Mapping class group actions from Hopf Monoids

[K-OS] Knot Online Seminar

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Motivation

- **understand mapping class group actions in Kitaev lattice models** [Kitaev,…]
  - from condensed matter physics, topological quantum computing
  - related to Turaev-Viro invariants

  ⇨ generalise some aspects of these models

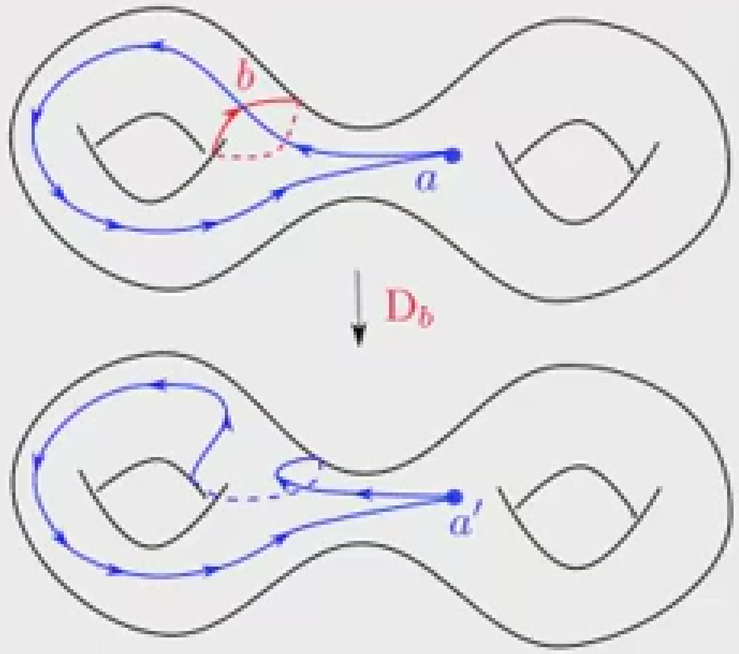
- **generalise mapping class group representations from Hopf algebras** [Lyubashenko,…]
  - data: Hopf algebras in rigid linear braided monoidal categories
  - related to Reshetikhin-Turaev TQFTs
  - arise in quantisation of moduli spaces of flat connections on surfaces [Alekseev, Grosse, Schomerus…]
  - well understood, need strong assumptions: [Buffenoir, Roche, Faitg…]
    $\mathcal{C}$ linear, abelian, with duals, rigid,….
  - projective mapping class group actions
  - recent generalisations to bimodule categories [Fuchs, Schweigert, Schaumann]

  ⇨ more general and different data, non-projective actions

- **flexible and explicit description**

1. Mapping class groups

- oriented surface Σ, genus $g \geq 1$, $n \geq 0$ boundary components

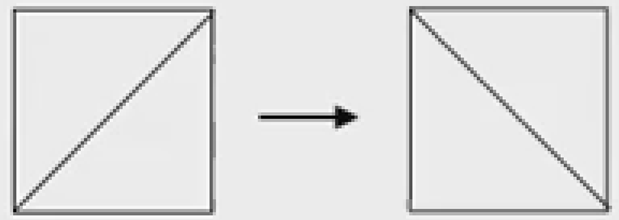


- **mapping class group**

$$\mathrm{Map}(\Sigma) = \frac{\mathrm{Homeo}^+(\Sigma)}{\mathrm{Homeo}_0^+(\Sigma)}$$

← fix $\partial\Sigma$ pointwise

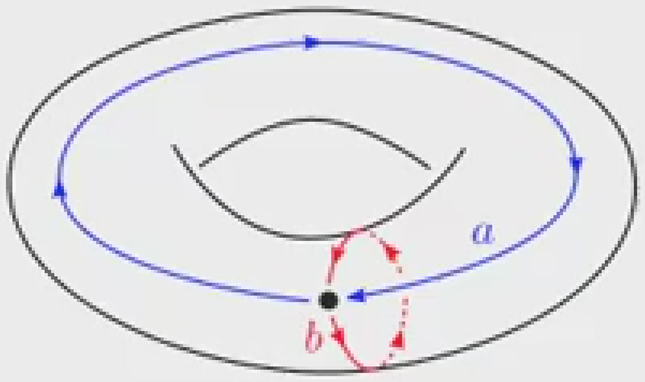
- presentation in terms of Dehn twists [Wajnryb, Gervais]
  simple generators and complicated relations



- action on triangulation: by flips [Penner,…]
  simple relations (involution+pentagon)

- action by chord slides [Bene,…]
  simple and more flexible

- Ex: Torus



$$\mathrm{Map}(T) = \mathrm{SL}(2,\mathbb{Z}) = \langle A, B \mid (BAB)^4 = 1, ABA = BAB \rangle$$

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Dehn twist along a — Dehn twist along b

Presentation by Dehn twists

oriented surface Σ genus $g \geq 1$
$n \geq 0$ boundary circles
$n + g > 1$

Theorem [Gervais]

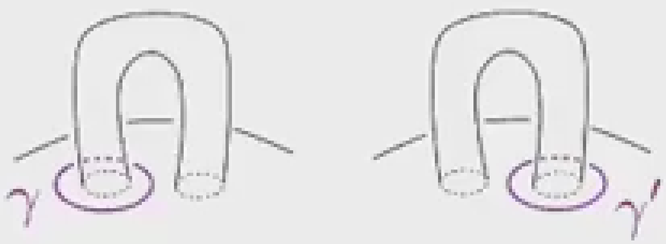
$\mathrm{Map}(\Sigma)$ generated by **Dehn twists** along $\alpha_i \quad 1 \leq i \leq g$
$\delta_j \quad 1 \leq j \leq n + 2g - 2$
$\gamma_{i,j} \quad 1 \leq i, j \leq n + 2g - 2 \quad i \neq j$

with relations

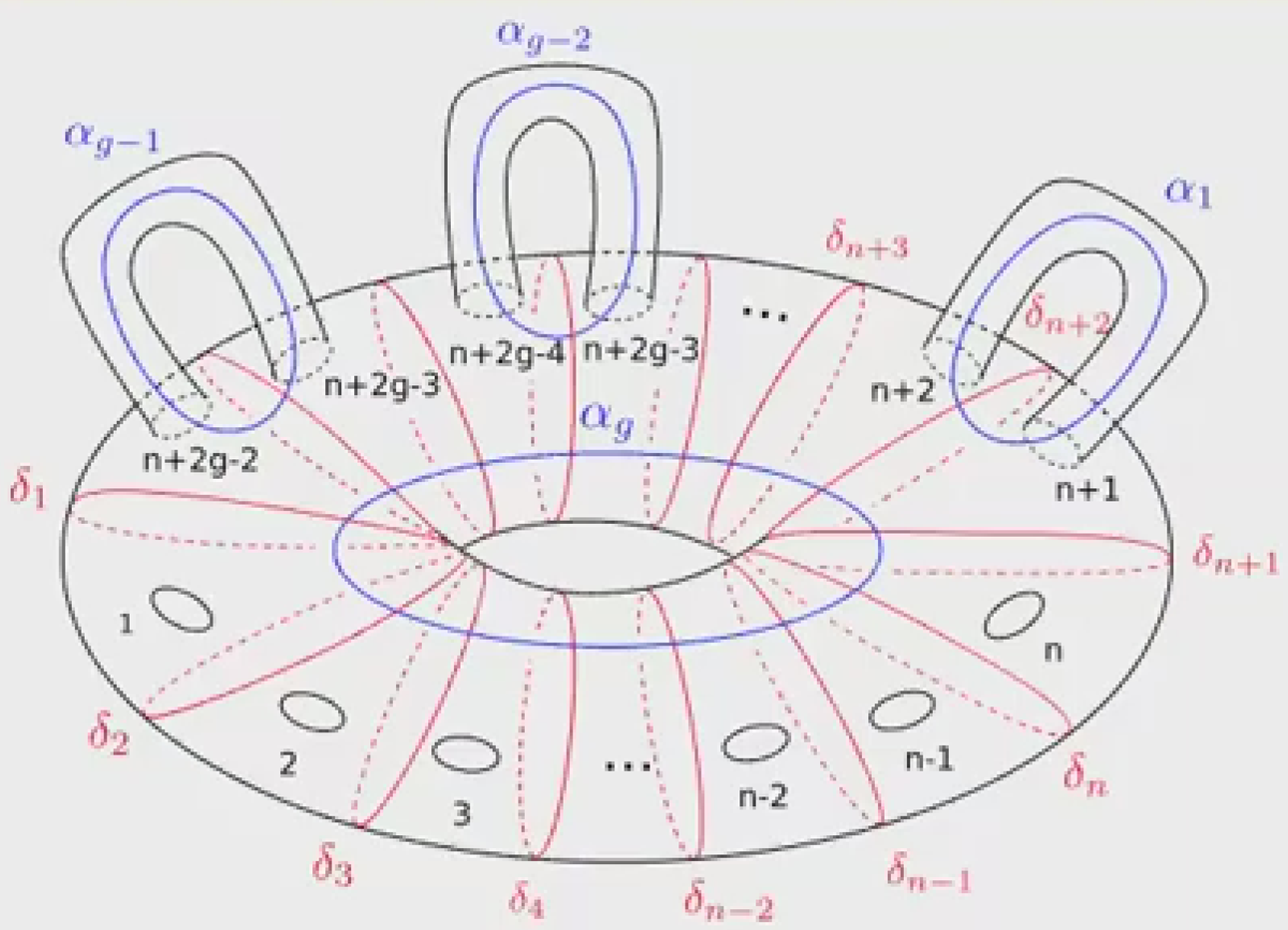
$D_\rho \circ D_\sigma = D_\sigma \circ D_\rho$ if $\rho \cap \sigma = \emptyset$

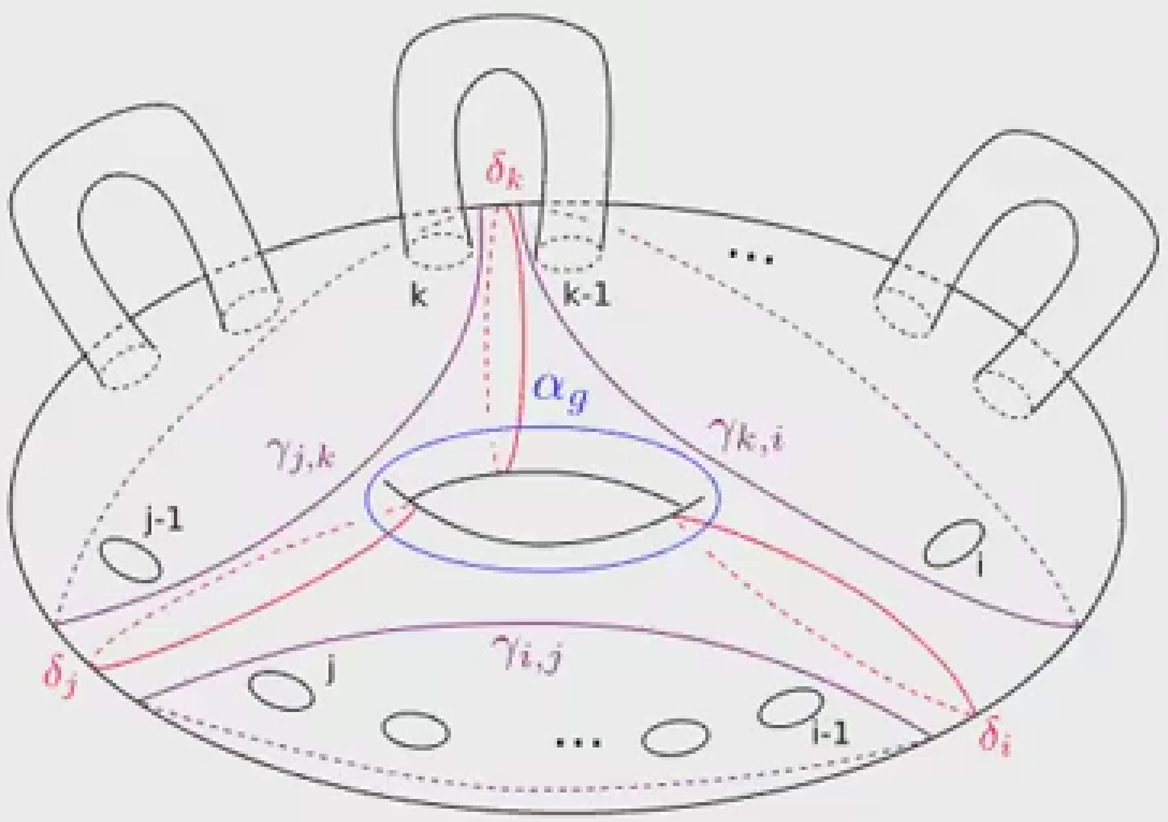
$D_\sigma \circ D_\rho \circ D_\sigma = D_\rho \circ D_\sigma \circ D_\rho$ if $|\rho \cap \sigma| = 1$

$D_\gamma = D_{\gamma'}$



$(D_{\delta_i} \circ D_{\delta_j} \circ D_{\delta_k} \circ D_{\alpha_g})^3 = D_{\gamma_{i,j}} \circ D_{\gamma_{j,k}} \circ D_{\gamma_{k,i}}$ for $(i,j,k) \neq (x,x,x)$ and $1 \leq k \leq i \leq j$ or $1 \leq i \leq j \leq k$ or $1 \leq j \leq k \leq i$





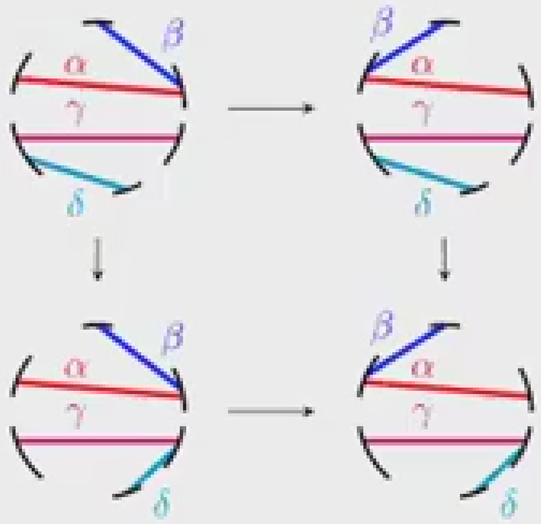
Presentation by chord slides

Theorem [Bene]

oriented surface Σ of genus $g \geq 1$ with 1 boundary component ⇨ chord diagram

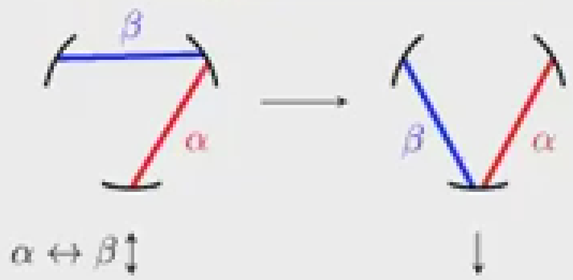
$\mathrm{Map}(\Sigma)$: finite sequences of **chord slides** that preserve diagram with relations

- **commutativity relation**

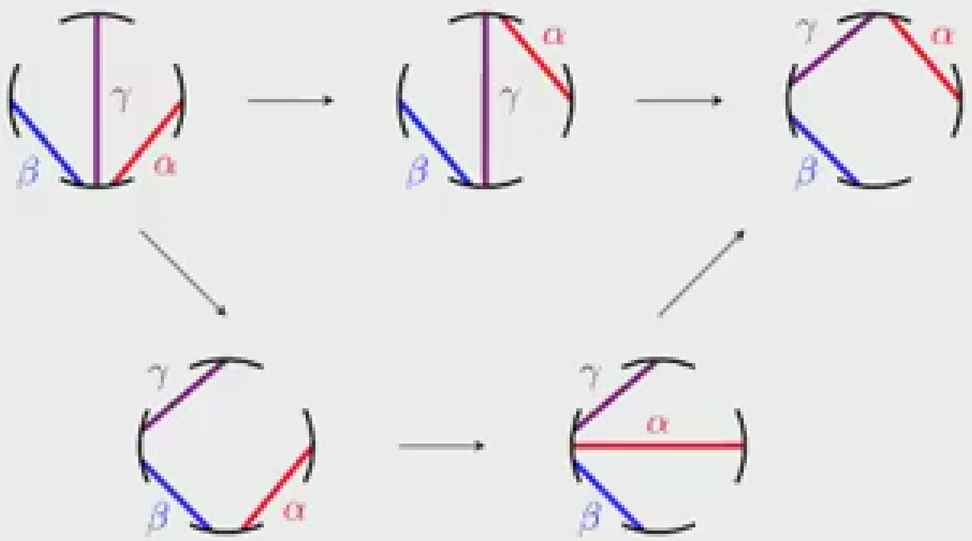


- **involutivity relation**

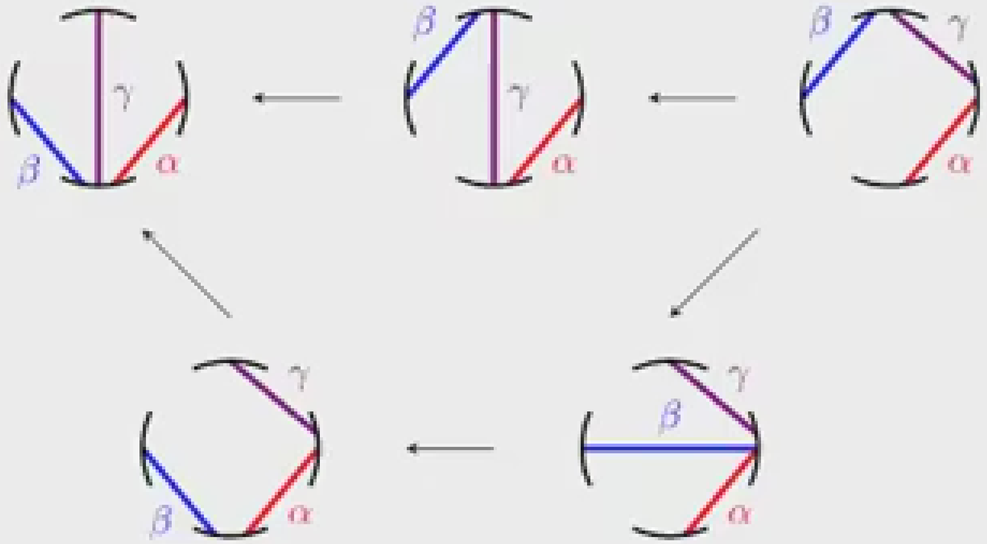
- **triangle relation**



- **left pentagon relation**



- **right pentagon relation**



2. Hopf Monoids

symmetric monoidal category

category $\mathcal{C}$

functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

tensor unit $e \in \mathrm{Ob}\mathcal{C}$

coherence data

symmetry constraint $c : \otimes \xrightarrow{\sim} \otimes^{op}$

- braided monoidal category
- $c_{A,B} = c^{-1}_{B,A}$

- **examples**
  - $\mathcal{C} = (\mathrm{Vect}_{\mathbb{F}}, \otimes, \mathbb{F})$
  - $\mathcal{C} = (\mathrm{Ch}_{k-\mathrm{Mod}}, \otimes_{\bullet}, 0 \to k \to 0)$
  - cartesian monoidal categories:

    tensor product = categorical product

    tensor unit = terminal object

    $\mathcal{C} = (\mathrm{Set}, \times, \{\bullet\})$

    $\mathcal{C} = (\mathrm{Top}, \times, \{\bullet\})$

    $\mathcal{C} = (\mathrm{Grp}, \times, \{e\})$

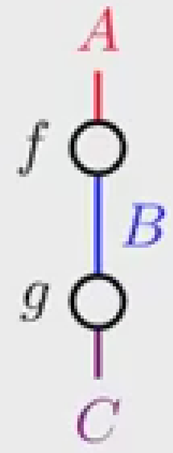
    $\mathcal{C} = (G\text{−Set}, \times, \bullet)$

    $\mathcal{C} = \mathrm{Fun}(\mathcal{A}, \mathrm{Set})$

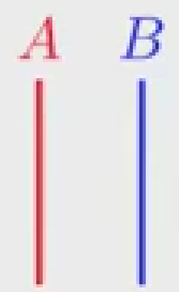
    $\mathcal{C} = \mathrm{SSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

- **graphical calculus**

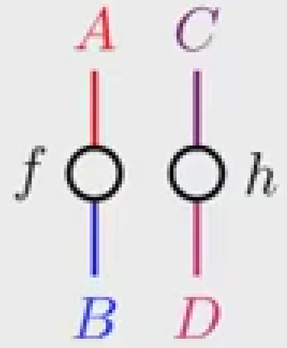
$f : A \to B$



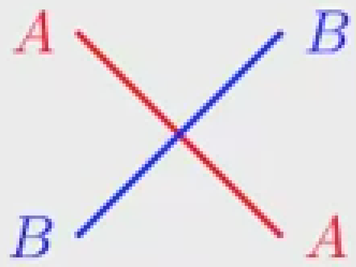
$g \circ f : A \to C$



$A \otimes B$



$f \otimes h : A \otimes C \to B \otimes D$



$c_{A,B} : A \otimes B \to B \otimes A$

Hopf monoid in symmetric monoidal category $\mathcal{C}$

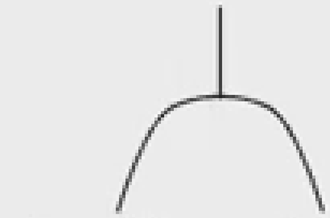
• **structure**

$m : H \otimes H \to H$

multiplication

$\eta : e \to H$

unit



$\Delta : H \to H \otimes H$

comultiplication

$\epsilon : H \to e$

counit

• **property**

$S : H \to H$

antipode

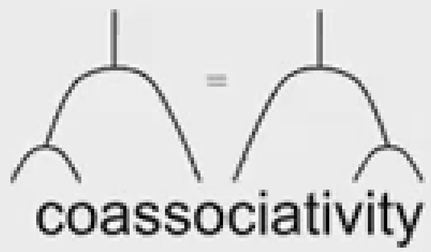
• **conditions**

• **monoid**

associativity

unitality

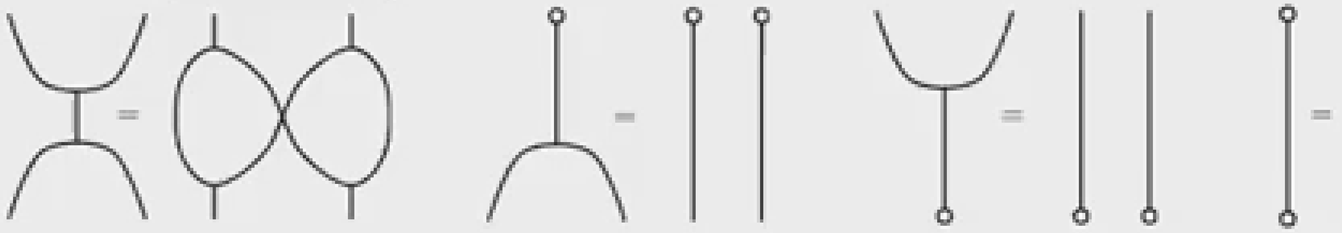
• **comonoid**



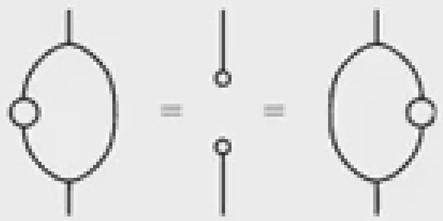
coassociativity

counitality

• **compatibility**

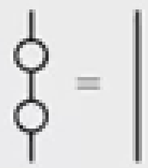


• **antipode**



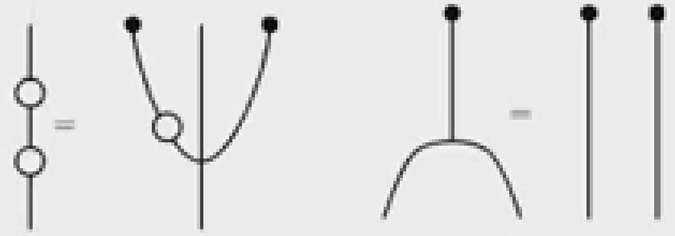
Δ, ϵ monoid morphisms $\quad m, \eta$ comonoid morphisms

involutive Hopf monoid



$S \circ S = 1_H$

pivotal Hopf monoid



$\sim \; g$ grouplike $\qquad g \cdot (S \circ S) \cdot g^{-1} = 1_H$

- **examples: involutive Hopf monoids**

- $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$: involutive Hopf algebras
  - group algebras $\mathbb{F}[G]$
  - universal enveloping algebras $U(\mathfrak{g})$
  - finite-dim semisimple Hopf algebras over $\mathbb{F} = \mathbb{C}$

- $\mathcal{C} = \mathrm{Ch}_{k\text{-Mod}}$: examples from algebraic topology

- cartesian monoidal category $(\mathcal{C}, \times, \bullet)$: group objects in $\mathcal{C}$

$\mathcal{C} = \mathrm{Set}$ groups

$$m : H \times H \to H, (h, h') \mapsto hh'$$
$$\eta : \{\bullet\} \to H, \bullet \mapsto e_H$$
$$\Delta : H \to H \times H, h \mapsto (h, h)$$
$$\epsilon : H \to \{\bullet\}$$
$$S : H \to H, h \mapsto h^{-1}$$

$\mathcal{C} = \mathrm{Top}$ topological groups

$\mathcal{C} = \mathrm{Cat}$ crossed modules

$\mathcal{C} = \mathrm{SSet}$ simplicial groups

$\mathcal{C} = G - \mathrm{Set}$ semidirect products with G

modules over Hopf monoid

object $M \in \mathrm{Ob}\,\mathcal{C}$

morphism $\triangleright : H \otimes M \to M$

comodules over Hopf monoid

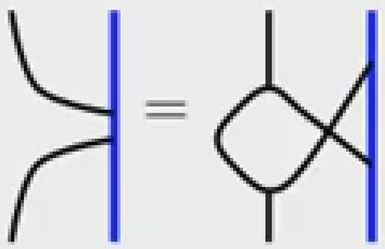
object $M \in \mathrm{Ob}\,\mathcal{C}$

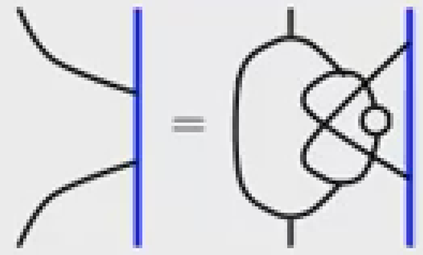
morphism $\delta : M \to H \otimes M$

- **examples**
  - $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$, H Hopf algebra: usual module and comodule
  - $\mathcal{C} = \mathrm{Set}$, H group: module = set M with H-action $\triangleright : H \times M \to M$

    comodule = map $f : M \to H$ ⇨ $\delta = (f, \mathrm{id}) : M \to H \times M$

Hopf modules module + comodule + compatibility condition **Yetter-Drinfeld modules**





- **examples**
  - $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$, H Hopf algebra: usual Hopf and Yetter-Drinfeld modules
  - $\mathcal{C} = \mathrm{Set}$, H group: set M with H-action $\triangleright : H \times M \to M$ and map $f : M \to H$

    with $f(h \triangleright m) = hf(m)$ with $f(h \triangleright m) = hf(m)h^{-1}$

(co)invariants

$\mathcal{C}$ finitely complete and cocomplete

H Hopf monoid in $\mathcal{C}$

- **invariants** of H-module $(M, \triangleright)$: coequaliser $H \otimes M \underset{\epsilon\otimes 1_M}{\overset{\triangleright}{\rightrightarrows}} M \xrightarrow{\pi} M^H$
- **coinvariants** of H-comodule (M, δ) : equaliser $M^{coH} \xrightarrow{\iota} M \underset{\eta\otimes 1_M}{\overset{\delta}{\rightrightarrows}} H \otimes M$
- **biinvariants** of H-module and H-comodule $(M, \triangleright, \delta)$: image of $\pi \circ \iota : M^{coH} \to M^H$

$$M_{inv} \overset{I}{\hookrightarrow} M^H \underset{i_2}{\overset{i_1}{\rightrightarrows}} M^H \amalg_{M^{coH}} M^H$$

- **fact:** $f : M \to M'$ isomorphism of H - modules and H - comodules
  ⇨ induces isomorphism $f' : M_{inv} \to M'_{inv}$

- **examples**
  - $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$, H finite-dim semisimple Hopf algebra: usual (co)invariants
  - $\mathcal{C} = \mathrm{Set}$, H group: set M with H-action $\triangleright : H \times M \to M$ and map $f : M \to H$

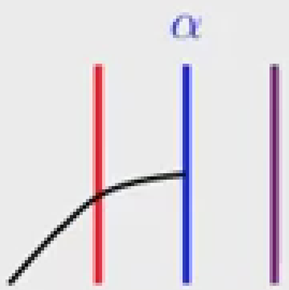
$M^H = M/\sim$ orbit space
$\pi : M \to M/\sim$ canonical surjection

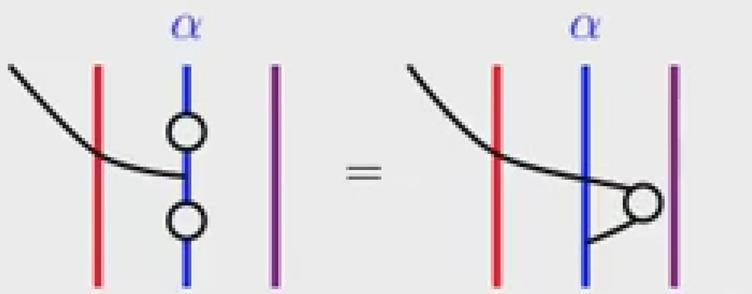
$M^{coH} = f^{-1}(1)$ preimage of unit
$\iota : M^{coH} \to M$ inclusion

$M_{inv} = \pi(M^{coH})$

3. Mapping class group actions from Hopf monoids and embedded graphs

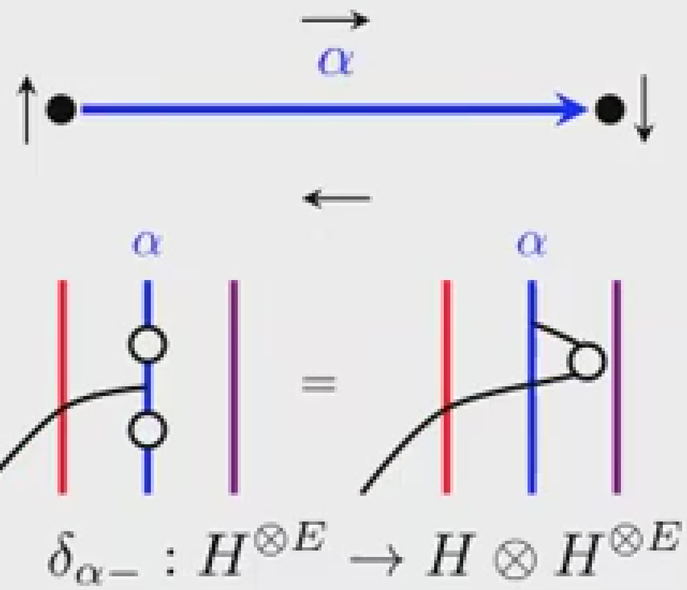
- **ingredients:**
  - involutive Hopf monoid H in symmetric monoidal category $\mathcal{C}$
  - directed graph Γ embedded in Σ , $\Sigma \setminus \Gamma$ disjoint union of discs and annuli
- **construction:**
  - **object** $H^{\otimes E} \in \mathrm{Ob}\,\mathcal{C}$
  - **directed edge ⇨ H -module and H -comodule structures on $H^{\otimes E}$**



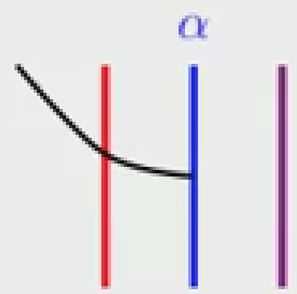


$\triangleright_{\alpha-} : H \otimes H^{\otimes E} \to H^{\otimes E}$

$\delta_{\alpha+} : H^{\otimes E} \to H \otimes H^{\otimes E}$



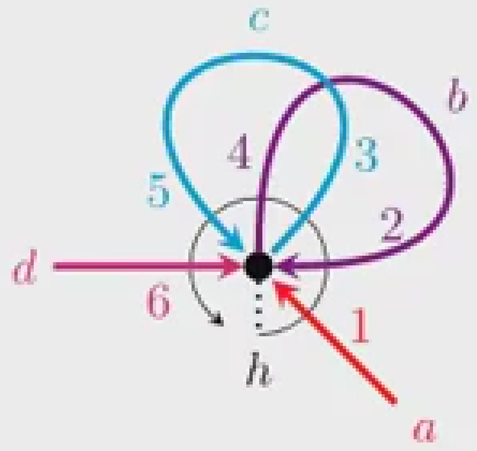
$\delta_{\alpha-} : H^{\otimes E} \to H \otimes H^{\otimes E}$

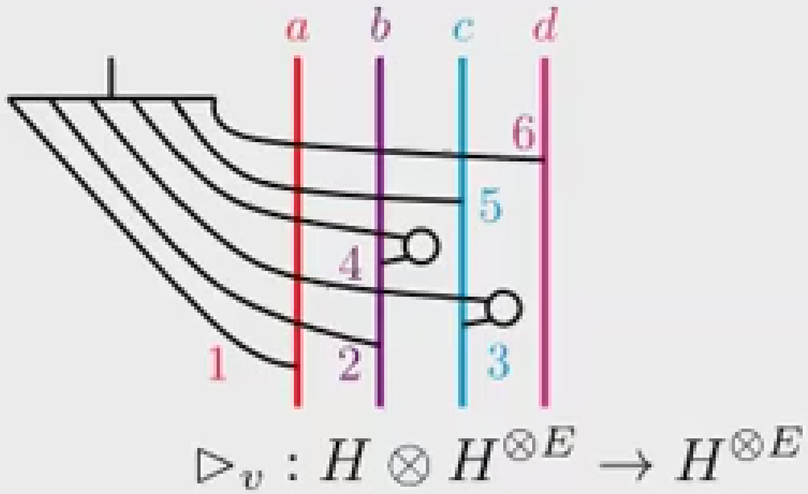


$\triangleright_{\alpha+} : H \otimes H^{\otimes E} \to H^{\otimes E}$

- Hopf bimodule structure on $H^{\otimes E}$
- edge orientation reversal via antipode (or involution constructed from pivotal structure)

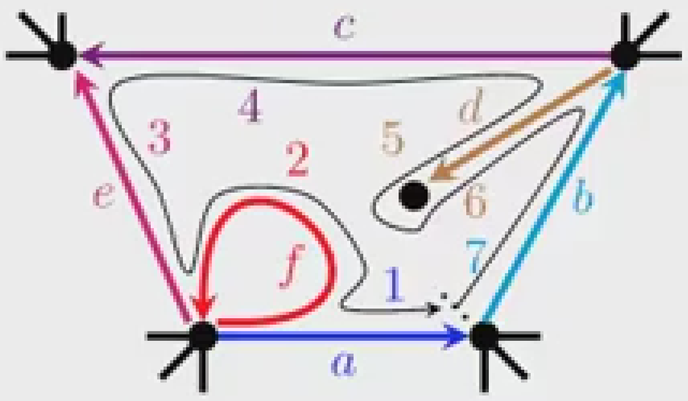
- **ciliated vertex ⇨ H - module structure**

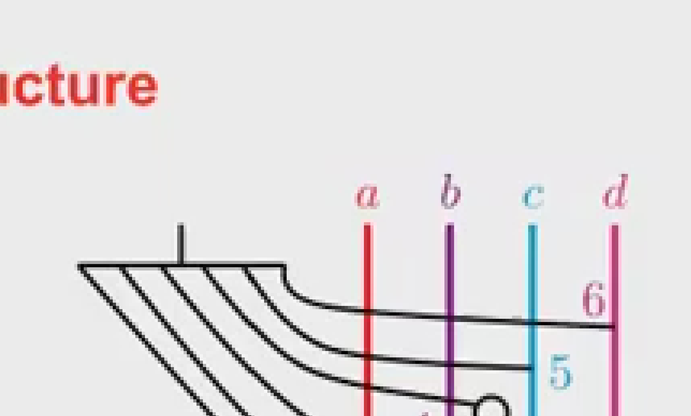




$$\triangleright_v : H \otimes H^{\otimes E} \to H^{\otimes E}$$

- **ciliated face ⇨ H - comodule structure**



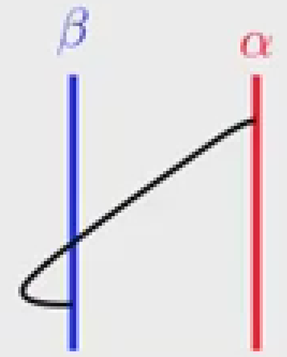


$$\delta_f : H^{\otimes E} \to H \otimes H^{\otimes E}$$

- module structures at different vertices commute ⇨ $H^{\otimes V}$-module structure on $H^{\otimes E}$
- comodule structures at different faces commute ⇨ $H^{\otimes F}$-comodule structure on $H^{\otimes E}$
- adjacent vertex + face at cilium ⇨ Yetter-Drinfeld module structure on $H^{\otimes E}$

3.a Mapping class group actions by edge slides

- **Def: edge slide** $S_{\alpha^L} : H^{\otimes E} \to H^{\otimes E}$



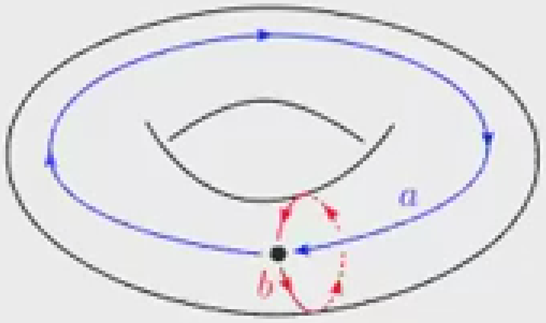
- other edge orientations: by applying antipode
  (or involution constructed from pivotal structure)

Theorem [T.V., C.M.]

- edge slides are isomorphisms $H^{\otimes E} \to H^{\otimes E}$
- isomorphisms of H- modules and H-comodules for $\triangleright_v, \delta_f$ if no edge ends slide over cilia
- satisfy involutivity, commutativity, triangle, pentagon relations
- **mapping class group action**
  - chord diagram with single vertex and face ⇨ surface Σ with one boundary component
    chord slides ⇨ group homomorphism $\rho : \mathrm{Map}(\Sigma) \to \mathrm{Aut}_{YD}(H^{\otimes E})$

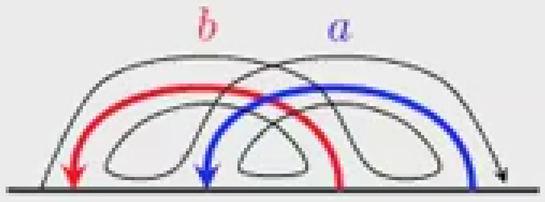
Proof: graphical computations with properties of involutive Hopf monoids
works more generally for pivotal Hopf monoids

3.b (1-holed) torus



- **torus** $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$

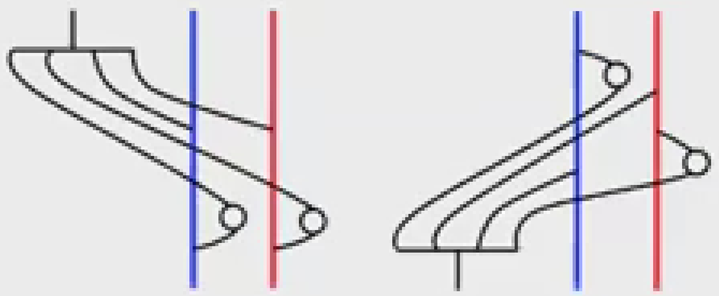
$$\mathrm{Map}(T) = \mathrm{SL}(2,\mathbb{Z}) = \langle A, B \mid (BAB)^4 = 1, ABA = BAB \rangle$$



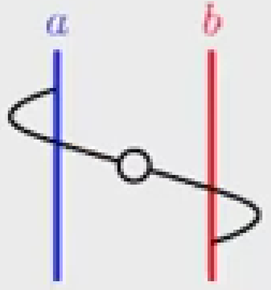
- **1-holed torus** $\pi_1(T_1) = F_2$

$$\mathrm{Map}(T_1) = \langle A, B \mid ABA = BAB \rangle$$

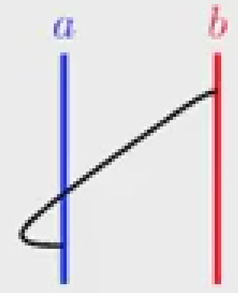
- **Yetter-Drinfeld module structure:**



- **Dehn twists**



$$D_a = S_{a^L} : H^{\otimes 2} \to H^{\otimes 2}$$



$$D_b = S_{b^L} : H^{\otimes 2} \to H^{\otimes 2}$$

Theorem [T.V., C.M.]

- Dehn twists define group homomorphism $\rho : \mathrm{Map}(T_1) \to \mathrm{Aut}_{YD}(H^{\otimes 2})$
- Dehn twists induce group homomorphism $\rho' : \mathrm{Map}(T) \to \mathrm{Aut}(H_{inv}^{\otimes 2})$

Proof:

- slides are isomorphisms of Yetter-Drinfeld modules
- relations $D_a D_b D_a = D_b D_a D_b$

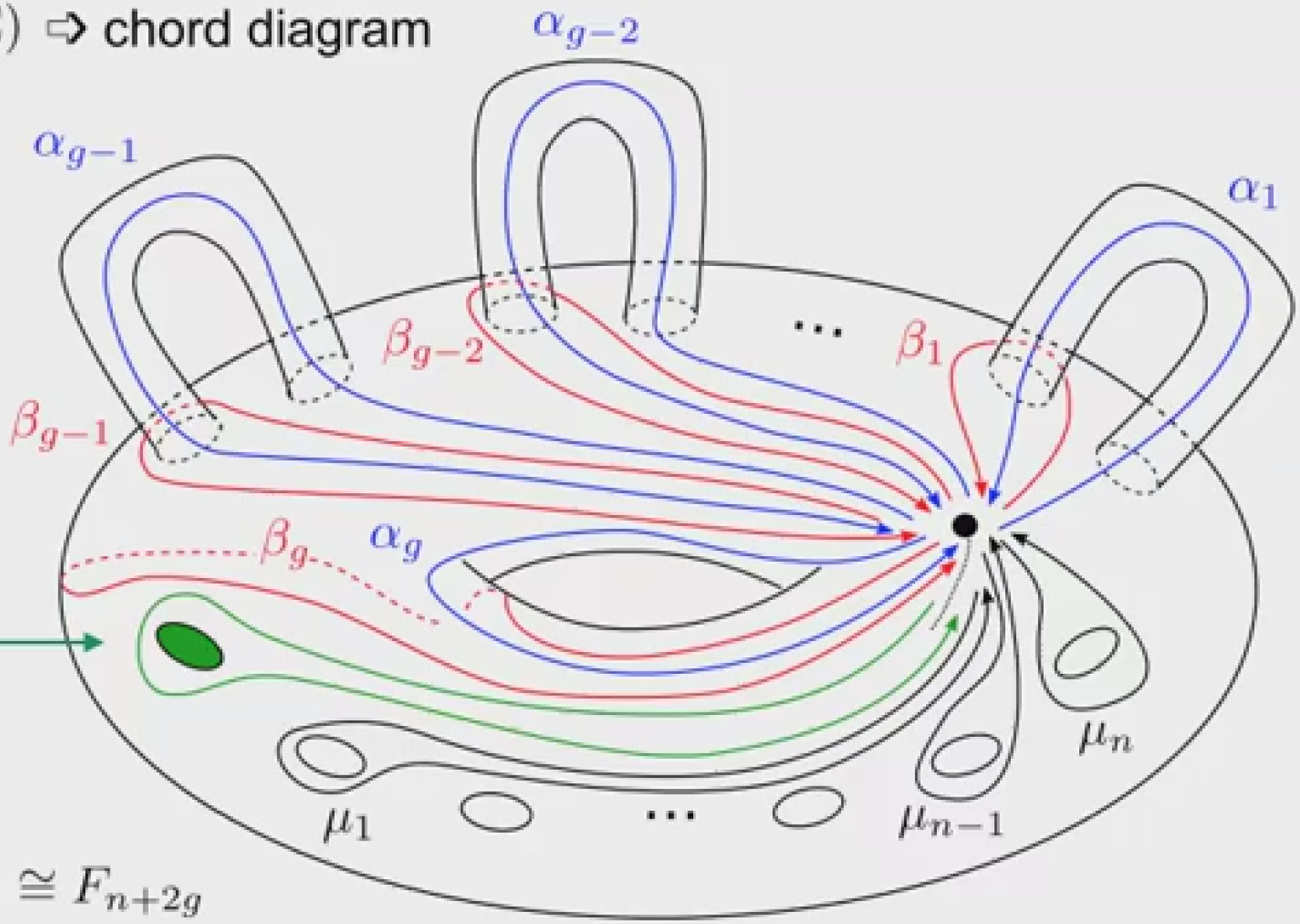
$$(D_b D_a D_b)^4 = \triangleright_v \circ (S \otimes 1_H) \circ \delta_f \Rightarrow (D'_b D'_a D'_b)^4 = 1_{H_{inv}^{\otimes 2}}$$

3.c Mapping class group actions by Dehn twists

- **setting:** surface Σ of genus $g \geq 1$, $n+1 \geq 1$ boundary components

  surface Σ' by attaching discs to boundary component of Σ

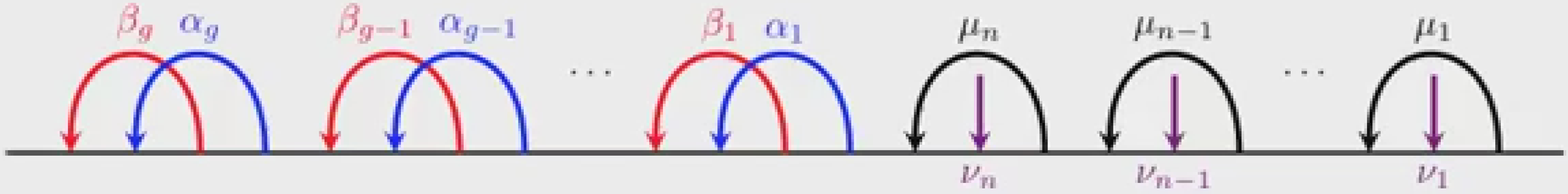
  set of generators of $\pi_1(\Sigma)$ ⇨ chord diagram



$$f = \mu_1^{-1} \cdots \mu_n^{-1} [\alpha_1^{-1}, \beta_1] \cdots [\alpha_g^{-1}, \beta_g]$$

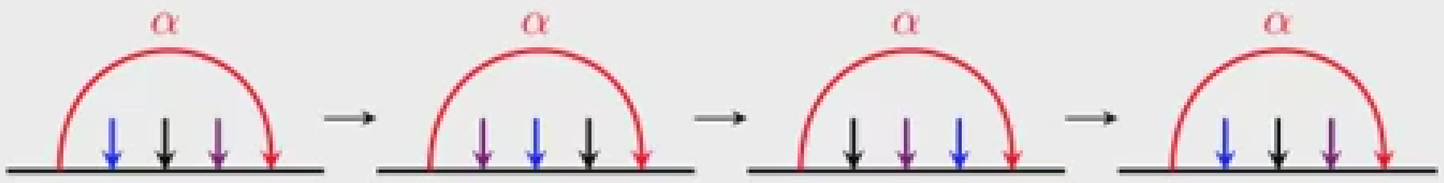
$$\pi_1(\Sigma) = \langle \mu_1, ..., \mu_n, \alpha_1, \beta_1, ..., \alpha_g, \beta_g \rangle \cong F_{n+2g}$$

$$\pi_1(\Sigma') = \langle \mu_1, ..., \mu_n, \alpha_1, \beta_1 ..., \alpha_g, \beta_g \mid f = 1 \rangle$$

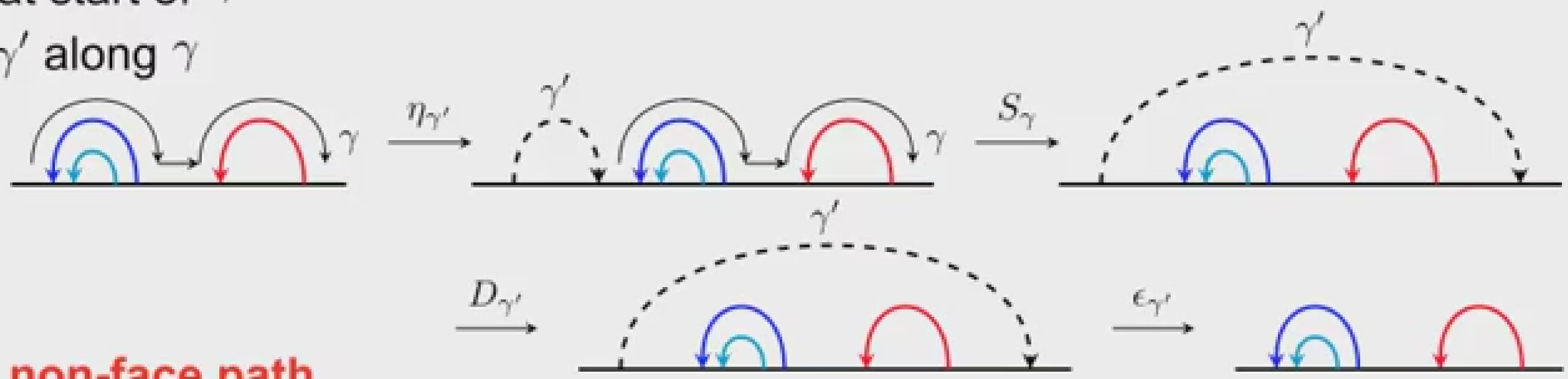


- **aim: explicit description of mapping class group action with Gervais' presentation**
  - realise curves α_i , δ_i, $\gamma_{i,j}$ as paths in the chord diagram
  - define Dehn twists along paths in chord diagram

- **Def: twist along loop** α
  - counterclockwise slide applied to all edge ends between ends of α



- **Def: twist along closed face path** = path that turns maximally left at each chord end
  - add a loop γ' at start of γ
  - slide target of γ' along γ
  - twist along γ'
  - destroy γ'



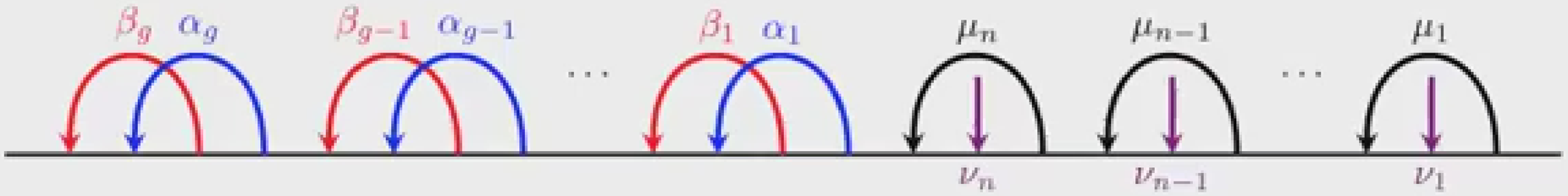
- **Def: twist along non-face path**
  - slide edge ends that interrupt γ out of the way
  - twist along face path γ
  - slide edge ends back

Theorem [T.V., C.M.]

- Dehn twists along α_i , δ_i , $\gamma_{i,j}$ define group homomorphism $\rho : \mathrm{Map}(\Sigma) \to \mathrm{Aut}_{YD}(H^{\otimes 2(n+g)})$
- induce group homomorphism $\rho : \mathrm{Map}(\Sigma') \to \mathrm{Aut}(H_{inv}^{\otimes 2(n+g)})$

⇨ also mapping class group actions for closed surfaces

Proof:



- express curves as paths in chord diagram, compute Dehn twists

α_i		$i = 1, ..., g$	loops
δ_i		$i = 0, ..., g$	loops or face paths
$\gamma_{i,j}$	$\gamma_{0,i}$	$0 \le j < i \le n + 2g - 2$	face paths
$\gamma_{i,j}$		$1 \le i < j \le n + 2g - 2$	non-face paths

⇨ automorphisms ot YD modules

$D_\rho : H^{\otimes 2(n+g)} \to H^{\otimes 2(n+g)}$

- verify relations by explicit diagrammatic computations

- additional relations in $\mathrm{Map}(\Sigma')$

$D_{\gamma_{1,0}} = 1, \quad D_{\delta_0} = D_{\delta_1}, \quad D_{\gamma_{0,1}} = (D_{\delta_0}^3 D_{\alpha_g})^3, \quad D_{\gamma_{0,k}} = D_{\gamma_{1,k}}, \quad D_{\gamma_{k,0}} = D_{\gamma_{k,1}}$

hold for induced isomorphisms of $H_{inv}^{\otimes 2(n+g)}$

Conclusions

- involutive Hopf monoid H in symmetric monoidal category $\mathcal{C}$, finitely (co)complete
- chord diagram ⇨ surface Σ by attaching annulus to each face
  ⇨ surface Σ' by attaching disc to boundary component f of Σ

⇨ **mapping class group action:**

- edge slides ⇨ automorphisms of Yetter-Drinfeld modules wrt $\triangleright_v$ and δ_f
- group homomorphism $\rho : \mathrm{Map}(\Sigma) \to \mathrm{Aut}_{YD}(H^{\otimes 2(n+g)})$
- group homomorphism $\rho : \mathrm{Map}(\Sigma') \to \mathrm{Aut}(H_{inv}^{\otimes 2(n+g)})$

- **features:**
  - rather general, not very restrictive on data (no duals needed)
  - description simple enough for explicit computations
  - mapping class group actions for surfaces with and without boundaries
  - surfaces with boundaries: also deformed actions with weaker assumptions
- **to dos:**
  - consider examples: crossed modules, …
  - applications to Kitaev lattice models: mapping class group action ~ quantum gates