[K-OS] 25/03/21

MILNOR INVARIANTS FOR KNOTTED SURFACES

(AND BEYOND)

J.B. MEILHAN
(Univ. Grenobles Alpes)

joint work in (slow) progress w./ B. AUDOUX & A. YASUHARA

PLAN

I. Review of Milnor's link invariants.

II. Some variants of Milnor inv.

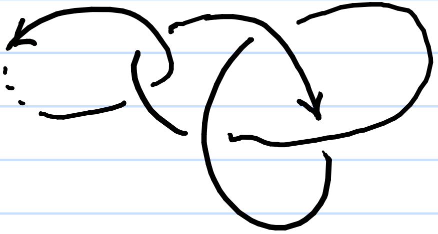
III. Cut-diagrams & Milnor inv. of knotted surfaces.

PLAN

I. Review of Milnor's Link invariants.

MILNOR'S LINK INVARIANTS

$L = L_1 \cup \ldots \cup L_n$ oriented / ordered link in S^3.



L_i

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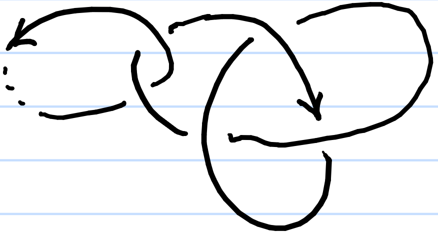

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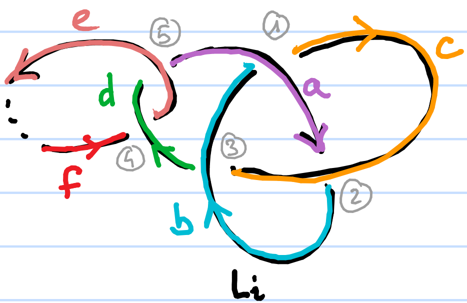

$$\pi_L = \left\langle \begin{matrix} a, b, c, \\ d, e, f, \\ \ldots \end{matrix} \;\middle|\; \begin{matrix} (1)\ ba = ac & (4)\ df = ed \\ (2)\ ac = ba & (5)\ ed = ae \\ (3)\ cb = bd & \ldots \end{matrix} \right\rangle$$

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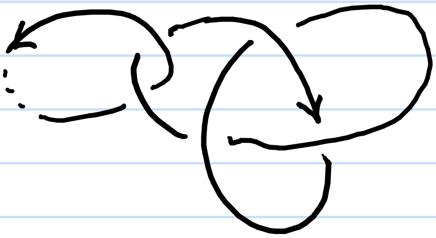

L_i

Given a group G

↳ LOWER CENTRAL SERIES of G : $(G_q)_q$

$G_1 := G$ and $G_{q+1} = [G, G_q]$

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qth NILPOTENT QUOTIENT : $N_q G = G/G_q$.

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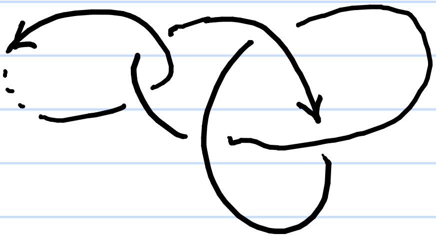

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[CHEN-MILNOR] For $q \geq 1$

$$N_q \pi_L = \left\langle m_1, \ldots, m_n \;\middle|\; \begin{array}{l} \cdot\ A_q = 1 \\ \cdot\ \forall i,\ [m_i ; \lambda_i] = 1 \end{array} \right\rangle$$

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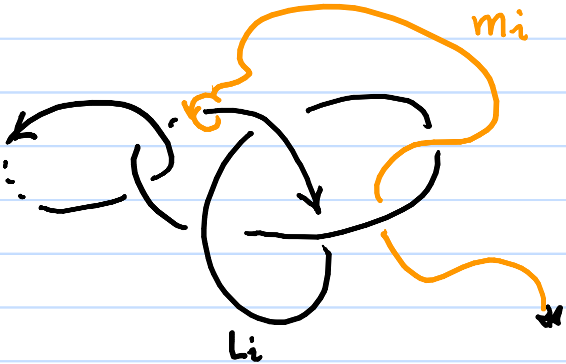

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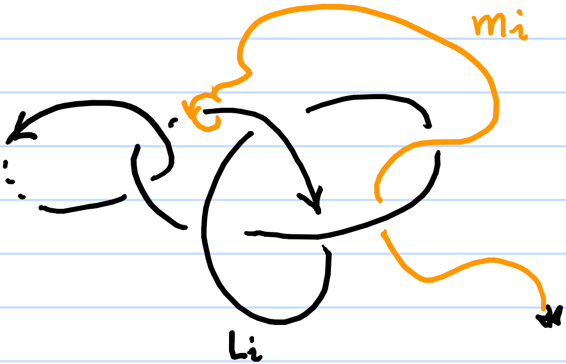

$A = F(m_1, \dots, m_n)$

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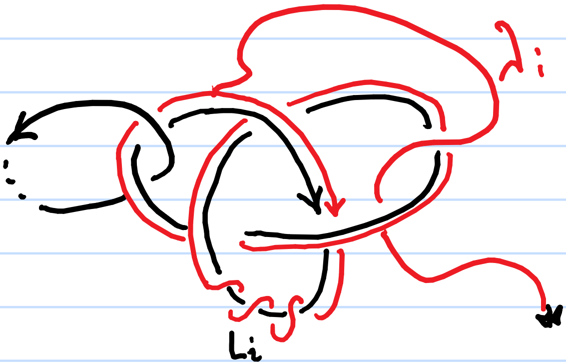

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is def. by $\begin{cases} m_i \longmapsto 1 + X_i \\ m_i^{-1} \longmapsto 1 - X_i + X_i^2 - X_i^3 \dots \end{cases}$

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Let I = sequ. in $\{1, \ldots, n\}$

$$\rightsquigarrow \quad \Delta_L(I) = \gcd \left\{ \mu_L(J) \;\middle|\; \begin{array}{c} J \text{ sequ. obt. from } I \text{ by DELETING} \\ \text{some index + possibly } \circlearrowright \end{array} \right\}$$

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<u>Ex</u>: $\quad \Delta_L(123) = \gcd \left\{ \begin{array}{l} \mu_L(13) \,;\; \mu_L(23) \;;\; \mu_L(12) \\ \mu_L(31) \,;\; \mu_L(32) \;;\; \mu_L(21) \end{array} \right\}$

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[Milnor; '57] $\forall$ sequence I of at most q integers in $\{1, \dots, n\}$,

$$\overline{\mu}_L(I) \equiv \mu_L(I) \mod \Delta_L(I)$$

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Examples.

- $\forall i$, $\mu_L(i) = 0$ [convention]
- $\forall i \neq j$, $\mu_L(ij) = \ell k(L_i, L_j) \in \mathbb{Z}$

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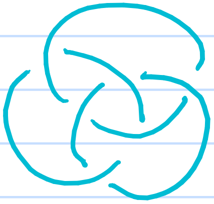
- $\forall i, \quad \mu_L(i) = 0 \quad$ [convention]
- $\forall i \neq j, \quad \mu_L(ij) = lk(L_i, L_j) \in \mathbb{Z}$.

$\hookrightarrow \bar{\mu}_L(ijk)$ def. modulo $\gcd\{lk(ij), lk(ik), lk(jk)\}$

&

$$\bar{\mu}_B(123) = 1$$

B



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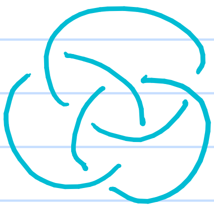
B 

[CASSON]. Milnor inv. are <u>concordance</u> invariants.

PLAN

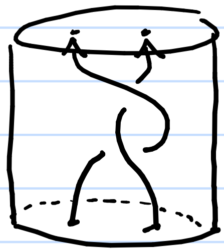
II. Some Variants of Milnor inv.

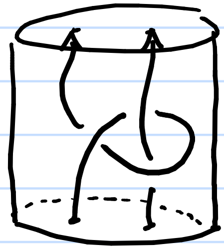
MILNOR STRING LINK INVARIANTS

[HABEGGER-LIN; '90]

MILNOR STRING LINK INVARIANTS

STRING LINKS :





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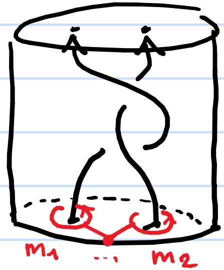
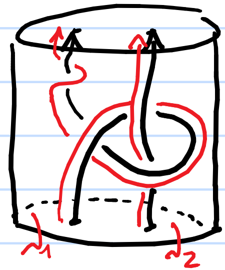


MERIDIANS

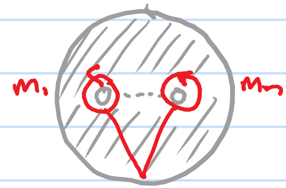


(pref.) LONGITUDES

Let $q \geq 1$

CHEN-MILNOR type presentation.

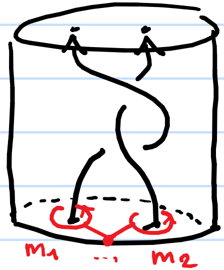
$$N_q \pi_L \simeq N_q A = \langle m_1, \dots, m_n \mid A_q \rangle$$



$D^2 \setminus \{\text{points}\}$

(Remark: uses "STALLINGS THEOREM")

MILNOR STRING LINK INVARIANTS



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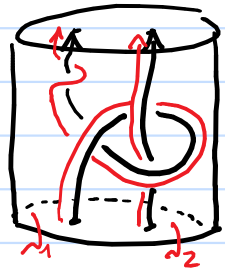

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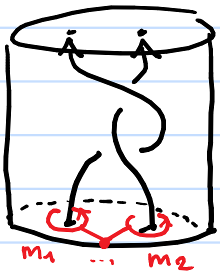

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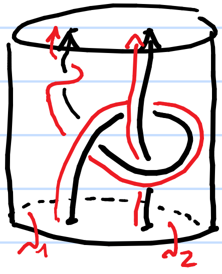

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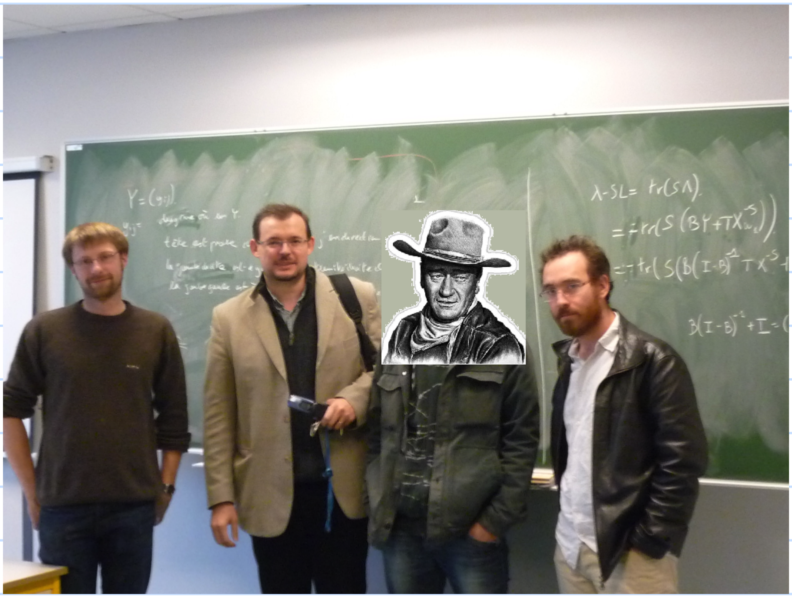
Milnor Invariant of Concordances

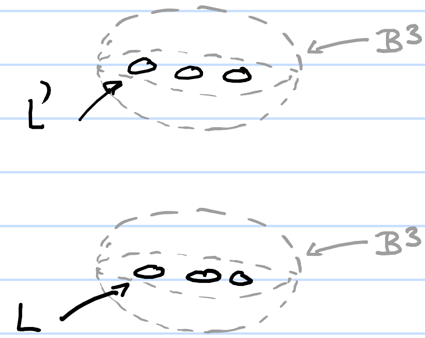
[M. - Yasuhara]

[Audoux - Bellingeri - M. - Wagner]



MILNOR INVARIANT OF CONCORDANCES

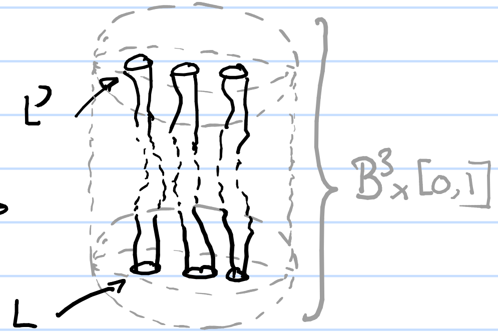
CONCORDANCE: Let L, L' be two n comp. links



MILNOR INVARIANT of CONCORDANCES

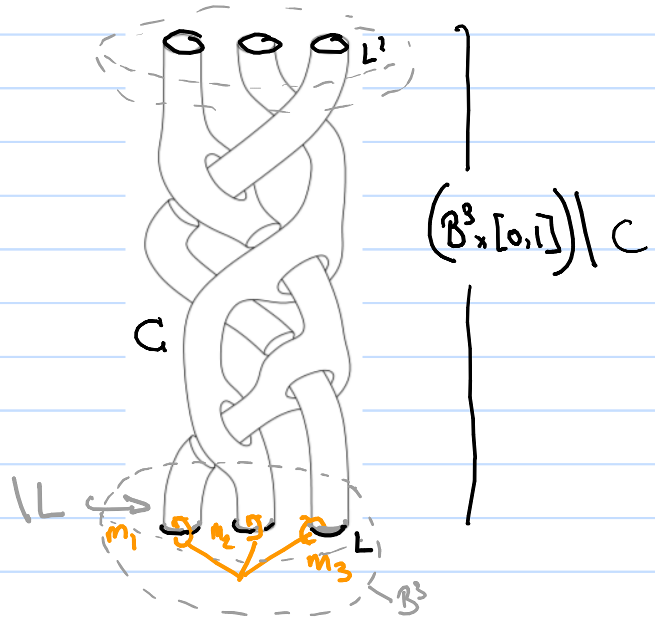
CONCORDANCE :

$$\begin{matrix} n \text{ annuli} \\ S^1 \times [0,1] \end{matrix} \xhookrightarrow[\text{smooth}]{C}$$



Milnor Invariant of Concordances

$\pi_C : \pi_1((B^3 \times [0,1]) \setminus C)$



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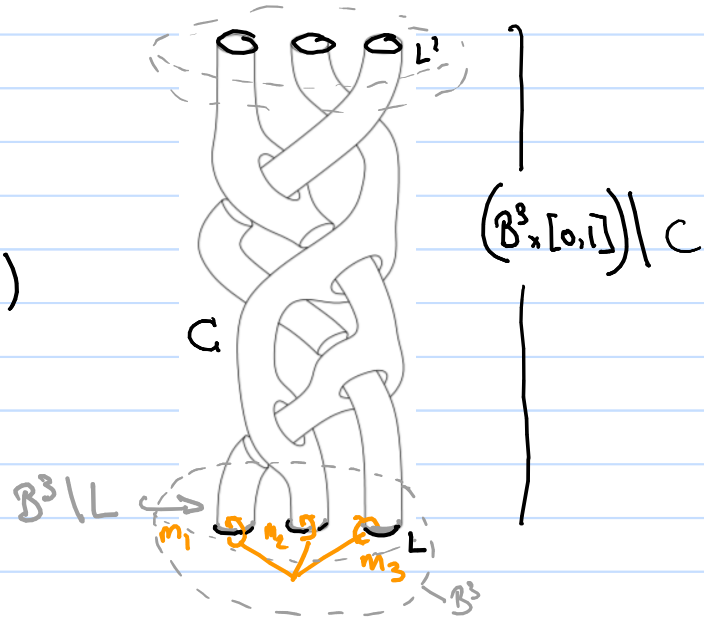
Chen-Milnor type presentation:

$\forall q \geq 1, \quad N_q \pi_C \simeq N_q \pi_L \quad (\simeq N_q \pi_{L'})$

$\langle m_1, \dots, m_n \mid A_q = 1 \, ; \, [m_i, \lambda_i] = 1 \rangle$

cf LINK CASE

(by "Stallings thm" again)



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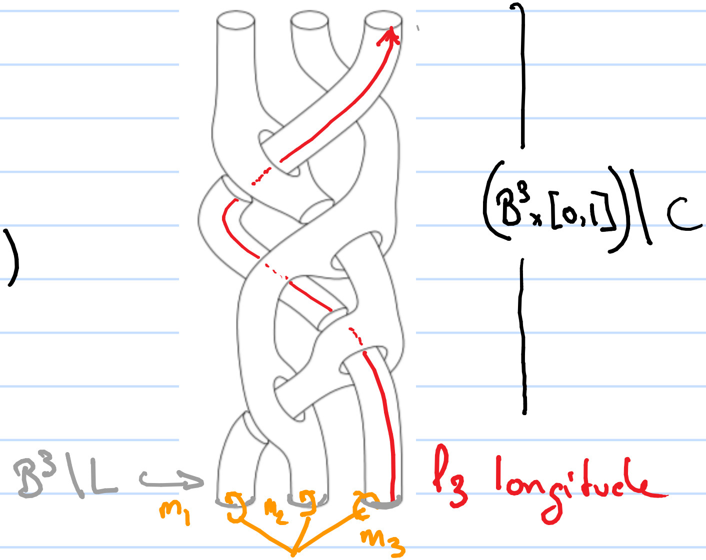

Can express longitudes $\ell_i \in N_q\pi_C$ as words in $\{m_j^{\pm 1}\}$, and extract Milnor-type invariants using E.

Milnor Invariant of Concordances

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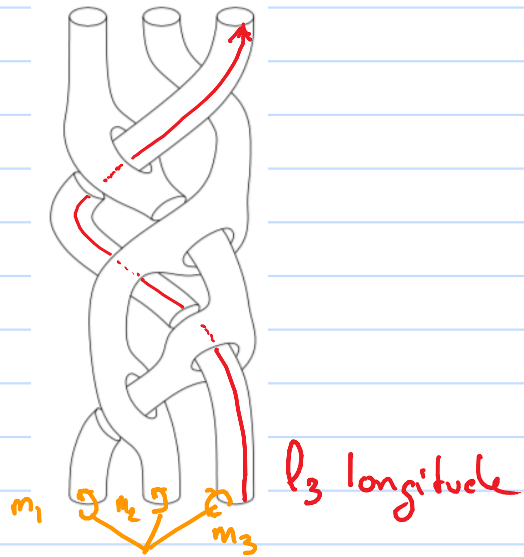


$\forall$ sequ. I of $\leq q$ indices,

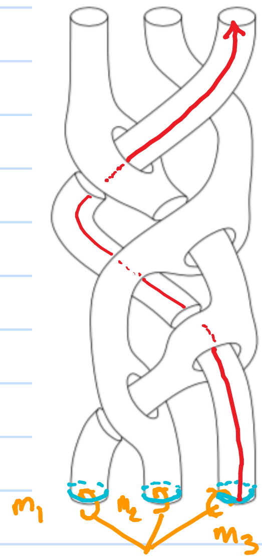
$$\overline{\mu}_C(I) \equiv \mu_C(I) \mod \overline{\Delta}_C(I)$$

is an (isotopy) invariant of C

Milnor Invariant of Concordances

Fix $q \geq 1$.

$$E(\ell_i) = 1 + \sum_{j_1 \cdots j_k} \mu_C(j_1 \cdots j_k; i) X_{j_1} \cdots X_{j_k}$$



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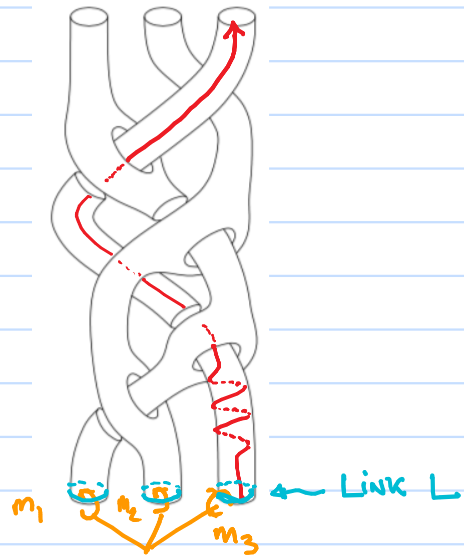

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SUMMARY

For links / string links / concordances,

→ express longitudes in $N_q\pi$ as word in $\{\text{meridians}^{\pm 1}\}$

→ extract numerical invariants from $E(\text{longitude})$ by modulo some 'Δ' that control choices / relations in $N_q\pi$.

NEED to know : • $N_q\pi$ generated by meridians

SUMMARY

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• relations in $N_q\pi$.

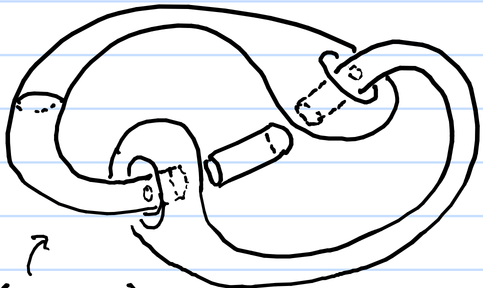
PLAN

III. Cut-diagrams & Milnor inv. of knotted surfaces.

BROKEN SURFACE DIAGRAMS

surface

Given $\Sigma \xhookrightarrow[\text{smooth}]{} \mathbb{B}^4$

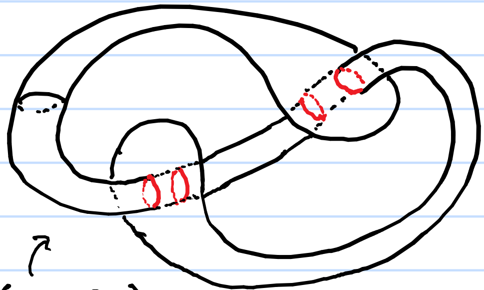


$(\Sigma = S^2)$

BROKEN SURFACE DIAGRAMS

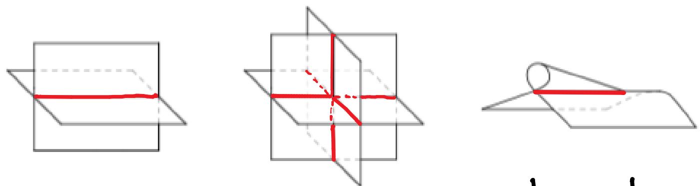
surface

Given $\Sigma \xhookrightarrow[\text{smooth}]{} \mathbb{B}^4$



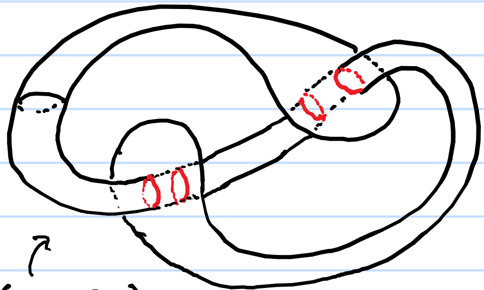
$(\Sigma = S^2)$

$=$ generic projection to $\mathbb{B}^3$



Broken Surface Diagrams

Given $\Sigma \xhookrightarrow[\text{smooth}]{} \mathbb{B}^4$ (surface)



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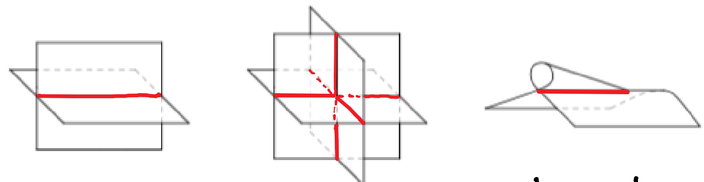
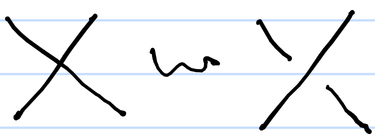


double triple branching

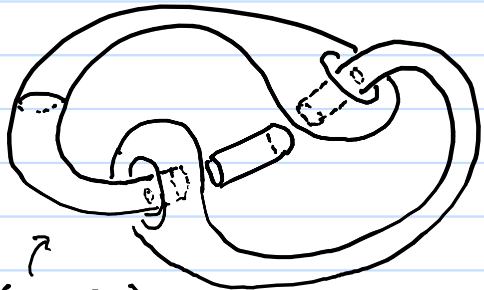
For Link diagrams:



Broken Surface Diagrams

surface

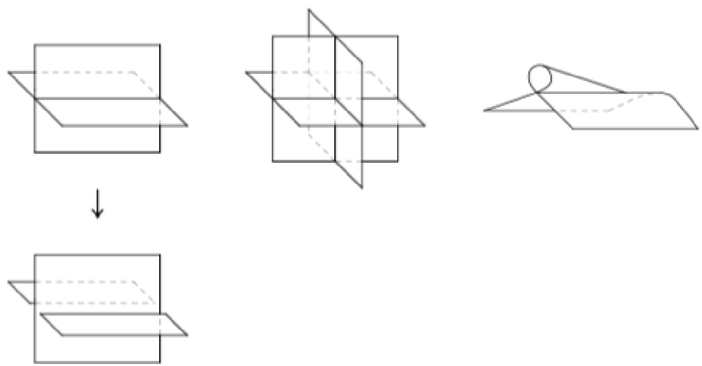
Given $\Sigma \underset{\text{smooth}}{\hookrightarrow} \mathbb{B}^4$



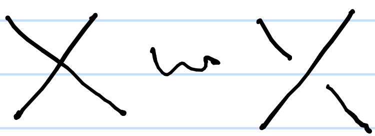
$(\Sigma = S^2)$

$=$ generic projection to $\mathbb{B}^3$

\+ OVER / UNDER information



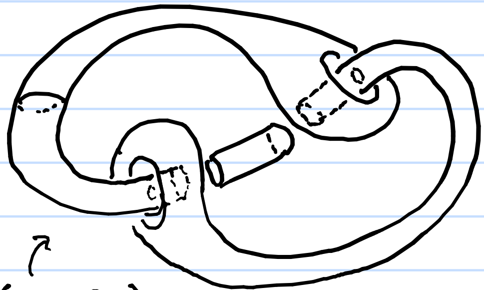
For Link diagrams:



BROKEN SURFACE DIAGRAMS

surface

Given $\Sigma \underset{\text{smooth}}{\hookrightarrow} \mathbb{B}^4$



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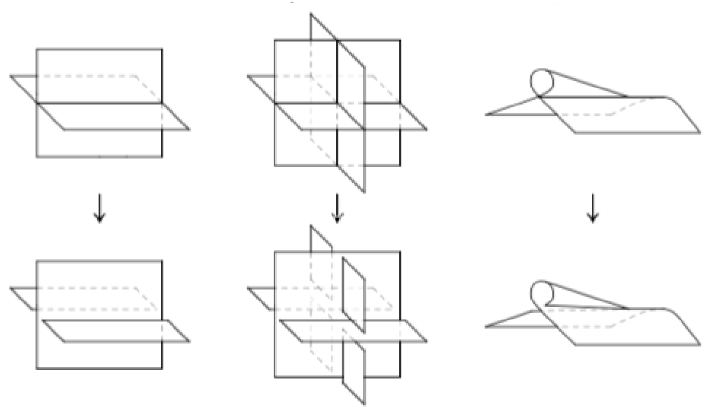
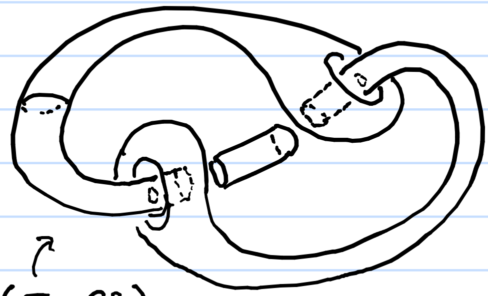


towards CUT-DIAGRAMS

surface

Given $\Sigma \xhookrightarrow{\text{smooth}}$ 4-space



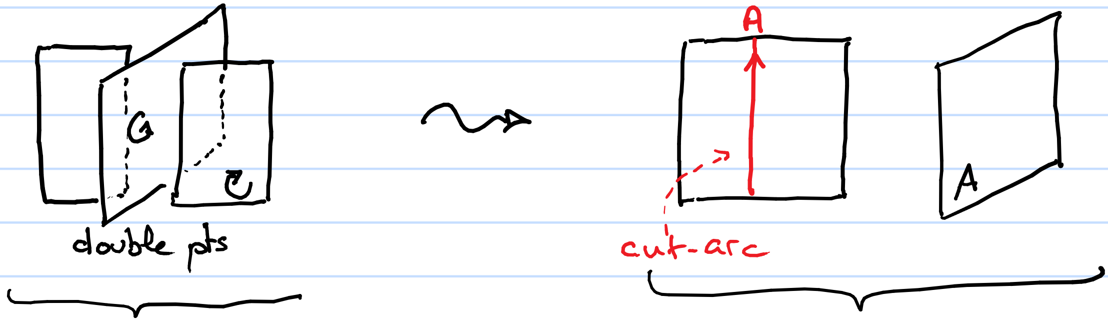
$(\Sigma = S^2)$

BROKEN SURFACE DIAGRAM

$\rightsquigarrow$ A diagram on the (abstract) surface Σ

CUT-DIAGRAM

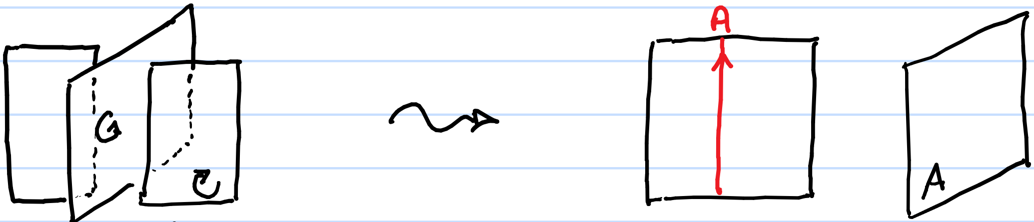
towards CUT-DIAGRAMS



Broken surface diag. of $\Sigma \hookrightarrow \mathbb{B}^4$

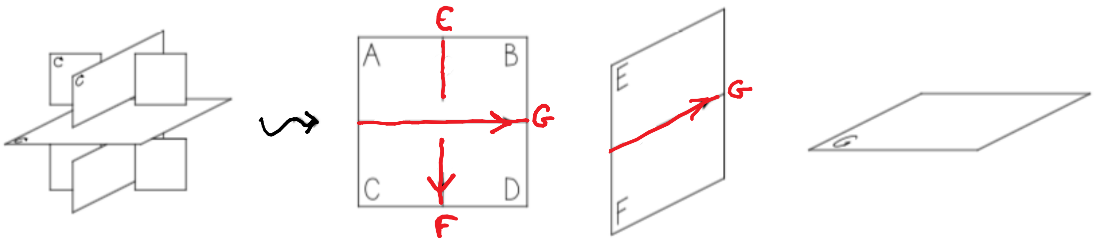
Cut-diagram over Σ

towards CUT-DIAGRAMS

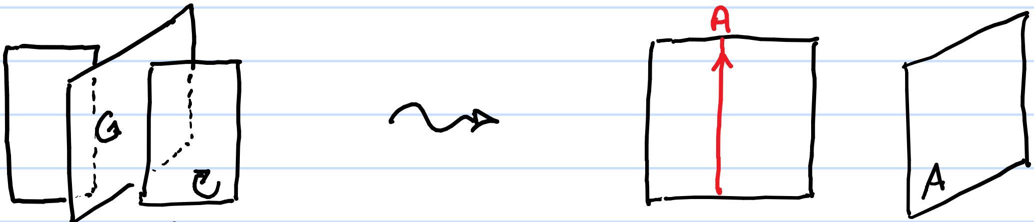


double pts

triple point



towards CUT-DIAGRAMS



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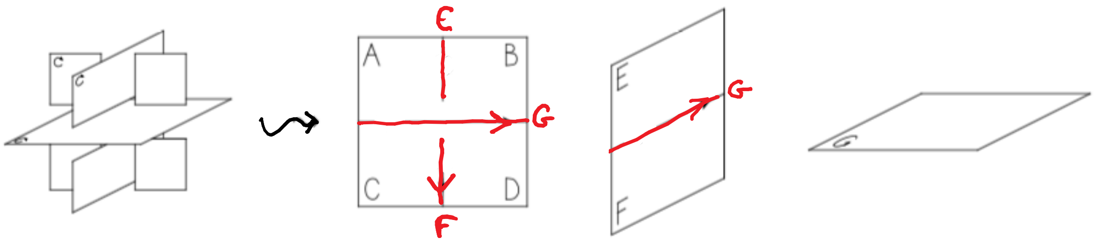
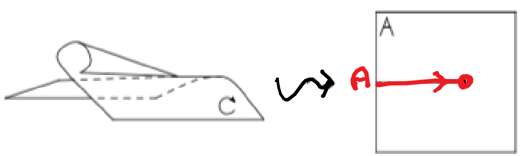
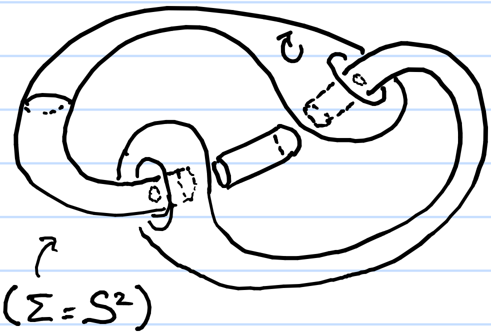


branch point



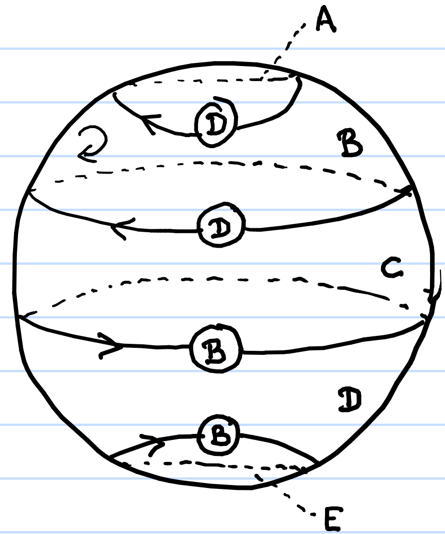
towards CUT-DIAGRAMS

$$\text{Given } \underset{\text{surface}}{\Sigma} \xhookrightarrow[\text{smooth}]{} \text{4-space}$$



BROKEN SURFACE DIAGRAM

$\rightsquigarrow$



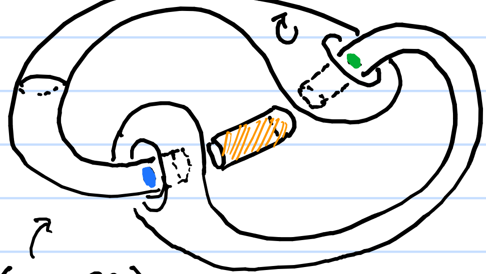
'CUT-DIAGRAM' over S^2

towards CUT-DIAGRAMS

surface

Given $\Sigma \xhookrightarrow[\text{smooth}]{} \text{4-space}$

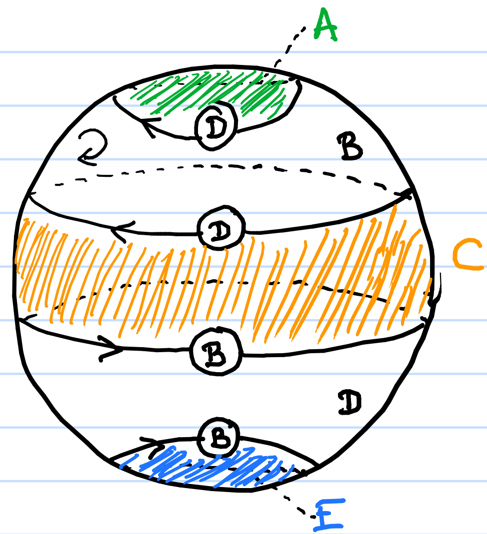
$\rightsquigarrow$



$(\Sigma = S^2)$

BROKEN SURFACE DIAGRAM

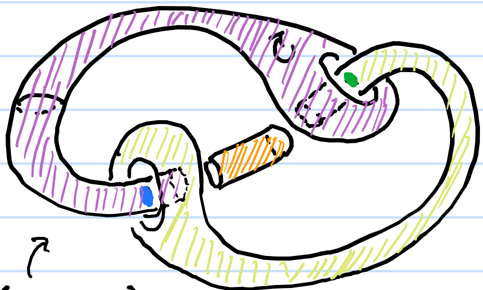
$\rightsquigarrow$



'CUT-DIAGRAM' over S^2

towards CUT-DIAGRAMS

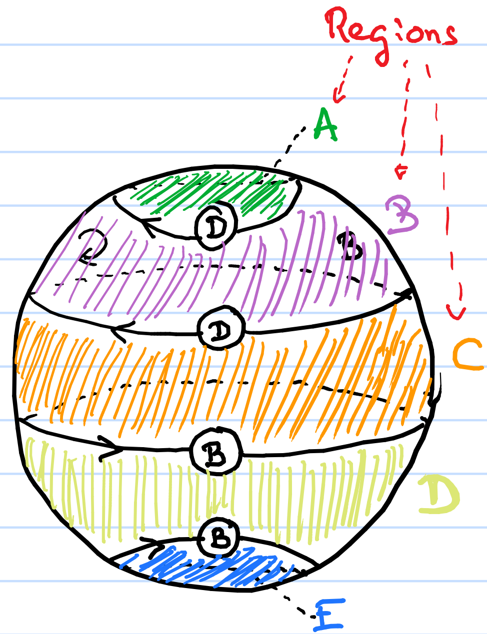
Given $\underset{\text{surface}}{\Sigma} \xhookrightarrow[\text{smooth}]{} \text{4-space}$



$(\Sigma = S^2)$

BROKEN SURFACE DIAGRAM

$\rightsquigarrow$

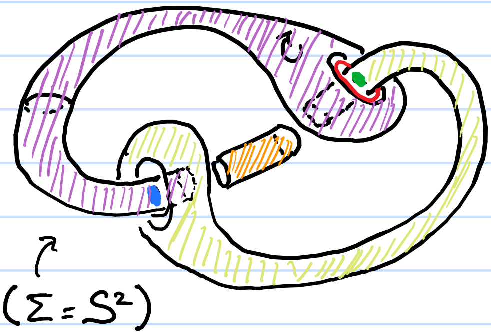


'CUT-DIAGRAM' over S^2

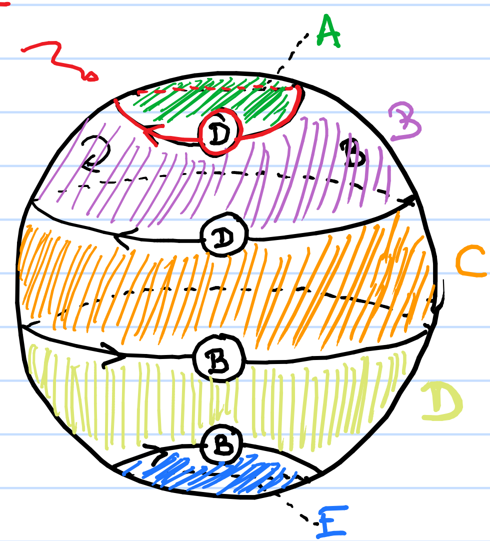
towards CUT-DIAGRAMS

Given $\Sigma \xrightarrow[\text{smooth}]{} $ 4-space

surface



BROKEN SURFACE DIAGRAM

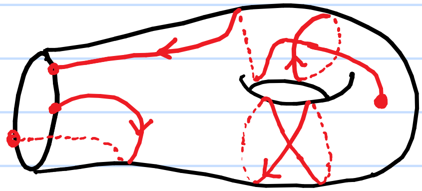


'CUT-DIAGRAM' over S^2

Cut-Diagrams

(abstract) surface

A **cut-diagram** over Σ = immersed oriented 1-man. $\mathcal{P} \looparrowright \Sigma$

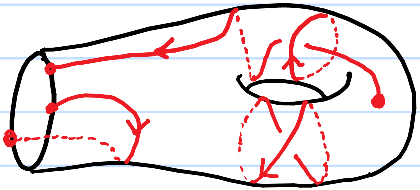


CUT-DIAGRAMS

(abstract) surface

A **CUT-DIAGRAM** over Σ = immersed oriented 1-man. $P \looparrowright \Sigma$

+ over/under info at each X



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splits Σ into

REGIONS

CUT-DIAGRAMS

(abstract) surface

A CUT-DIAGRAM over Σ = immersed oriented 1-man. $\mathcal{P} \looparrowright \Sigma$

+ over/under info at each $\times$

splits Σ into REGIONS

splits $\mathcal{P}$ into CUT ARCS

CUT-DIAGRAMS

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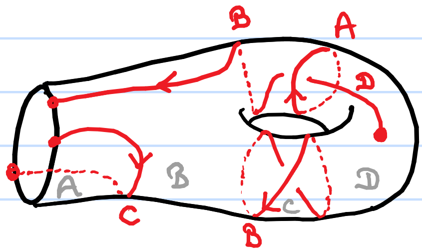


+ over/under info at each $\times$

splits Σ into **REGIONS**

splits $\mathcal{P}$ into **CUT ARCS**

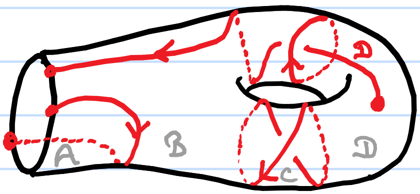
+ **LABELING** of each cut arc by a region

CUT-DIAGRAMS

(abstract) surface

A **CUT-DIAGRAM** over Σ = immersed oriented 1-man. $P \looparrowright \Sigma$

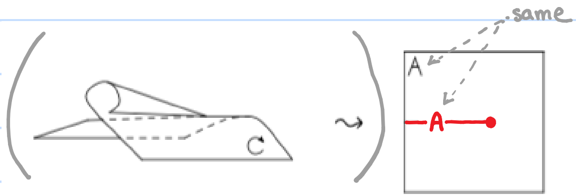
+ over/under info at each $\times$



splits Σ into **REGIONS**

splits P into **CUT ARCS**

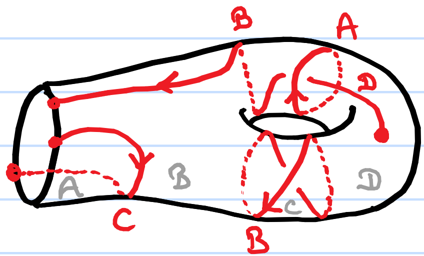
+ **LABELING** of each cut arc by a region satisfying 'natural' conditions:



CUT-DIAGRAMS

(abstract) surface

A CUT-DIAGRAM over Σ = immersed oriented 1-man. $\mathcal{P} \looparrowright \Sigma$



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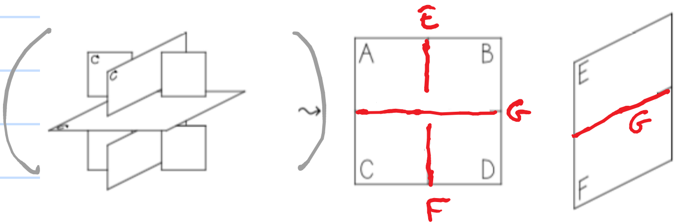
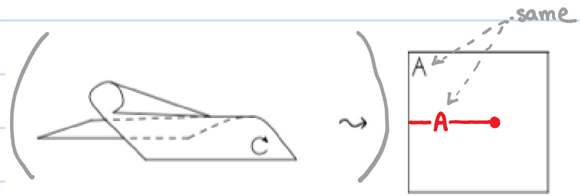
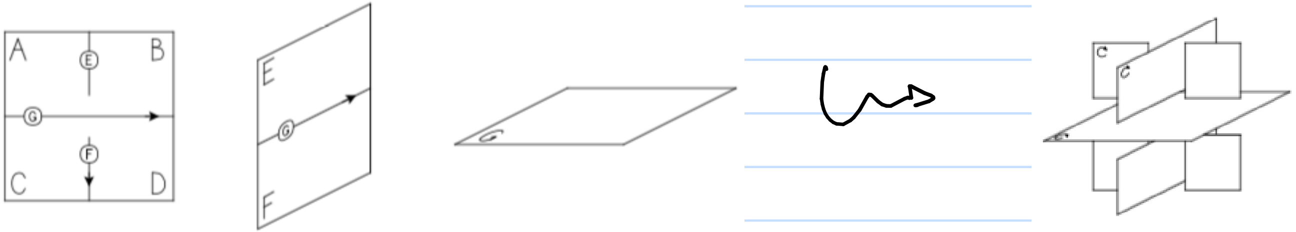


&



Cut-Diagrams



"Build your own broken surface diagram with paper and scissors!"

GROUP of a CUT-DIAGRAM

$\mathcal{C}$ = cut-diagram over Σ

$\rightsquigarrow$ group $G(\mathcal{C})$

GROUP of a CUT-DIAGRAM

$\mathcal{C}$ = cut-diagram over Σ

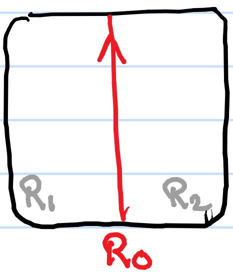
$\leadsto$ group $G(\mathcal{C})$: ▸ a generator R for each region
(MERIDIANS)

GROUP of a CUT-DIAGRAM

$\mathcal{C}$ = cut-diagram over Σ

$\rightsquigarrow$ group $G(\mathcal{C})$: ▷ a generator R for each region (MERIDIANS)

▷ a relation at each cut arc :



$\rightsquigarrow$ $\boxed{R_1 = R_0^{-1} . R_2 . R_0}$

GROUP of a CUT-DIAGRAM

$\mathcal{C}$ = cut-diagram over Σ

$\rightsquigarrow$ group $G(\mathcal{C})$: ▷ a generator R for each region (MERIDIANS)

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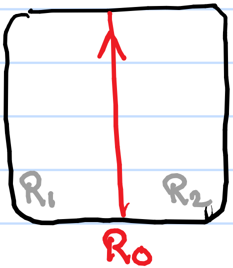


$\rightsquigarrow$ $\boxed{R_1 = R_0^{-1} \cdot R_2 \cdot R_0}$

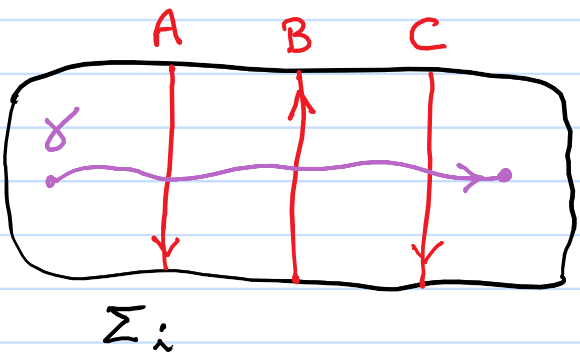
FACT. If $\mathcal{C}$ comes from the broken diagram of some knotted surf. K, then $G(D) \simeq \pi_1(B^4 \setminus K)$.

LONGITUDES

Given $\mathcal{C}$ cut-diag. on Σ

and γ = oriented path on Σ_i ith comp. of Σ

$\rightsquigarrow$ $\widetilde{W}_\gamma =$ in $G(\mathcal{C})$



LONGITUDES

Given $\mathcal{C}$ cut-diag. on Σ

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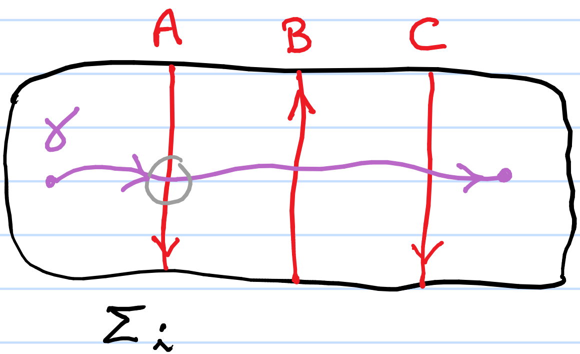
$\leadsto \tilde{W}_\gamma$ = A in $G(\mathcal{C})$

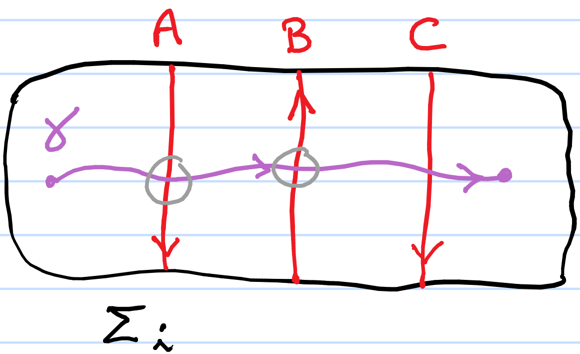


LONGITUDES

Given $\mathcal{C}$ cut-diag. on Σ

and γ = oriented path on Σ_i ith comp. of Σ

$\rightsquigarrow \widetilde{W}_\gamma = \quad A.B^{-1} \quad$ in $G(\mathcal{C})$



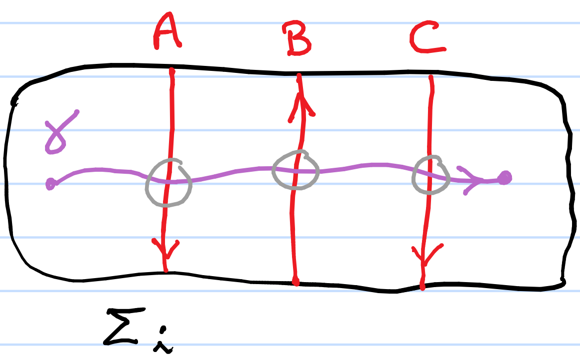
LONGITUDES

Given $\mathcal{C}$ cut-diag. on Σ

and γ = oriented path on Σ_i ---- ith comp. of Σ

$\rightsquigarrow \quad \widetilde{W}_\gamma = \qquad A.B^{-1}.C \qquad$ in $G(\mathcal{C})$

ith LONGITUDE assoc. to γ



LONGITUDES

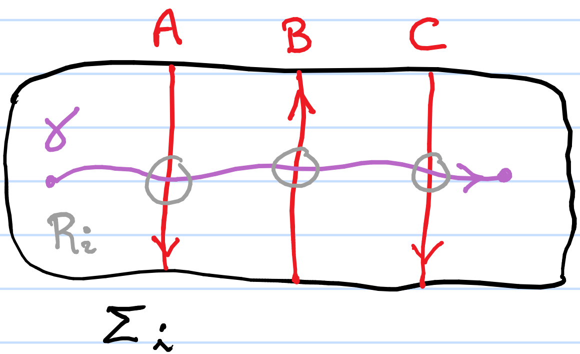
Given $\mathcal{C}$ cut-diag. on Σ

and γ = oriented path on Σ_i ← ith comp. of Σ

$$\rightsquigarrow \quad W_\gamma = R_i^{w(\gamma)} \cdot A . B^{-1} . C \qquad \text{in } G(\mathcal{C})$$

PREFERRED ith LONGITUDE

assoc. to γ



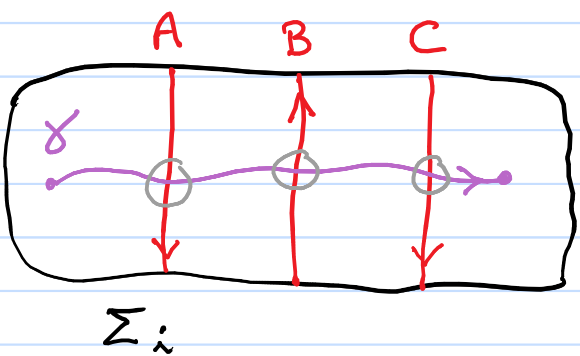
LONGITUDES

Given $\mathscr{C}$ cut-diag. on Σ ith comp. of Σ

and γ = oriented path on Σ_i

$$\rightsquigarrow \quad W_\gamma = R_i^{\omega(\gamma)} . A . B^{-1} . C \qquad \text{in } G(\mathscr{C})$$

PREFERRED ith LONGITUDE assoc. to γ



Fact. $W_\gamma \in G(\mathscr{C})$ only depends on the HOMOTOPY class of γ!

Chen-Milnor-type Presentation

Let $\mathcal{C}$ = cut-diagram on $\Sigma = \Sigma_1 \sqcup \dots \sqcup \Sigma_n$.

Let R_i = choice of meridian for Σ_i

& $\{\ell_{ij}\}_j$ = loops forming a basis for $\pi_1(\Sigma_i)$

then $\forall q \geq 1$,

$$N_q G(\mathcal{C}) = \left\langle R_1, \dots, R_n \;\middle|\; \begin{array}{l} A_q = 1 \\ [R_i ; W_{ij}^q] = 1, \ \forall i, j \end{array} \right\rangle$$

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then $\forall q \geqslant 1$,

$A = F(R_1, \dots, R_e)$

$$N_q G(\mathcal{C}) = \left\langle R_1, \dots, R_n \;\middle|\; \begin{array}{l} A_q = 1 \\ [R_i ; W_{ij}^q] = 1, \ \forall i,j \end{array} \right\rangle$$

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ith preferred longitude

$A = F(R_1, \dots, R_\ell)$

then $\forall q \geq 1$,

$$N_q G(\mathcal{C}) = \left\langle R_1, \dots, R_n \;\middle|\; \begin{array}{l} A_q = 1 \\ [R_i ; w_{ij}^q] = 1, \ \forall i,j \end{array} \right\rangle$$

Chen-Milnor-type Presentation

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& $\{l_{ij}\}_j$ = loops forming a basis for $\pi_1(\Sigma_i)$

then $\forall q \geq 1$,

$$N_q G(\mathcal{C}) = \left\langle R_1, \dots, R_n \;\middle|\; \begin{array}{l} A_q = 1 \\ [R_i ; W_{ij}^q] = 1, \ \forall i,j \end{array} \right\rangle$$

$$\left(\begin{array}{l} \text{[Chen-Milnor] For } q \geq 1 \\ L : \text{Link} \end{array} \quad N_q \pi_L = \left\langle m_1, \dots, m_n \;\middle|\; \begin{array}{l} \cdot\ A_q = 1 \\ \cdot\ \forall i,\ [m_i ; \lambda_i] = 1 \end{array} \right\rangle \right)$$

Chen-Milnor-type Presentation

Let $\Sigma \hookrightarrow B^4$ knotted surface ; $\pi_\Sigma := \pi_1(B^4 \setminus \Sigma)$

Let m_i = choice of meridian for Σ_i

& $\{\ell_{ij}\}_j$ = loops forming a basis for $\pi_1(\Sigma_i)$

then $\forall q \geq 1$,

$$N_q \pi_\Sigma = \left\langle m_1, \dots, m_n \;\middle|\; \begin{array}{l} A_q = 1 \\ [m_i ; w_{ij}^q] = 1, \ \forall i,j \end{array} \right\rangle$$

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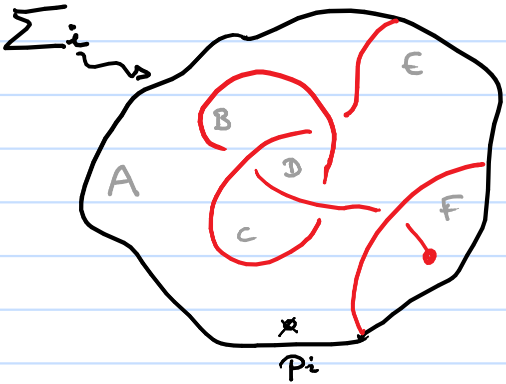
Idea of Proof



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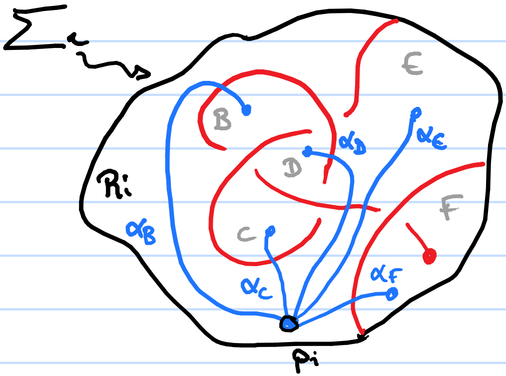


α ROAD SYSTEM for $\mathcal{C}$

$\rightsquigarrow \quad \forall q$

$$\eta^\alpha_q : F\langle \text{Regions}\rangle \longrightarrow F\langle R_1,\dots,R_n\rangle$$

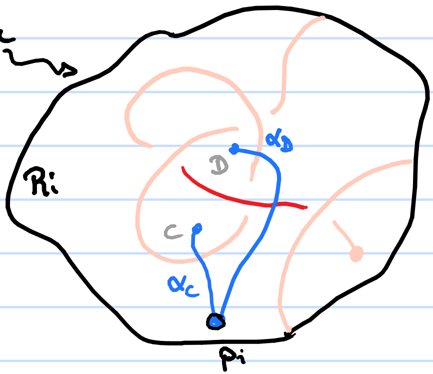
def by: $\eta^\alpha_1(i\text{th regions}) = R_i$

and

$$\eta^\alpha_{q+1} : \underset{\Sigma_i}{\underset{\cap}{R}} \longmapsto \eta^\alpha_q(\omega_{\alpha_R})^{-1}\cdot R_i \cdot \eta^\alpha_q(\omega_{\alpha_R})$$

IDEA of PROOF

Σ_c



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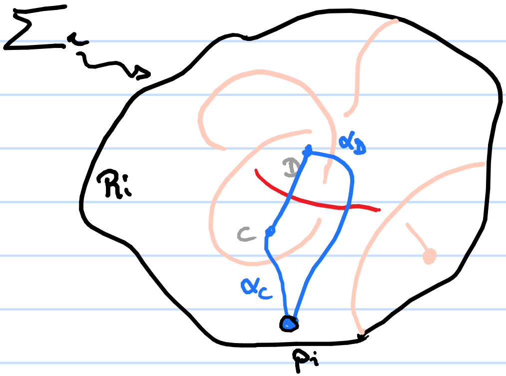
'WIRTINGER' relations

Idea of Proof



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'Wirtinger' relations $\rightsquigarrow$ $[R_i, \eta_q(\underbrace{i\text{th loop-longitude}}_{\text{product of } \rho_{ij}\text{'s}})] = 1$

MILNOR INVARIANTS of CUT-DIAGRAMS

Let $q \geq 1$

Let $\mathcal{C}$ = cut-diagram on Σ + choice of meridians R_i
+ $\{\ell_{ij}\}_j$ basis for $\pi_1(\Sigma_i)$.

$\hookrightarrow$ any elem. of $N_q G(\mathcal{C})$ is given by a word in $\{R_i^{\pm 1}\}$.

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$\hookrightarrow$ $\forall$ path γ on Σ_i $\rightsquigarrow$ W_γ^q = word in $\{R_j^{\pm 1}\}$

MAGNUS Exp : $E(W_\gamma^q) = 1 + \sum_{j_1 \cdots j_k} \mu_{\mathcal{C}}(j_1, \ldots, j_k; \gamma) X_{j_1} \cdots X_{j_k}$

$\in \mathbb{Z}\langle\langle X_1, \ldots, X_\ell \rangle\rangle$

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Milnor invariants of cut-diagrams

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$\triangleright$ $\forall$ sequ. I in $\{1,\dots,n\}$, $\quad \Delta_{\mathcal{C}}(I) = \gcd\left\{ \mu_{\mathcal{C}}(I') \,\middle|\, \begin{array}{l} I' \text{ obt. by deleting} \\ \text{index and } \circlearrowleft \end{array} \right\}$

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[AUDOUX-M.-YASUHARA] $\forall$ sequ. J of $< q$ indices,

MILNOR NUMBERS of $\mathcal{C}$ $\dashrightarrow$ $\overline{\mu}_{\mathcal{C}}(J;\gamma) \equiv \mu_{\mathcal{C}}(J;\gamma) \mod \Delta_{\mathcal{C}}(Ji)$

is Well-defined.

MAIN RESULTS

- $\mathcal{C}$ cut-diagram over Σ , γ path on Σ

  $\rightsquigarrow$ $\overline{\mu}_{\mathcal{C}}(J;\gamma)$ well-def. $\forall$ sequence J.

[AUDOUX-M.-YASUHARA]

If $\mathcal{C}$ = cut-diag. associated w/ diag. of $K : \Sigma \hookrightarrow \mathbb{B}^4$

$\overline{\mu}_{\mathcal{C}}(J;\gamma)$ is an invariant of K.

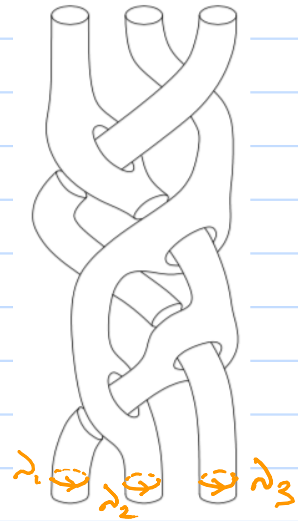
Moreover,

this is a concordance invariant.

EXAMPLE

If $\Sigma = \ell$ annuli ; $K : \Sigma \hookrightarrow \mathbb{B}^4$ 'concordance'

$\forall i$, $\pi_1(i\text{th annulus}) = \langle \lambda_i \rangle$



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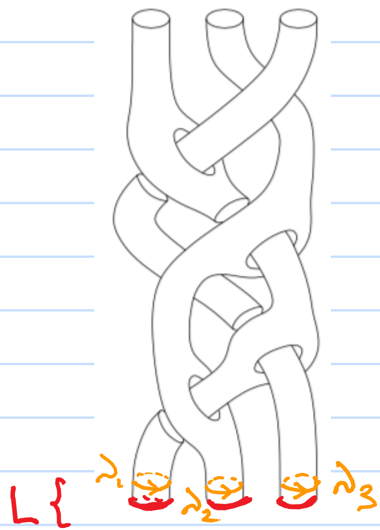

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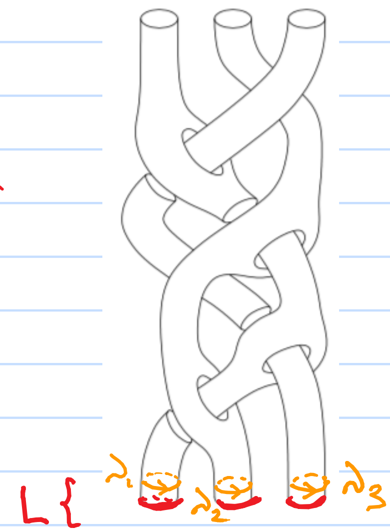
$\forall i$, $\pi_1(i\text{th annulus}) = \langle \lambda_i \rangle$

$\rightsquigarrow \forall$ sequ. J, $\Delta_{\mathcal{C}}(J;i) = \underbrace{\Delta_L(J;i)}_{\text{det. by Milnor inv. of } L}$

$\rightsquigarrow \bar{\mu}_{\mathcal{C}}(J;i)$ recovers [M-YASUHARA]



MORE ?

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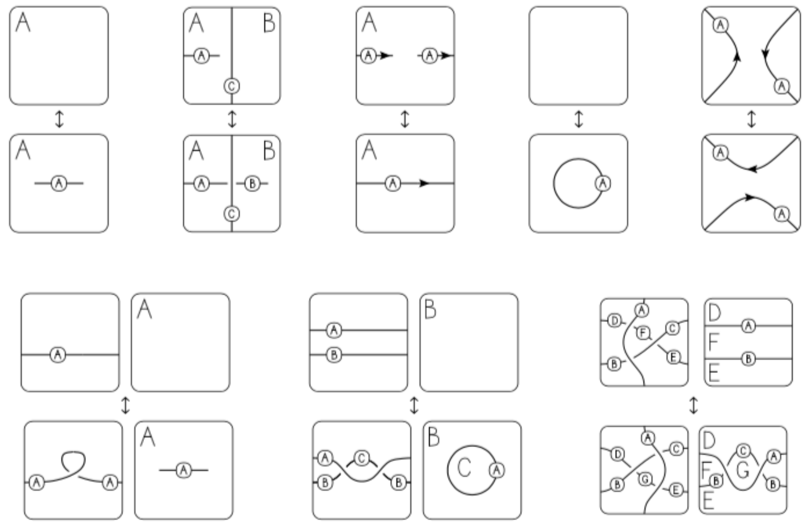
'Roseman' moves for cut-diagrams :



More ?

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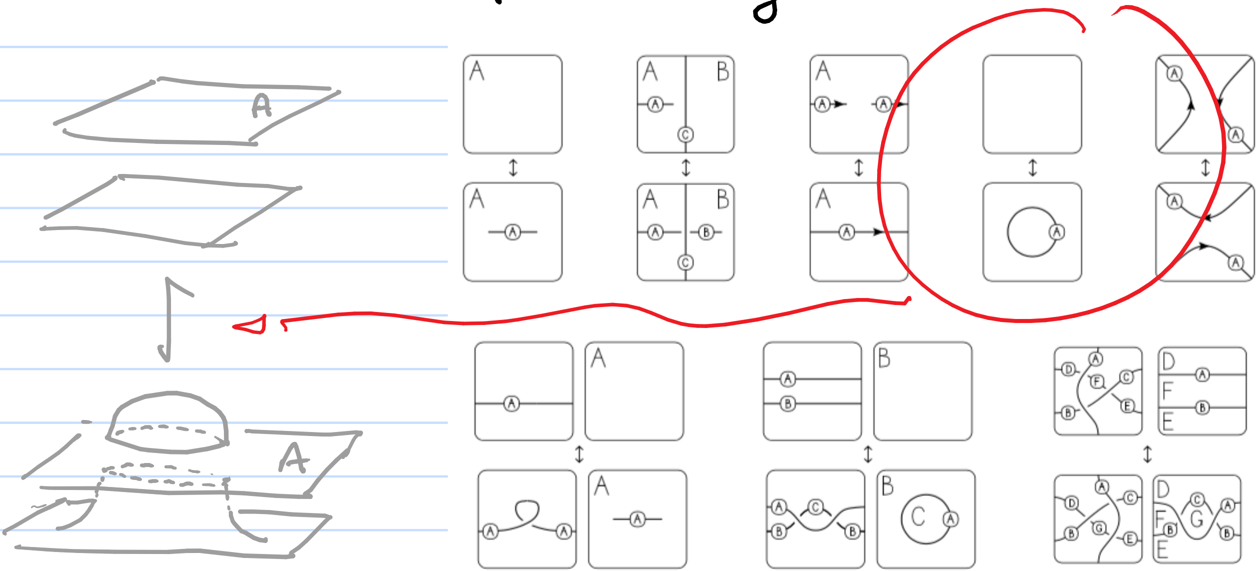

↝ Milnor numbers are (isot.) 'inv. of knotted surfaces'

MORE ?

▷ Cut-diagrams in <u>any</u> dimension + notion of concordance

↳ Milnor numbers of codim. 2 embeddings
(+ concordance inv.)

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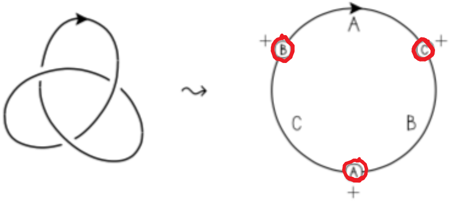
- Cut-diagrams in any dimension + notion of concordance

↳ Milnor numbers of codim. 2 embeddings
(+ concordance inv.)

(In dim. 1, recovers Milnor inv of links/string links
in fact, of welded tangles)



1-dim. cut diagrams / 'Reidemeister' moves.