Convex Hyperbolic 4-manifolds

joint with S. Riolo & L. Slavich

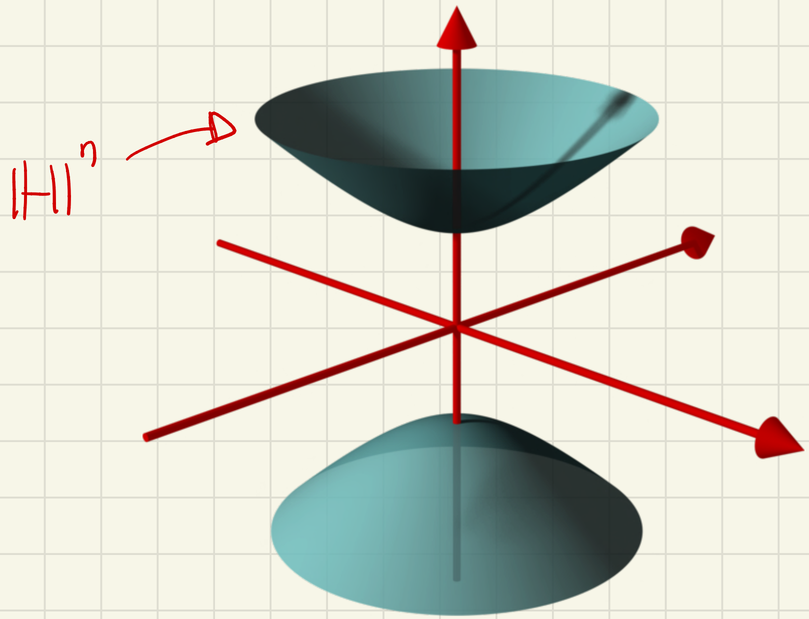
A **HYPERBOLIC n-MANIFOLD** is (equivalently):

a) a Riemannian manifold with $K \equiv -1$

b) a Riemannian manifold locally isometric to $\mathbb{H}^n$

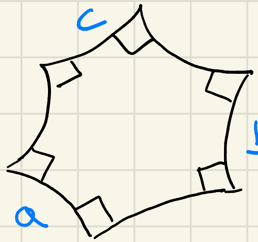
c) (if complete) $\mathbb{H}^n / \Gamma$ with $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acting freely (i.e. no elliptics) and properly discontinuously (i.e. Γ discrete)

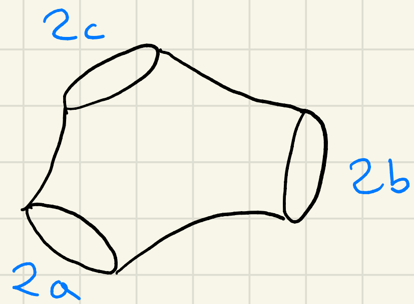
$$\mathrm{Isom}(\mathbb{H}^n) \cong O^+(n,1)$$

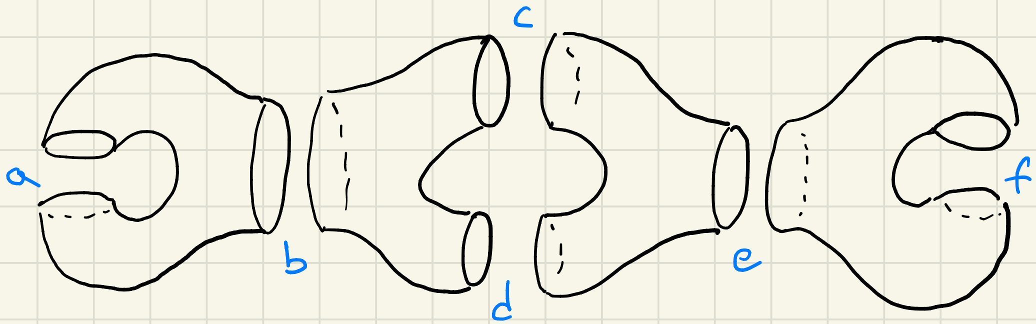


How can we construct hyperbolic n-manifolds?

$\underline{n=2}$: $\forall a,b,c > 0 \quad \exists!$



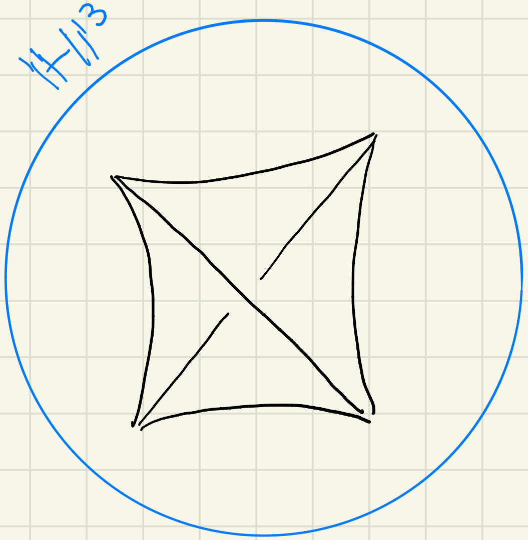




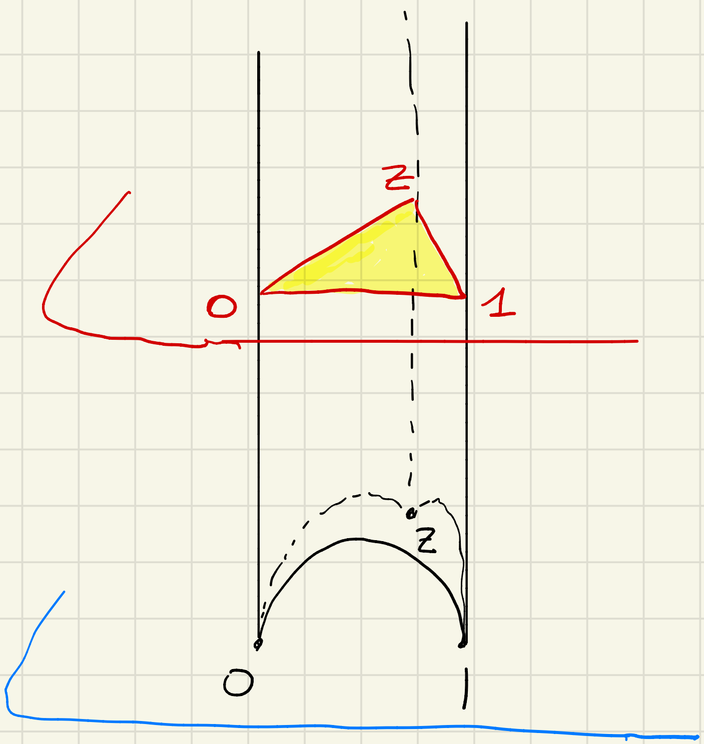
$3g-3$ lengths

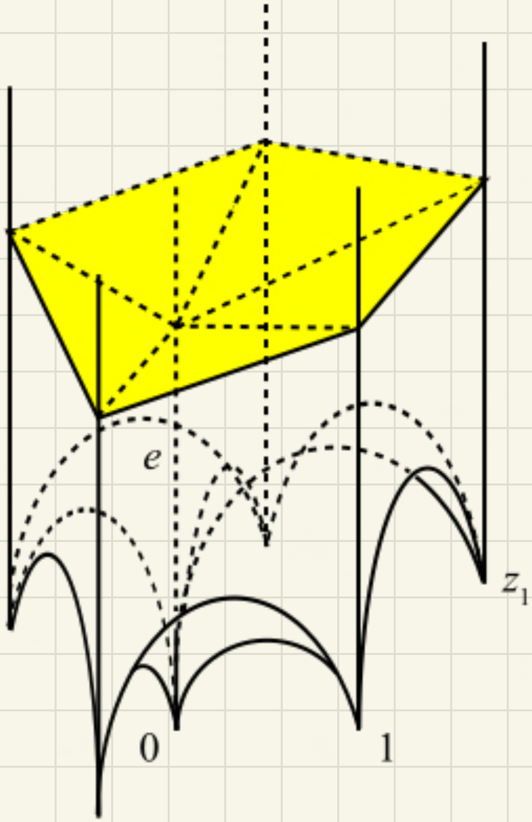
$3g-3$ torsion parameters

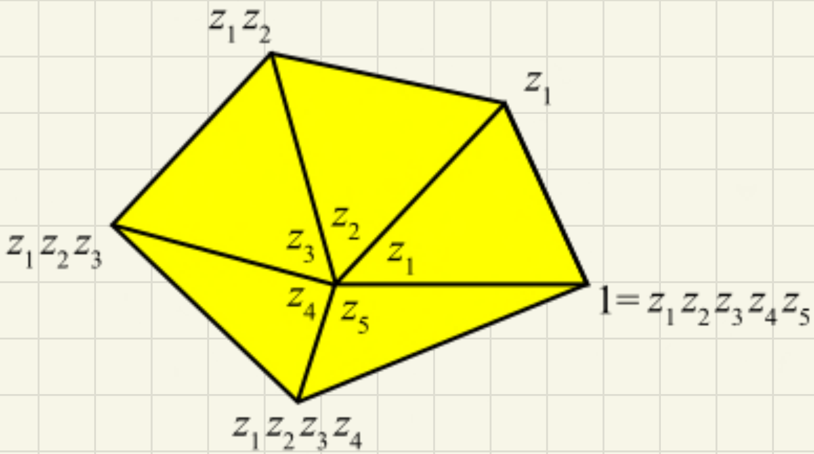
$$\text{Teichmüller Space} \cong \mathbb{R}^{6g-6}$$



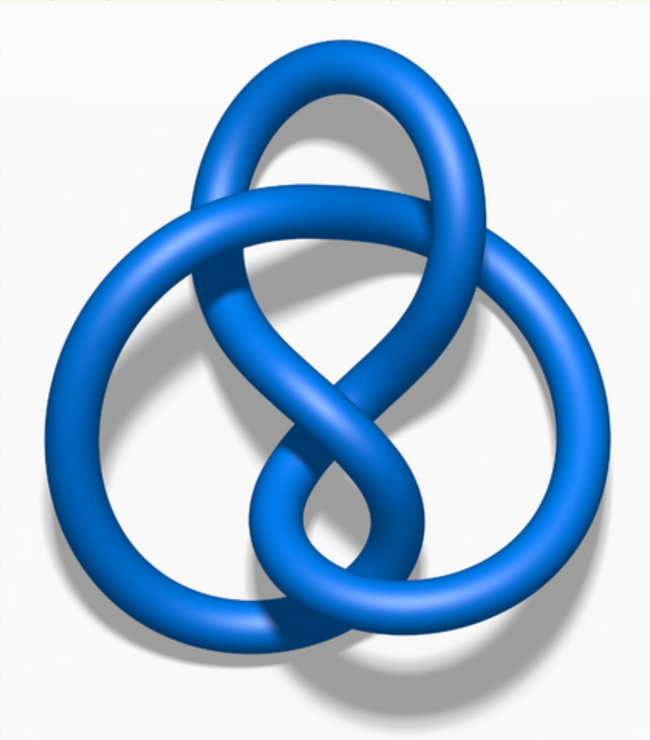
$\underline{n=3}$:

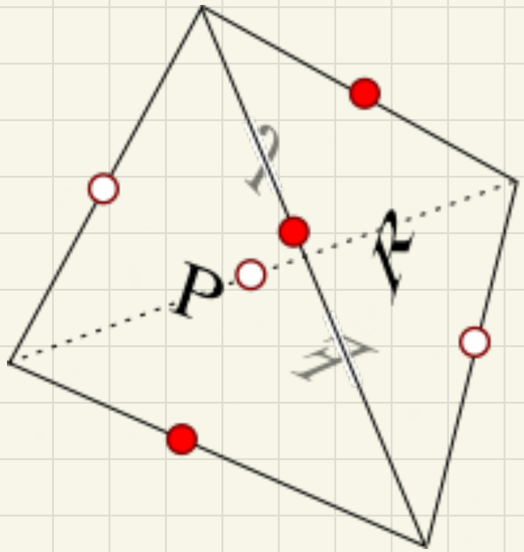


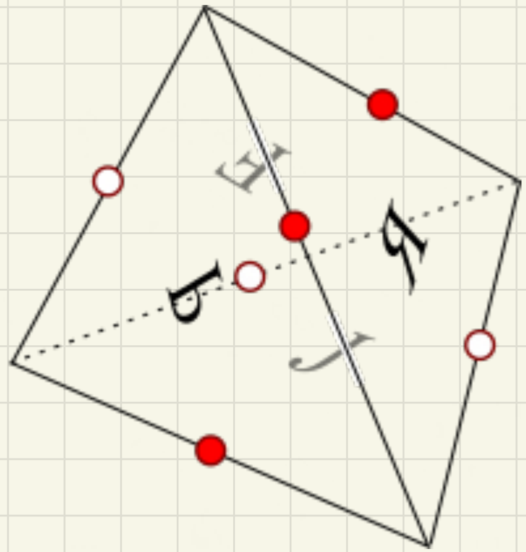




Compatibility equations : $z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot z_5 = 1$



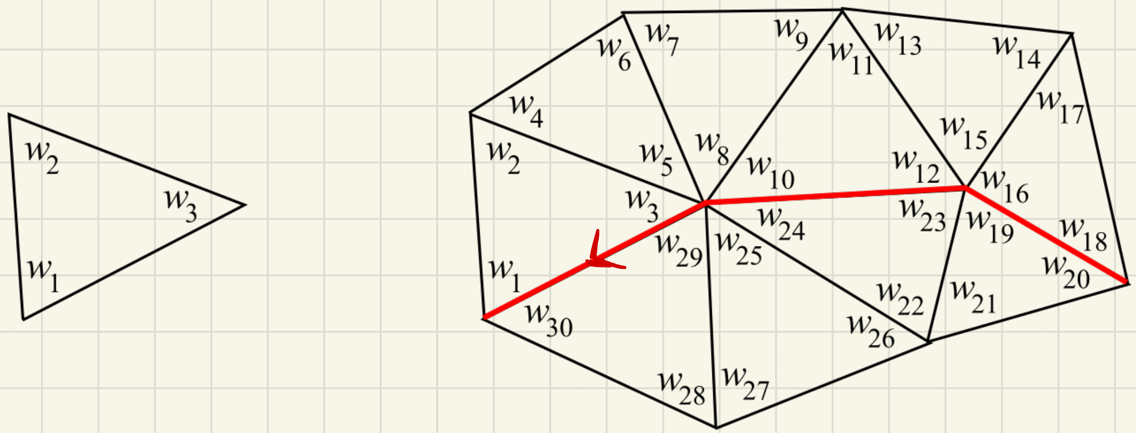




Edges have valence 6

$z = e^{\frac{2\pi i}{3}}$ IDEAL REGULAR TETRAHEDRON

Completeness equations:



\# vertices of γ

$$\cdot \cdots w_1 \cdot w_3 \cdot w_5 \cdot w_8 \cdot w_{10} \cdot w_{12} \cdot w_{15} \cdot w_{16} \cdot w_{18} \cdots\cdots = e^{(2+|\gamma|)\pi i}$$

Thurston's geometrization:

1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

Bull AMS 1982

Thurston proved it for Haken manifolds. His proof uses hyperbolic 3-manifolds with infinite volume.

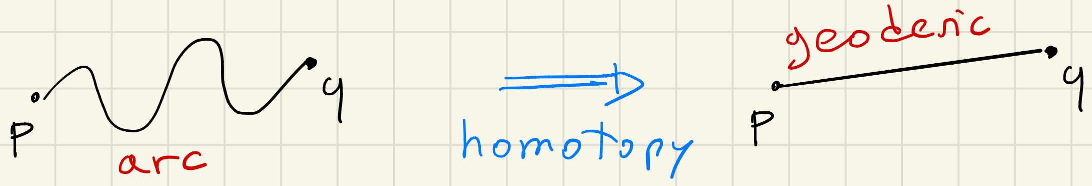
Later proved by Perelman with Ricci flow for all 3-manifolds.

A hyperbolic n-manifold WITH BOUNDARY is equivalently

a) a Riemannian manifold with boundary and $K \equiv -1$

b) a Riemannian manifold locally isometric to a n-submanifold with boundary of $\mathbb{H}^n$

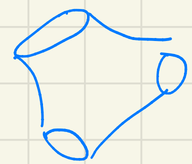
It is CONVEX if



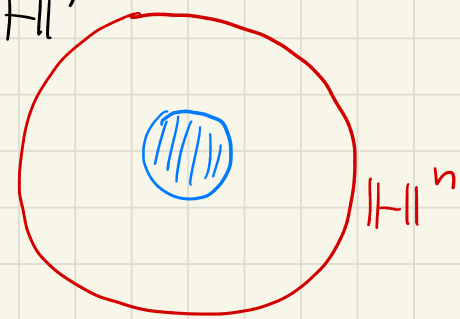
If M complete, then M convex $\Longleftrightarrow$ M locally convex

Examples:

- M hyperbolic complete with geodesic boundary



- A closed ball in $\mathbb{H}^n$

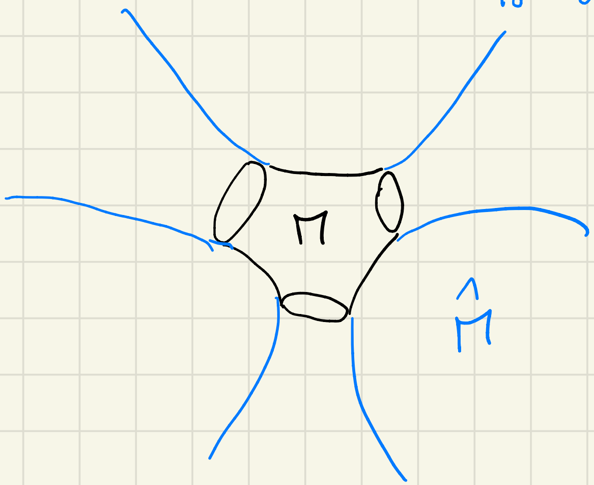


Nice properties :

M hyperbolic complete & convex $\longrightarrow$ Has a CANONICAL EXTENSION $M \subseteq \hat{M}$ complete hyperbolic & without boundary

$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Uparrow$

$D : \tilde{M} \to \mathbb{H}^n$ developing map is a diffeo onto a convex submanifold $C \subseteq \mathbb{H}^n$ $\Longrightarrow$ $M = C/\Gamma \subseteq \hat{M} = \mathbb{H}^n/\Gamma$

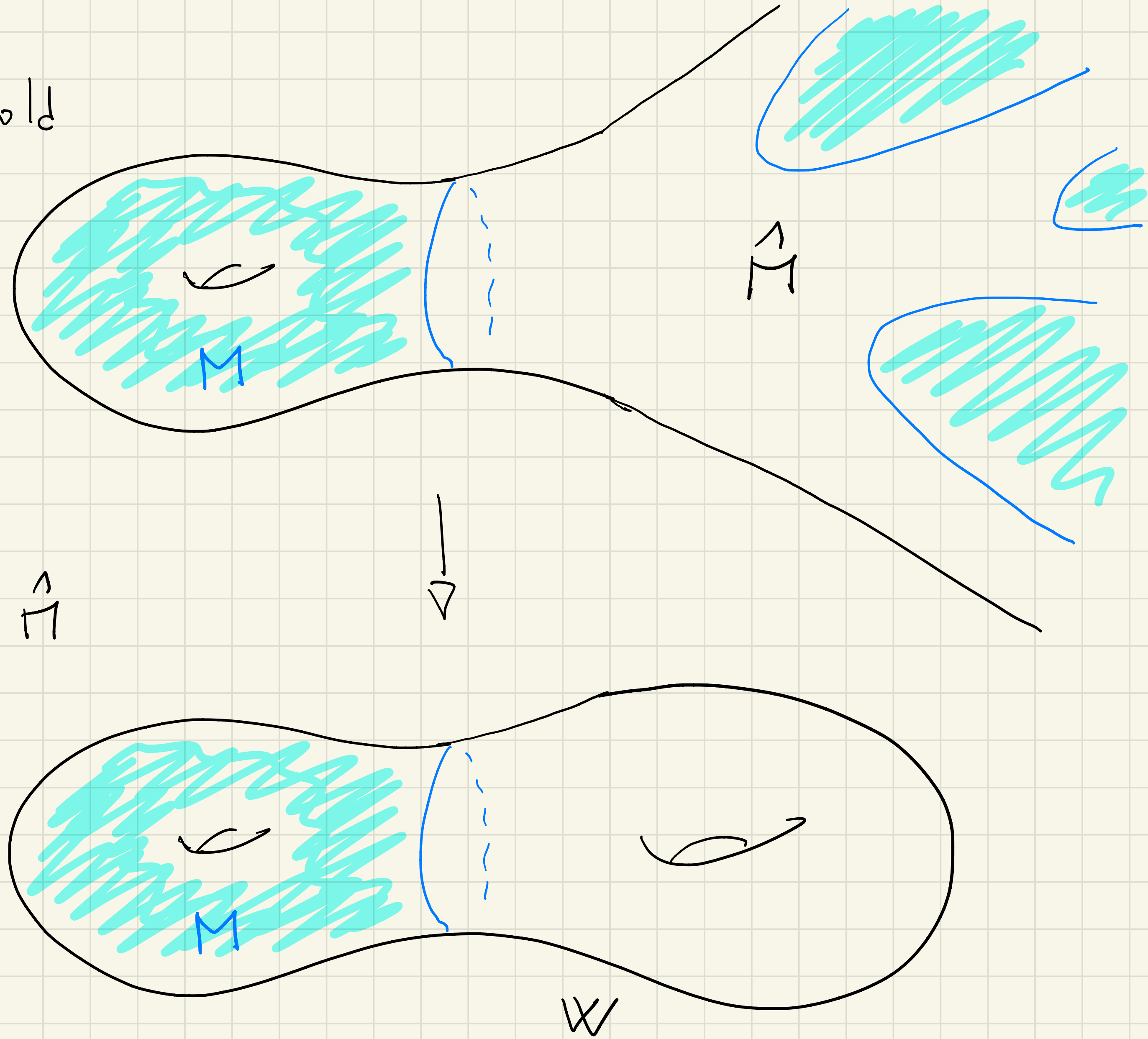


If M is compact, then $\hat{M}$ is GEOMETRICALLY FINITE and diffeomorphic to $\mathrm{int}(M)$

If $M \subseteq W$ compact convex submanifold in complete hyperbolic W without boundary

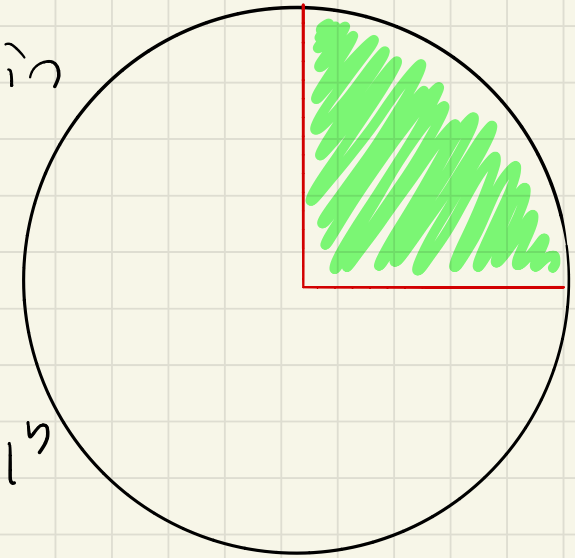
Then $\pi_1 M \hookrightarrow \pi_1 W$ injective

and covering is isometric to $\hat{M}$



A hyperbolic n-manifold with RIGHT-ANGLED CORNERS

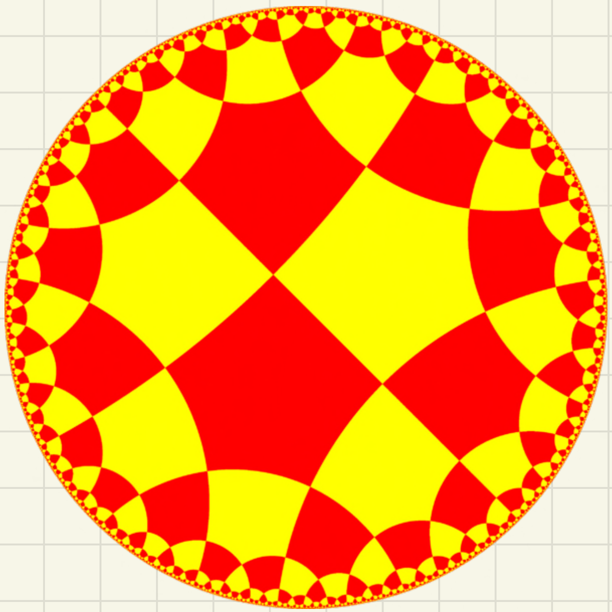
is a topological manifold with an atlas in



and isometries as transition maps

<u>Examples</u>:

- Hyperbolic n-mfold with geodesic boundary
- Right-angled polyhedron $P \subseteq \mathbb{H}^n$



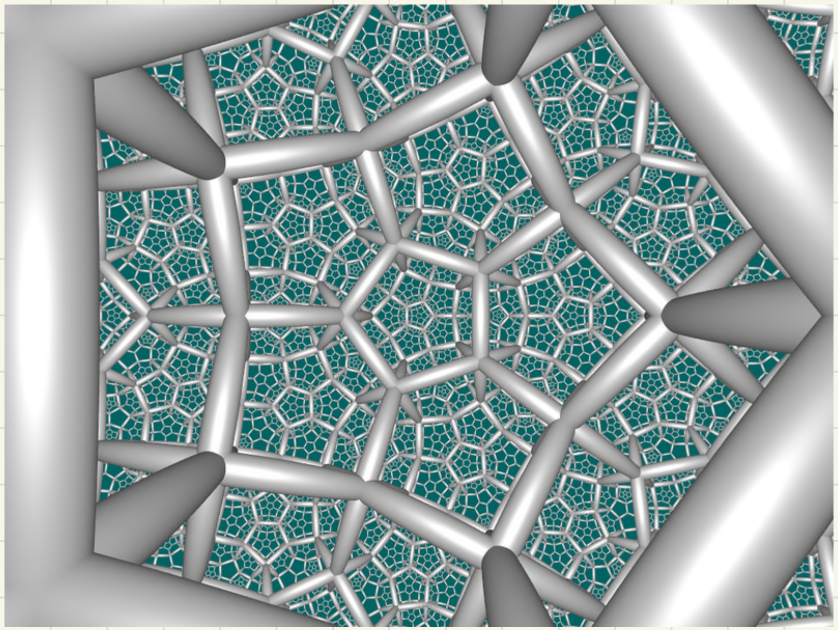
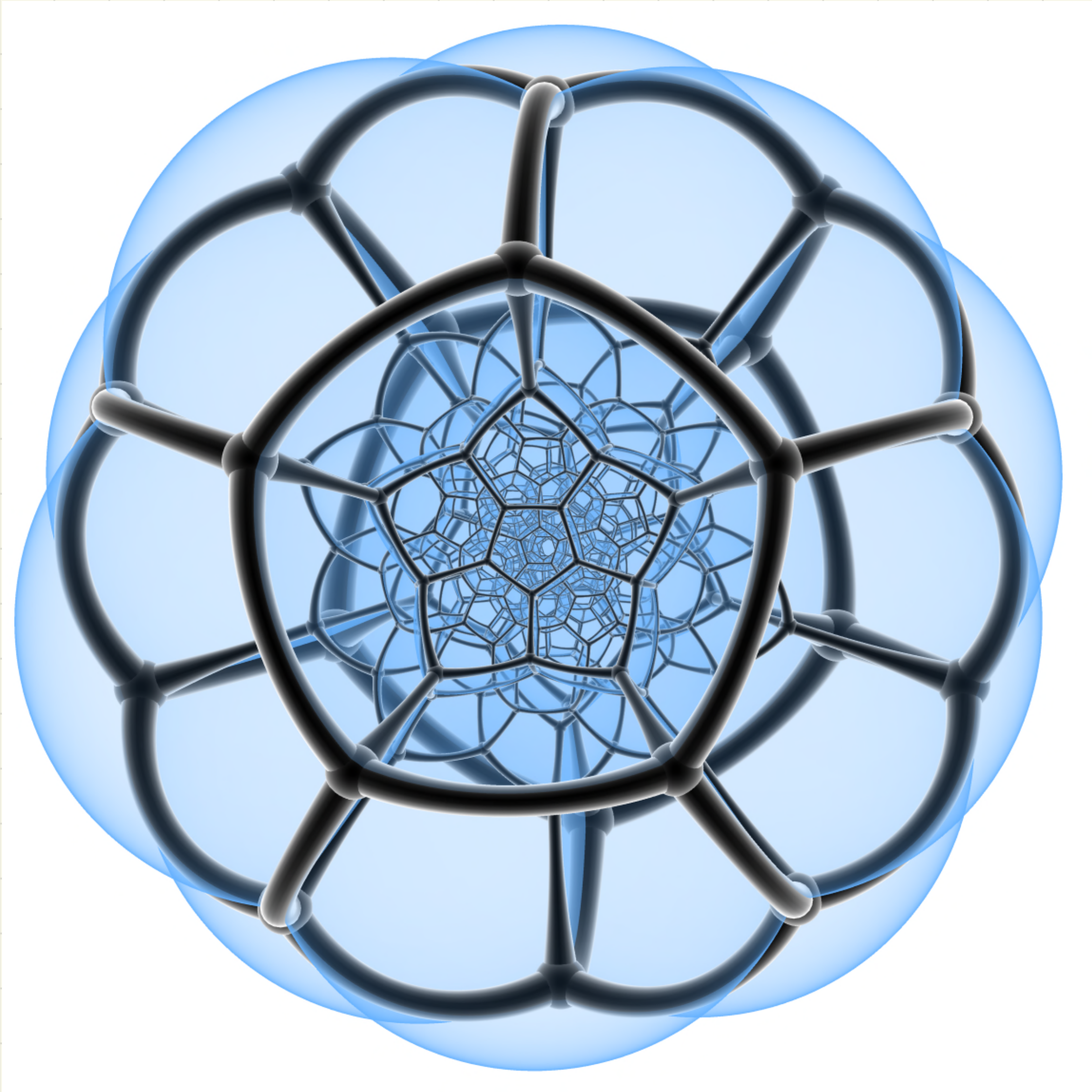
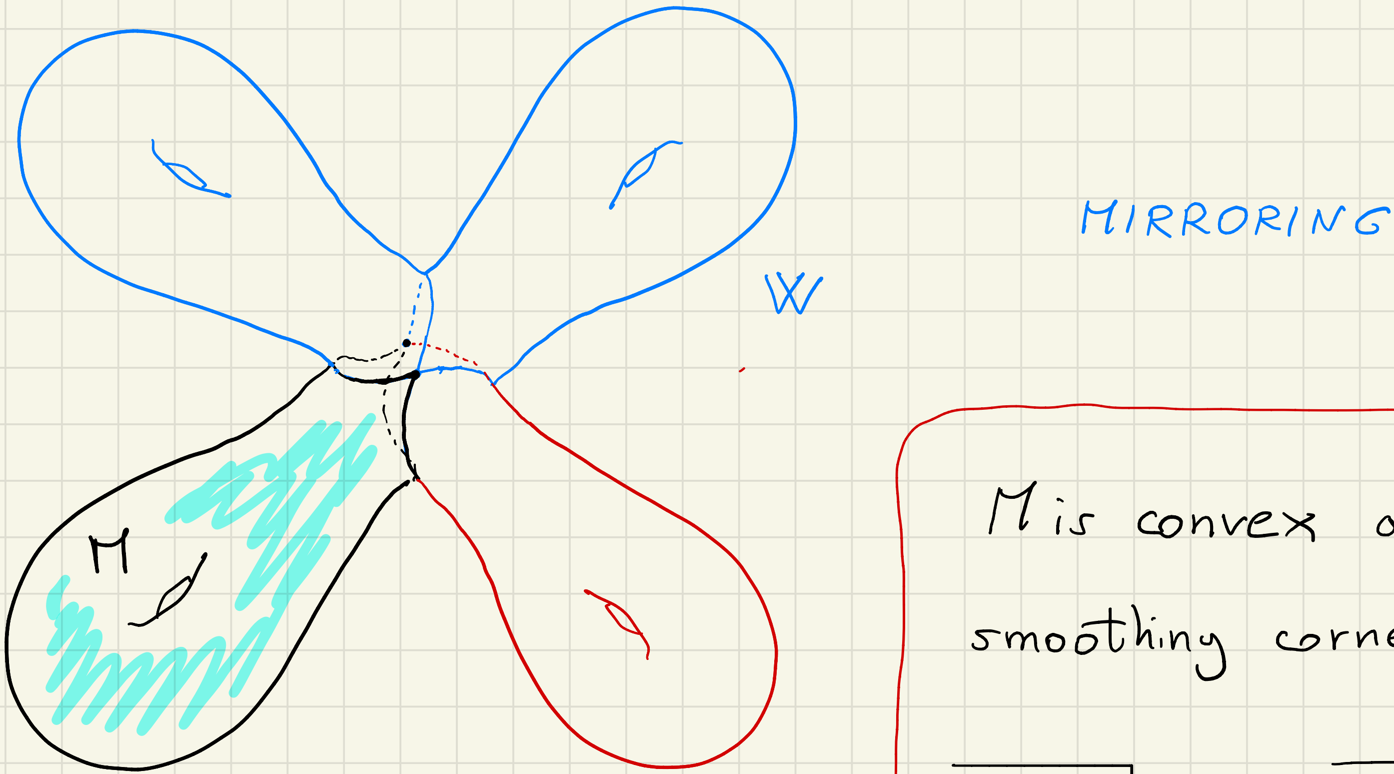


figure by Roice3
CC-BY-SA
Wikipedia Commons

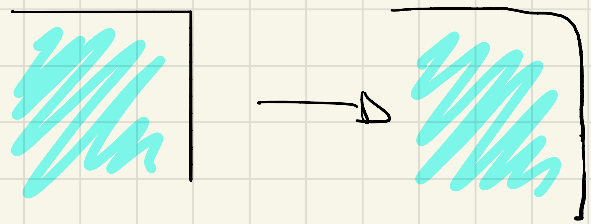
120-cell



Prop: Every compact hyperbolic M with right-angled corners and embedded faces is contained in a closed hyperbolic W



M is convex after smoothing corners



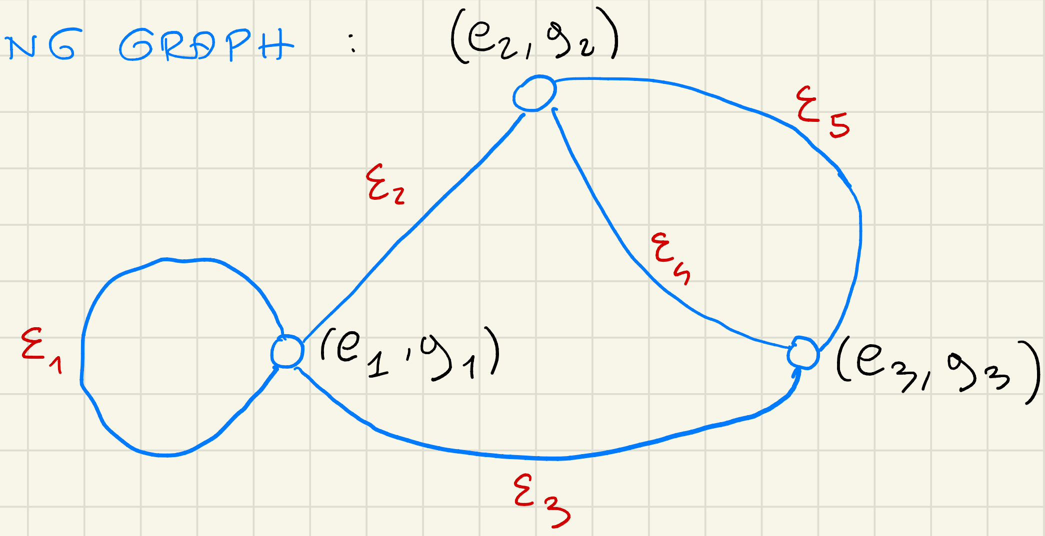
Question: Which compact n-manifolds with (non-empty) boundary admit a hyperbolic structure with right-angled corners? (and embedded faces)

$n=2$: All surfaces with boundary

$n=3$: Irreducible and algebraically atoroidal 3-manifolds
no essential spheres
$\mathbb{Z}\times\mathbb{Z} \not\subset \pi_1 M$
i.e. no immersed essential tori

$n=4$: [MRS] Many plumbings do

A PLUMBING GRAPH :

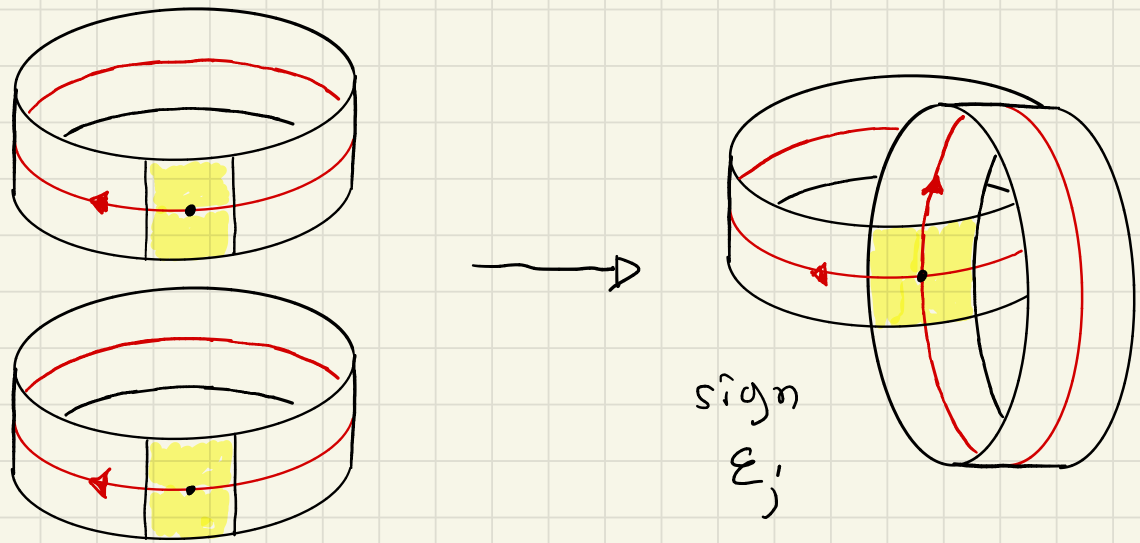


$g_i, e_i \in \mathbb{Z} \quad g_i \geq 0$

$\varepsilon_j = \pm 1$

$(e_i, g_i) \longrightarrow$ Disc bundle over oriented genus-g_i surface with Euler number e_i

$\varepsilon_j \longrightarrow$ Plumbing with sign ε_j :



Thm: If $g_i \geq 2(|e_i| + v_i + 1)$ at every vertex

genus Euler valence of vertex

then M has hyperbolic structure with right-angled corners (and embedded faces)

Cor: For every symmetric $\mathbb{Z}$-matrix Q there is a convex compact hyperbolic M with $Q_M = Q$ that embeds in a closed hyperbolic 4-manifold W

Cor: For every symmetric $\mathbb{Z}$-matrix Q there is a geometrically finite hyperbolic M with $Q_M = Q$ that covers a closed hyperbolic 4-manifold W

Cor: There are (many) closed hyperbolic 4-manifolds
orientable
that are not spin.

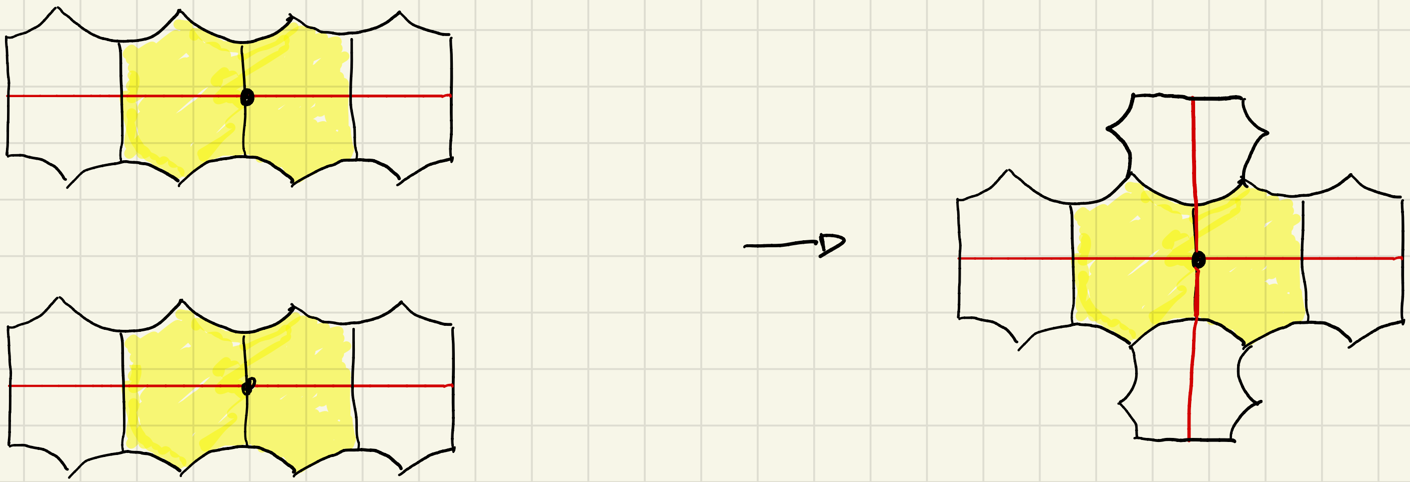
Cor: There are (many) closed hyperbolic n-manifolds
orientable
that are not spin for all $n \geq 4$.

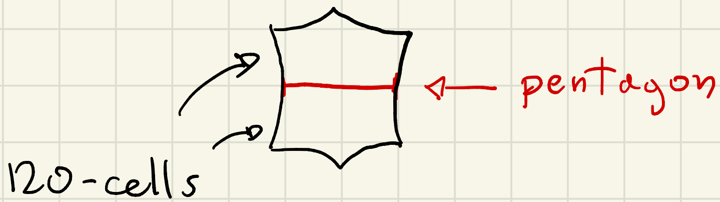
[use embedding theorem for arithmetic manifolds of simple type from Kolpakov - Ried - Slavich]

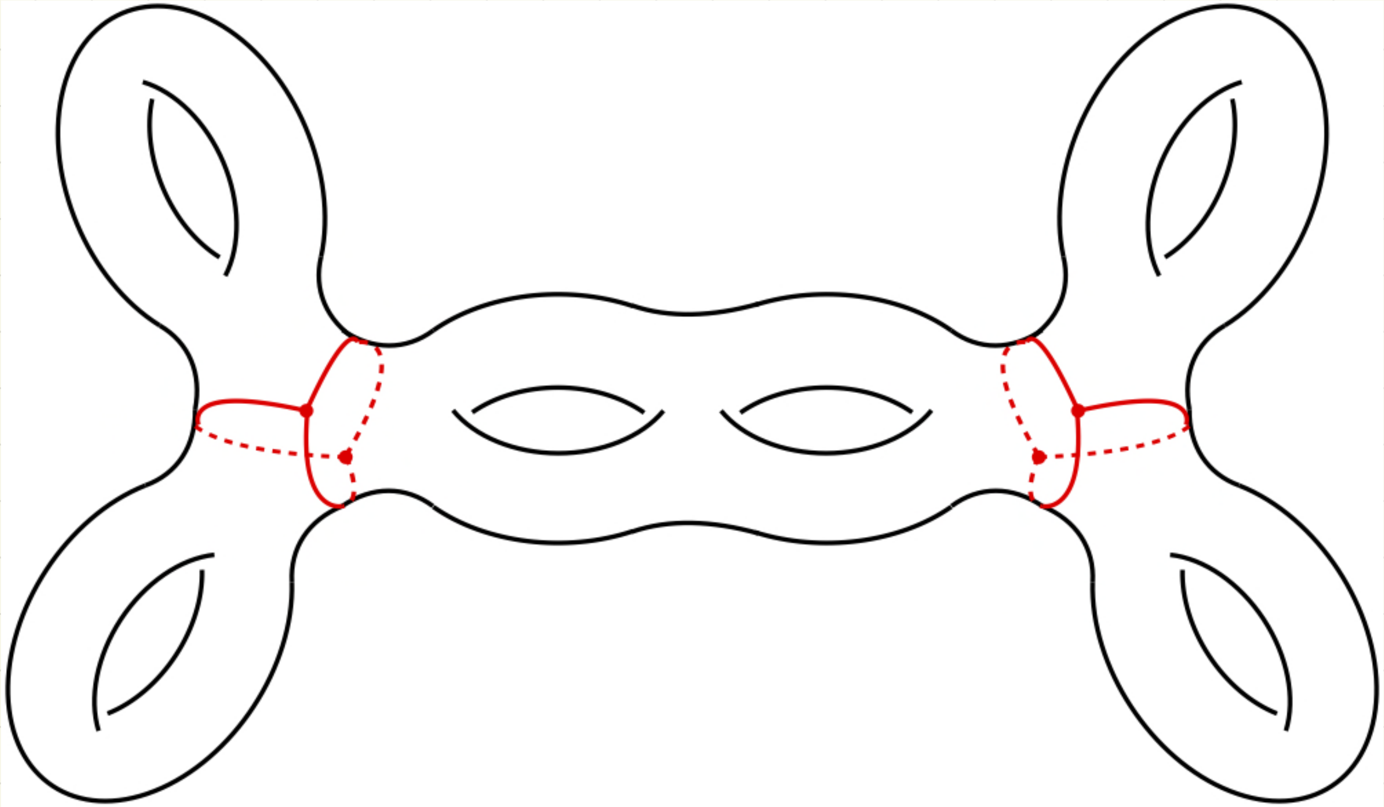
Cor: There are closed hyperbolic 4-manifolds such that

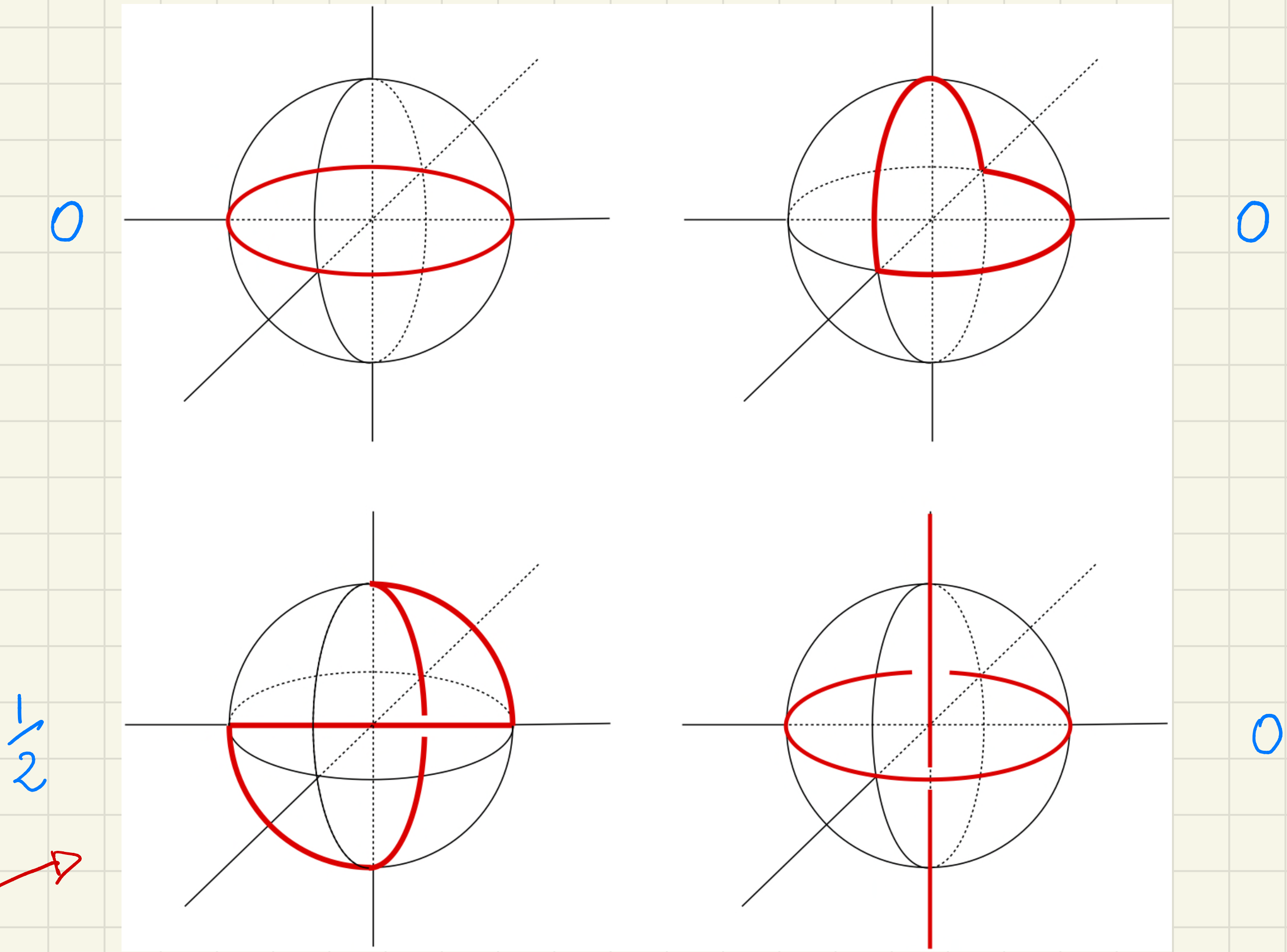
- $H_2(M, \mathbb{Z})$ is not generated by immersed surfaces
- are covered by non-trivial bundles over surfaces

Idea of the proof:









inspired by trisection of $\mathbb{CP}^2$

Gromov - Lawson - Thurston formula for e