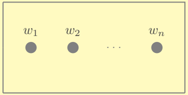

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Proposition (Hopf algebra structure)

Endowed with the coproduct structure (+ extra algebraic structure), $U_q\mathfrak{sl}(2)$ *becomes a Hopf algebra*

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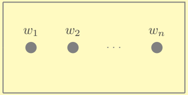

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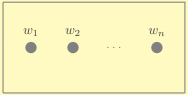

Integral version

Let $\mathcal{R}_0 = \mathbb{Z}\left[q^{\pm 1}\right]$ be the ring of integral Laurent polynomials in the variable q.

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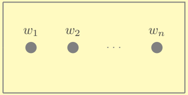

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Definition (Half integral version)

Let $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ be the $\mathcal{R}_0$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by E, $K^{\pm 1}$ and $F^{(n)}$ for $n \in \mathbb{N}^*$,

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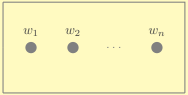

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$$F^{(n)} = \frac{(q-q^{-1})^n}{[n]_q!} F^n$$

are the *divided powers of* F.

Notations for quantum numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \text{and } [n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

Verma modules

Let $\mathcal{R}_1 = \mathbb{Z}\left[q^{\pm 1}, q^{\pm \alpha}\right]$.

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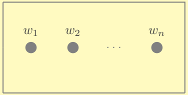

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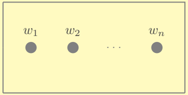

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$$K \cdot v_j = q^{\alpha - 2j} v_j \text{ and } E \cdot v_j = v_{j-1} \; (v_{-1} := 0)$$

$$F^{(n)} \cdot v_j = \left(\begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} \left(q^{\alpha-k-j} - q^{-\alpha+j+k} \right) \right) v_{j+n}.$$

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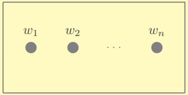

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K is diagonal, v_j is said to have <u>weight</u> $\alpha - 2j$.

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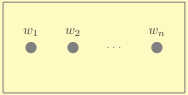

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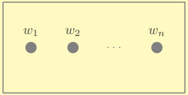

Braid representations

Proposition (Quantum braid action)

There exists a representation:

$$Q : \mathcal{R}_1\left[\mathcal{B}_n\right] \to \operatorname{End}_{\mathcal{R}_1}\left(\left(V^{\alpha}\right)^{\otimes n}\right)$$

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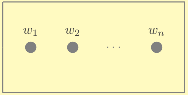

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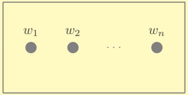

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For $r \in \mathbb{N}$, following spaces are sub-representations of $\mathcal{B}_n$.

- *"sub-weight r"*: $W_{n,r} = Ker(K - q^{n\alpha - 2r})$.
- *"highest weight"*: $Y_{n,r} = W_{n,r} \cap Ker\, E \subset W_{n,r}$.

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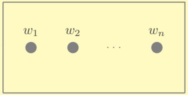

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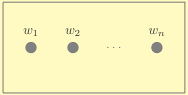

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$$\begin{tikzcd}
W_{n,r-1} \arrow[r, bend right, "F^{(1)}"'] & W_{n,r} \arrow[l, bend right, "E"'] & Y_{n,r} \arrow[l, hook']
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- *"highest weight"*: $Y_{n,r} = W_{n,r} \cap Ker\, E \subset W_{n,r}$.

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W_{n,r-1} \arrow[r, bend right, "F^{(1)}"'] & W_{n,r} \arrow[l, bend right, "E"']
\end{tikzcd}$$

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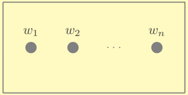

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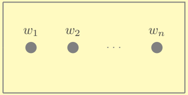

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Definition (Configuration space of points in the punctured disk)

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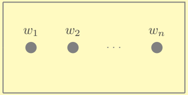

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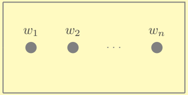

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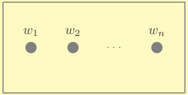

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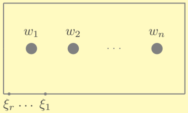

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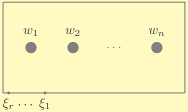

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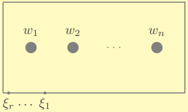

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Definition (Local system)

Let L_r be the *maximal abelian local system* associated with the Hurewicz map:

$$\rho_r : \mathbb{Z}\left[\pi_1(X_r, \boldsymbol{\xi_r})\right] \to \mathcal{R}_{\text{max}}.$$

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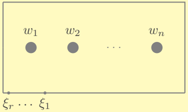

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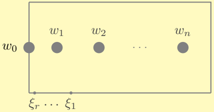

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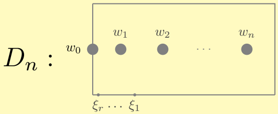

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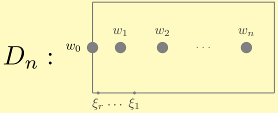

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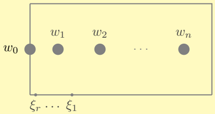

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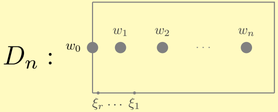

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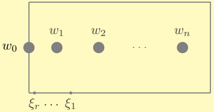

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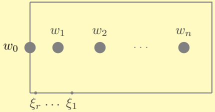

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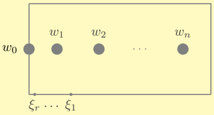

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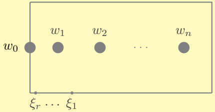

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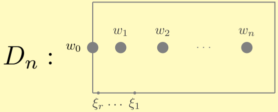

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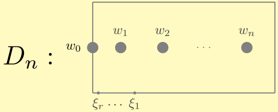

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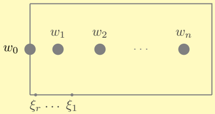

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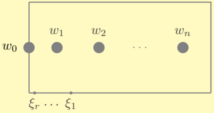

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Theorems (History of Lawrence's representations)

- *The modules* $\mathcal{H}_r^{abs}$ *are representations of* $\mathcal{B}_n$ *(Lawrence, 1990)*
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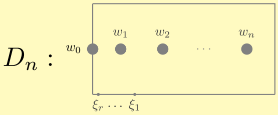

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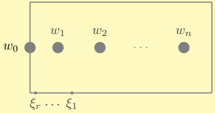

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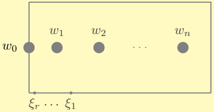

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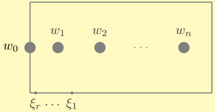

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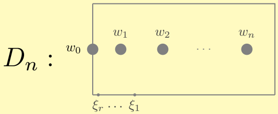

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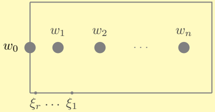
Colored version for Lawrence representations (arXiv:2004.00977)

This preprint provides matrices for colored version of Lawrence representations.

$D_n :$



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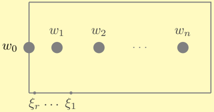

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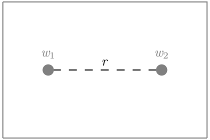

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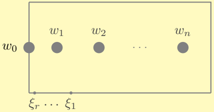

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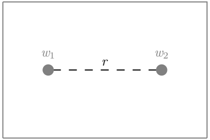

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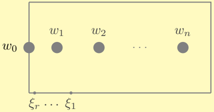

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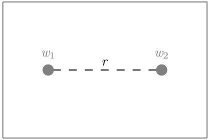

$\in X_r$

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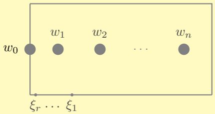

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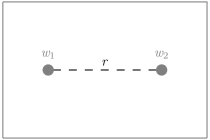

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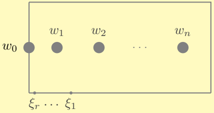

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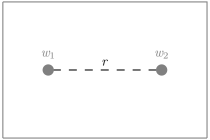

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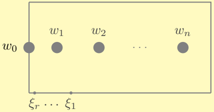

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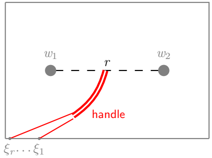

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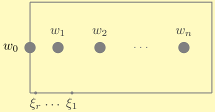

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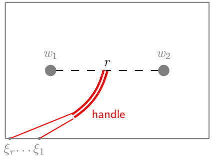

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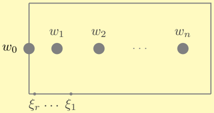

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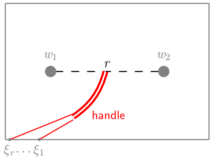

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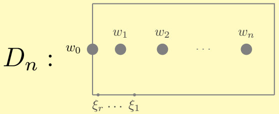

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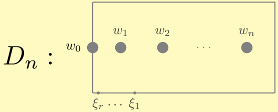

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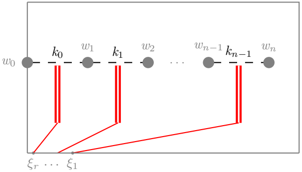
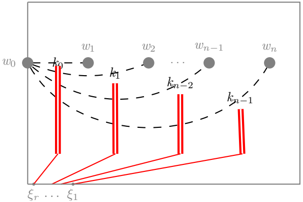
For $(k_0, \ldots, k_{n-1})$ such that $\sum k_i = r$, following drawings correspond to classes in $\mathcal{H}_r^{\text{rel}}$:

$U(k_0, \ldots, k_{n-1}) =$



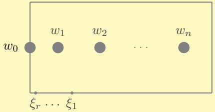
, and

$A(k_0, \ldots, k_{n-1}) =$



"multi - arcs"

D_n :



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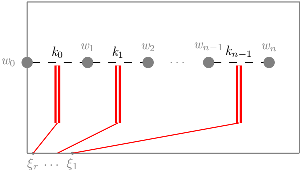
$$\mathcal{H}_r^{\text{abs}} := \mathrm{H}_r^{lf}\left(X_r; L_r\right)$$

$$\mathcal{H}_r^{\text{rel}} := \mathrm{H}_r^{lf}\left(X_r, X_r^-; L_r\right)$$

Examples of homology classes

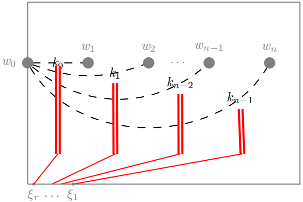
For $(k_0, \ldots, k_{n-1})$ such that $\sum k_i = r$, following drawings correspond to classes in $\mathcal{H}_r^{\text{rel}}$:

$U(k_0, \ldots, k_{n-1}) =$



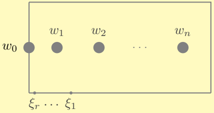
, and

$A(k_0, \ldots, k_{n-1}) =$



“multi - arcs”

D_n :



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$$(V^\alpha)^{\otimes n} \simeq_{\mathcal{B}_n} \bigoplus_{r \in \mathbb{N}} W_{n,r}$$

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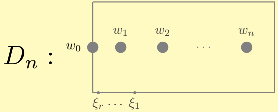
Structure of the homology

Proposition (Structural result)



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Structure of the homology

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- *$\mathcal{H}_r^{\mathit{rel}}$ is a free $\mathcal{R}_{\mathit{max}}$-module,*

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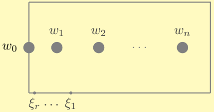

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Proposition (Structural result)

- $\mathcal{H}_r^{\mathit{rel}}$ *is a free* $\mathcal{R}_{\mathit{max}}$*-module,*
- *For which the set* $\{U(k_0, \ldots, k_{n-1})$ *s.t.* $\sum k_i = r\}$ *is a basis,*

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Proposition (Structural result)

- $\mathcal{H}_r^{rel}$ is a free $\mathcal{R}_{max}$-module,
- For which the set $\{U(k_0, \ldots, k_{n-1}) \text{ s.t. } \sum k_i = r\}$ is a basis,
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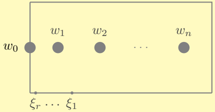

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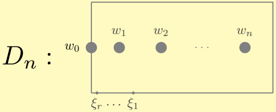
Idea: If X is a configuration space, let $X^{\mathbb{R}}$ be configurations restricted to the real line.



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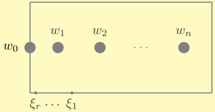

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Structure of the homology

Proposition (Structural result)

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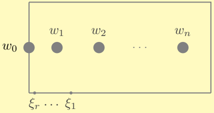

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2) $X_r^{\mathbb{R}} = \bigsqcup_{\sum k_i = r} U(k_0, \ldots, k_{n-1})$

D_n :

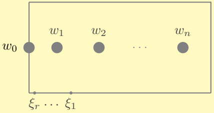

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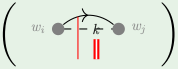
Diagram rules and multi-arcs basis

Example of homological rule



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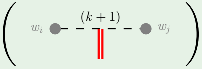

(with $t = q^{-2}$).

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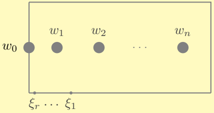

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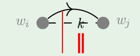
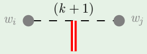
$\Big($  $\Big) = q^{-k} [k+1]_q \Big($  $\Big)$ (with $t = q^{-2}$).

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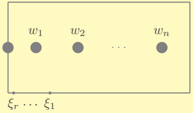

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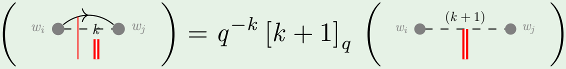
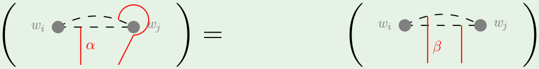


$$= q^{-k}\,[k+1]_q$$ (with $t = q^{-2}$).

Handle rule



$$=$$

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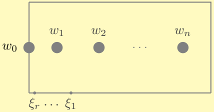

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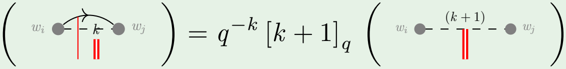

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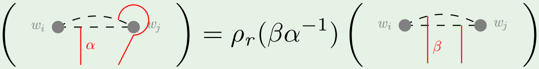

$= \rho_r(\beta\alpha^{-1})$

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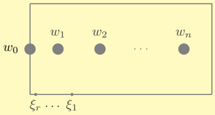

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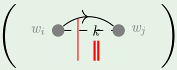

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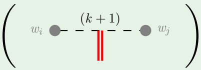

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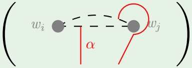

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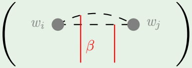
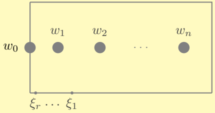


Corollary

The multi-arcs form an integral basis of $\mathcal{H}_r^{rel}$ (i.e. over $\mathcal{R}_{max}$).

Idea: express multi-arcs in the U-basis.

D_n :



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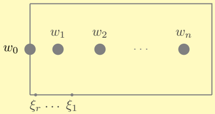

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Homological operators

General idea: mimic the quantum weight structure.

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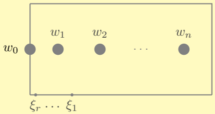

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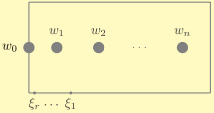
plus a diagonal action of K measuring the weight:

Definition (K)

$$K = q^{\sum \alpha_i} t^r \mathrm{Id}_{\mathcal{H}_r^{\text{rel}}} \in \mathrm{End}\left(\mathcal{H}_r^{\text{rel}}\right)$$

D_n :



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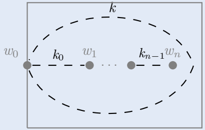
Homological operators

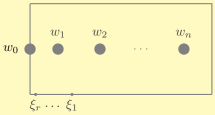
Definition (Divided powers of F)

$$F^{(k)} : \begin{cases} \mathcal{H}_r^{\mathsf{rel}} \to \mathcal{H}_{r+k}^{\mathsf{rel}} \\ U(k_0, \ldots, k_{n-1}) \mapsto \end{cases}$$



(+ choice of lift) .

$D_n :$



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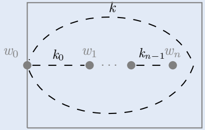

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$$E : \mathcal{H}_r^{\text{rel}} \xrightarrow{-\partial_*} H_{r-1}(X_r^-; L_r) \simeq \mathcal{H}_{r-1}^{\text{rel}}$$

Homological representation of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$

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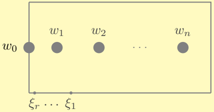

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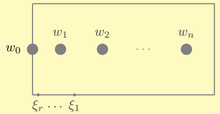

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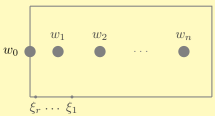

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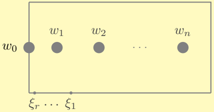

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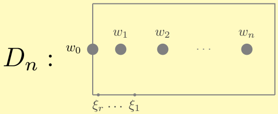

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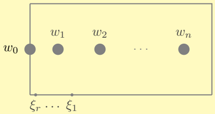
Homological interpretation of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ presentation

The proof relies on checking homologically:

$$[l]_q! F^{(l)} = \left(F^{(1)}\right)^l \text{ and } \left[E, F^{(1)}\right] = K - K^{-1}.$$

D_n :



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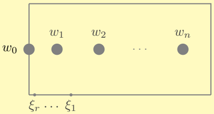
It gives a homological interpretation for the following set of relations:

$$\left[E, F^{(l+1)}\right] = F^{(l)}\left(q^{-l}K - q^l K^{-1}\right) \text{ and } F^{(l)}F^{(m)} = \begin{bmatrix} l+m \\ l \end{bmatrix}_q F^{(l+m)}$$

that completes the presentation of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

D_n :



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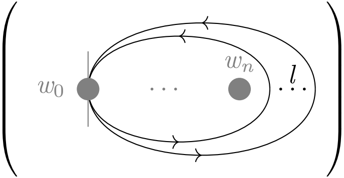
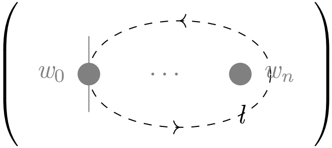
Pieces of computations

- Idea for the divided power property:



$\propto [l]_t!$



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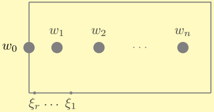

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Recovering monoidality of Verma modules

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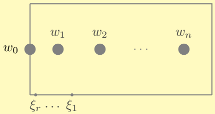

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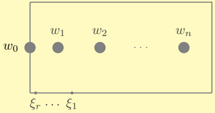

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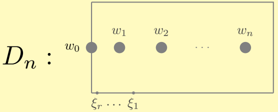

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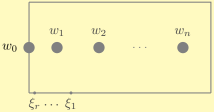
Idea of the proof.

Compute the action in the multi-arcs basis using homological calculus. $\square$

Recovering the quantum braid action

The action of $\mathcal{B}_n$ on D_n extends to X_r coordinate by coordinate.

D_n :



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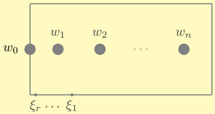

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The action of $\mathcal{B}_n$ on D_n extends to X_r coordinate by coordinate.

Lemma

The action of $\mathcal{B}_n$ lifts to $\mathcal{H}_r^{\text{rel}}$.

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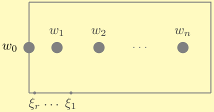

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Let $\alpha_1 = \cdots = \alpha_n = \alpha$. The isomorphism:

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is an isomorphism of $\mathcal{B}_n$ representations,

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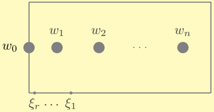

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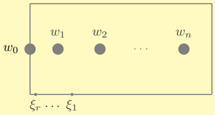

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Recovering the quantum braid action

The action of $\mathcal{B}_n$ on D_n extends to X_r coordinate by coordinate.

Lemma

The action of $\mathcal{B}_n$ lifts to $\mathcal{H}_r^{\text{rel}}$.

Theorem (M.)

Let $\alpha_1 = \cdots = \alpha_n = \alpha$. The isomorphism:

$$\mathcal{H}^{\alpha,\ldots,\alpha} \to (V^{\alpha})^{\otimes n}$$

is an isomorphism of $\mathcal{B}_n$ representations, such that $\mathcal{H}_r^{\text{rel}} \xrightarrow[\mathcal{B}_n]{\sim} W_{n,r}$.

Remark

There is a multi-color version but restricting representations to the *pure braid group*.

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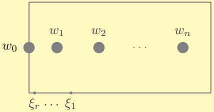

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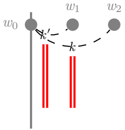
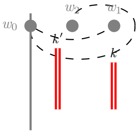
Pieces of computations

$\sigma_1 \cdot \Bigg($

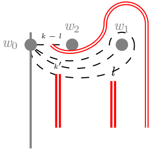


$\Bigg) = \Bigg($



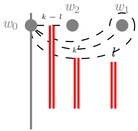
$\Bigg)$

$= \sum_{l=0}^{k} \Bigg($



$\Bigg)$

$= \sum_{l=0}^{k} t^{-k'(k-l)} t^{-l(k-l)} q^{-2(k-l)\alpha_1} \Bigg($



$\Bigg).$

$= \dots$

Summary

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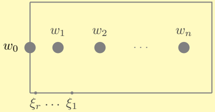

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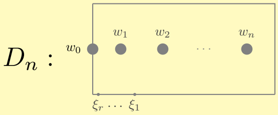

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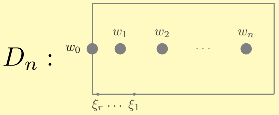

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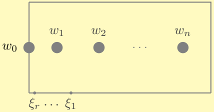

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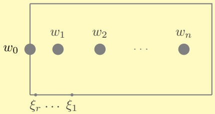

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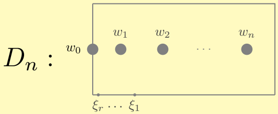

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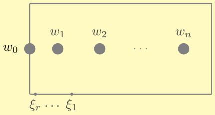

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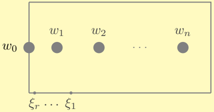

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Application to knot theory (in progress)

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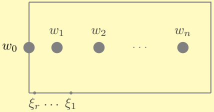

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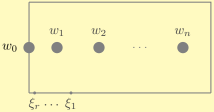

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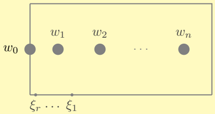

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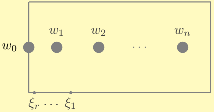

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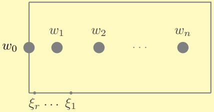

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$$\begin{tikzcd} W_{n,r-1} \arrow[r, bend right, "F^{(1)}"'] & W_{n,r} \arrow[l, bend right, "E"'] & Y_{n,r} \arrow[l, hook'] \end{tikzcd}$$

Homological representations

$$\mathcal{H}_r^{\text{abs}} := \mathrm{H}_r^{lf}\left(X_r; L_r\right)$$

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Application to knot theory (in progress)

Notations

- Let $\mathrm{J}_K(l)$ be the *l-colored Jones polynomial* of a knot K (computed using the l-dimensional simple module of $U_q\mathfrak{sl}(2)$).
- Let f be a mapping class acting on a topological space X, we denote the *abelianized Lefschetz number* of f by $\mathcal{L}_H\left(f, X\right)$.

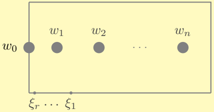
Theorem (M.)

Let K be the closure of a braid $\beta \in \mathcal{B}_n$, then:

$$\mathrm{J}_K(l+1) = \Lambda q^{-nl} \sum_{r=0}^{nl} (-1)^r \mathcal{L}_H\left(\beta, (X_r, X_r^-)\right)_{|\alpha_i = l}\, q^{2r},$$

where Λ is an invertible coefficient.

D_n :



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Thank you for your attention

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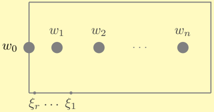

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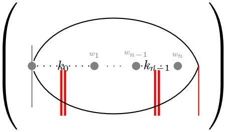
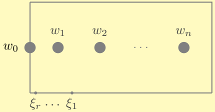
- Idea for $\left[E, F^{(1)}\right]$:

$-E \cdot$



D_n :



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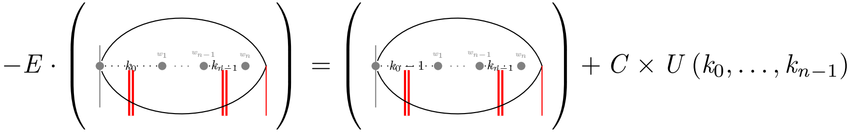
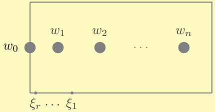


$$-E \cdot \left(\quad \right) = \left(\quad \right) + C \times U\left(k_0, \ldots, k_{n-1}\right)$$

D_n :



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