

Skeins and algebras

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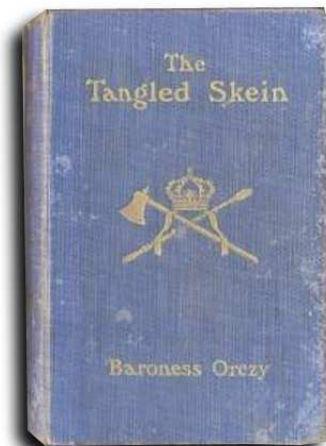
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Algebras are familiar objects.
What about skeins?



skein (noun): *[skane]* a loosely coiled length of wool





John Conway had the knack of finding simple memorable terms for his many innovative ideas.

Among these were the use of the words *Skein* and *Tangle* in knot theory.

Algebras that lend themselves to pictorial representations

A common relation in algebras

$$aba = bab$$

brings to mind the Reidemeister III relation for knot diagrams. It lies at the heart of diagrammatic representations of many algebras - sometimes in the guise of the Yang-Baxter equations.

Artin's n -string braid group and its algebraic representation is a prime example of such an interface between algebraic and geometric representations which is familiar to most of us in these seminars.

The Hecke algebra $H_n(z)$ of type A is the group algebra of the braid group B_n over $\mathbb{Z}[z]$ with the relation

$$a^2 = 1 + za$$

for each $a = \sigma_i$, or some rescaled variant of this. It is a deformation of the group algebra of S_n , where $z = 0$.

The quadratic relation, in the form

$$a - a^{-1} = z$$

suggests the diagrammatic picture

The diagram shows the equation $\sigma_i - \sigma_i^{-1} = z(\text{right arc} - \text{left arc})$. On the left, there are two crossings of two strands. The first crossing has both strands with arrows pointing upwards and to the right. The second crossing has both strands with arrows pointing upwards and to the left. These are separated by a minus sign. To the right of the minus sign is the expression $= z$ followed by two arcs. The first arc is a rightward-curving arc with an arrow pointing to the right. The second arc is a leftward-curving arc with an arrow pointing to the left.

which lies at the heart of the Homfly polynomial for knots and links.

Motto

Where there is a quadratic relation in an algebra look for some Homfly-based model.

Algebras with suitable relations include

- Hecke algebras $H_n(z)$
- Affine Hecke algebras $\dot{H}_n(z)$
- Double affine Hecke algebras $\ddot{H}_n(z, q)$

These algebras come with presentations by generators and relations.

We may want to adapt the coordinate ring, for instance to factorise the quadratic relation as

$$(a - s)(a + s^{-1}) = 0,$$

with $z = s - s^{-1}$, or rescale the roots a bit.

The skein side of the picture

This comes from diagrammatic constructions based on framed oriented curves in 3-manifolds, which give some neat models for the algebras.

These models also highlight certain elements in the algebras that have memorable pictures.

Here is the basic skein framework for these and some further interesting examples of algebras.

Skeins

For a 3-manifold M the (HOMFLYPT) *skein* $\text{Sk}(M)$ is based on oriented framed curves in M up to isotopy.

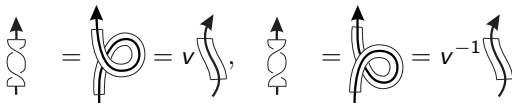
- Take Λ -linear combinations of curves
- Factor by three local relations

Quadratic



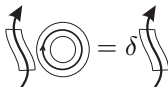
A diagram showing the quadratic relation in skein theory. It consists of two crossings of two strands with arrows pointing upwards. The first crossing is a right-handed crossing, and the second is a left-handed crossing. They are separated by a minus sign. This is followed by an equals sign and the variable z , which is then followed by two parallel strands with arrows pointing upwards.

Framing



Two equations illustrating framing relations. The first equation shows a vertical strand with a full twist (two crossings) equal to a strand with a full twist equal to v times a single strand. The second equation shows a vertical strand with a full twist equal to a strand with a full twist equal to v^{-1} times a single strand. In both cases, the strands have arrows pointing upwards.

Unknot



A diagram showing the unknot relation. It consists of a single strand with an arrow pointing upwards that forms a full twist (two crossings). This is equal to δ times a single strand with an arrow pointing upwards.

For compatibility we need $\delta z = v^{-1} - v$ in Λ .

Algebras turn up naturally when $M = F \times [0, 1]$ for a surface F .

Stacking copies of $F \times I$ defines a product on $\text{Sk}(F \times I)$, making it an algebra over Λ .

We can fix n points $J \subset F$ and include n arcs from $J \times \{0\}$ to $J \times \{1\}$, to give an algebra for each n .

Write $\text{Sk}_n(F)$ for this algebra - its elements are linear combinations of n -tangles in (F, J) .

When $n = 0$ we just have the skein of closed curves in $F \times I$ - write simply as $\text{Sk}(F)$.

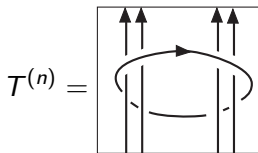
In $\text{Sk}_n(F)$ the n -braids play an important role - these are tangles with just n monotonic arcs. If we stick to these, and just impose the Quadratic relations then the known presentations of surface braid groups give a presentation of the resulting algebra.

The simplest case is $F = D^2$, where this leads directly to the Hecke algebra $H_n(z)$, having a nice basis of $n!$ elements.

Theorem (M - Traczyk 1986 [2])

$$\text{Sk}_n(D^2) \cong H_n(z) \otimes \Lambda$$

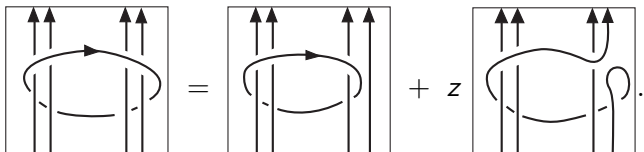
It is worth noting this element $T^{(n)}$ in $Sk_n(D^2)$ (and so in $H_n(z)$).



It is clearly central - any tangle can slide down from the top to the bottom.

The disc

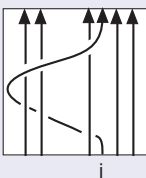
Use the skein relations to write it in terms of braids.



The diagram shows a skein relation for a disc with four strands. On the left, a strand from the second position crosses over the first strand. This is equal to the sum of two diagrams: the first is the same as the left but with the crossing removed, and the second is the same as the left but with a loop on the second strand. The variable z is placed between the two diagrams on the right.

We finish up with the sum of the Jucys-Murphy elements, T_j .

Theorem (M, 2000 [5])

$$T^{(n)} = \delta + v^{-1}z \sum T_j = \text{[Diagram]} \in H_n.$$


The diagram shows a braid with n strands. The j -th strand from the left crosses over the $(j-1)$ -th strand. The label j is placed below the crossing.

Symmetric functions of the Jucys-Murphy elements are known algebraically to be central in H_n - diagrammatically we can put more complicated curves around the loop in $T^{(n)}$ to get all the central elements.

The annulus

Look now at the skeins based on the annulus $A = S^1 \times I$.
The case $\text{Sk}(A) := \mathcal{C}$ with closed curves only, is a commutative algebra.

Theorem (Turaev 1987 [7])

\mathcal{C} is a free polynomial algebra on generators $\{A_m, m \in \mathbb{Z} - \{0\}\}$.

\mathcal{C} is very useful in constructing extra skein invariants of knots K by placing curves in A around a neighbourhood of K to give more complicated elements of a skein.

Interpretations of \mathcal{C} as a ring of symmetric polynomials in commuting variables $\{y_i^{\pm 1}\}$ have proved fruitful, [1]. In particular I will use the elements $P_m \in \mathcal{C}$ which correspond to the power sum $\sum y_i^m$.

The annulus

In the annulus there are some simple 1-braids, giving elements

$$Id = \boxed{\begin{array}{c} \uparrow \\ | \\ \downarrow \end{array}}, \quad x = \boxed{\begin{array}{c} \nearrow \\ \searrow \end{array}}, \quad x^3 = \boxed{\begin{array}{c} \nearrow \\ \nearrow \\ \searrow \\ \searrow \end{array}}$$

in $Sk_1(A)$. When we include closed curves in the skein the quadratic relation gives a relation

$$\boxed{\begin{array}{c} \uparrow \\ \hline \downarrow \end{array}} - \boxed{\begin{array}{c} \uparrow \\ \hline \downarrow \end{array}} = zx = (s - s^{-1})x$$

between 1-tangles.

The annulus

If we put the element P_m on the closed curve it has the property that

$$\begin{array}{|c|} \hline \uparrow \\ \hline P_m \\ \hline \rightarrow \\ \hline \end{array} - \begin{array}{|c|} \hline \uparrow \\ \hline P_m \\ \hline \rightarrow \\ \hline \end{array} = (s^m - s^{-m})x^m.$$

The annulus

This leads to the result

$$\begin{array}{|c|} \hline \uparrow \uparrow \uparrow \uparrow \uparrow \\ \hline \xrightarrow{P_m} \\ \hline \end{array}
 -
 \begin{array}{|c|} \hline \uparrow \uparrow \uparrow \uparrow \uparrow \\ \hline \xrightarrow{P_m} \\ \hline \end{array}
 = (s^m - s^{-m}) \sum Z_i^m$$

where

$$Z_i = \begin{array}{|c|} \hline \uparrow \uparrow \uparrow \uparrow \uparrow \\ \hline \xrightarrow{\quad} \\ \hline \end{array}$$

i

in $\text{Sk}_n(A)$.

The skein $\text{Sk}_n(A)$, restricted to using n -braids, is isomorphic to the affine Hecke algebra $\dot{H}_n(z)$. The full skein gives $\dot{H}_n(z)$ also, with extended coefficients.

Theorem

$$\text{Sk}_n(A) \cong \dot{H}_n(z) \otimes \mathcal{C}$$

Peter Samuelson and I worked out the structure of the algebra $\text{Sk}(T^2)$.

Theorem (M, Samuelson 2015 [3])

$\text{Sk}(T^2)$ is generated by elements $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 - \{0,0\}$ represented by simple closed curves in T^2 decorated by P_m . It is not commutative but

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^k - s^{-k})P_{\mathbf{x}+\mathbf{y}}$$

where $k = \det(\mathbf{x}\mathbf{y})$.

The torus

The *double affine Hecke algebra* \ddot{H}_n of Cherednik is an algebra over $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$ generated by

$$\{T_i\}, 1 \leq i \leq n-1, \{X_j\}, \{Y_j\}, 1 \leq j \leq n$$

with relations

$$\begin{aligned}(T_i + s)(T_i - s^{-1}) &= 0 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ [T_i, T_j] &= 0, |i - j| > 1 \\ X_{i+1} &= T_i X_i T_i, \\ Y_{i+1} &= T_i^{-1} Y_i T_i^{-1} \\ [T_i, X_j] = [T_i, Y_j] &= 0, j \neq i, i + 1 \\ [X_i, X_j] = [Y_i, Y_j] &= 0 \\ X_1^{-1} Y_2 &= Y_2 X_1^{-1} T_1^{-2} \\ Y_1 X_1 \cdots X_n &= q X_1 \cdots X_n Y_1\end{aligned}$$

This suggests a comparison with $\text{Sk}_n(T^2)$ based on n -braids in T^2 .

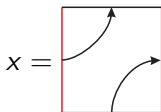
The comparison is partially successful, taking $T_i = \sigma_i^{-1}$, $z = s - s^{-1}$ and X_i, Y_i to be pure braids where string i moves once around the x or y direction in T^2 .

However this only works with braids and with $q = 1$.

If we go beyond braids and include closed curves there is a further immediate problem, even for $n = 1$.

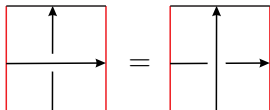
The torus

In T^2 the 1-braids have the form $x^k y^l$, with $yx = xy$.
Our picture of



in the annulus gives us a view of x in the torus, while y runs similarly round the y - direction.

If we include a closed curve in the torus we realise that



since the closed curve at the back can be moved all the way round the torus in the y -direction to reappear at the front.

The torus

The equation

The diagram shows an equation between three square boxes. The first box contains a horizontal line with an arrow pointing right, and a vertical line with an arrow pointing up. The second box is identical to the first. A minus sign is between the two boxes. To the right of the minus sign is the expression $(s - s^{-1})$. To the right of this expression is a third square box containing two curved arrows: one starting from the top-left corner and curving towards the top-right, and another starting from the bottom-left corner and curving towards the bottom-right.

then becomes $0 = (s - s^{-1})x$.

In a plan view of the torus we can see this as

The diagram shows an equation between three square boxes. The first box has a horizontal line with an arrow pointing right, and a vertical line with an arrow pointing up. The second box is identical to the first. A minus sign is between the two boxes. To the right of the minus sign is the expression $(s - s^{-1})$. To the right of this expression is a third square box containing a horizontal line with an arrow pointing right, and a vertical line with an arrow pointing up. The boxes are colored: the first and second boxes have red outlines and black arrows, while the third box has a green outline and green arrows.

The torus

Samuelson and I have found a fix for both these problems which introduces q and avoids this sort of collapse.

Put in an extra fixed string $* \times I \subset T^2 \times I$. Others can pass through it at the expense of multiplication by q .

Viewed from above

The diagram shows an equality between three square diagrams. Each square has a red border and a horizontal black line with an arrow pointing right. The top and bottom edges of the square have green arrows pointing right. A dot is located in the lower-left quadrant of each square. The first square has a dot on the horizontal line. The second square has a dot below the horizontal line and a horizontal grey line with an arrow pointing right above the horizontal line. The third square has a dot above the horizontal line and a horizontal grey line with an arrow pointing right below the horizontal line. The equations are: $\text{Square 1} = q \cdot \text{Square 2} = q \cdot \text{Square 3}$.

Then

The diagram shows an equality between four square diagrams. Each square has a red border and a horizontal black line with an arrow pointing right. The top and bottom edges of the square have green arrows pointing right. A dot is located in the lower-left quadrant of each square. The first square has a dot on the horizontal line. The second square has a dot below the horizontal line and a horizontal grey line with an arrow pointing right above the horizontal line. The third square has a dot above the horizontal line and a horizontal grey line with an arrow pointing right below the horizontal line. The fourth square has a dot on the horizontal line. The equation is: $\text{Square 1} - \text{Square 2} = (s - s^{-1}) \cdot \text{Square 3} = (1 - q) \cdot \text{Square 4}$.

With braids only we get the relation $yx = qxy$ when $n = 1$, and more generally we identify $\ddot{H}_n(z, q)$ with $B\text{Sk}_n(T^2, *)$ using braids only, along with the fixed string $*$. (In a recent arXiv preprint [4]).

We then have a homomorphism

$$\varphi_n : \ddot{H}_n(z, q) \rightarrow \text{Sk}_n(T^2, *).$$

So long as $q \neq 1$ we avoid the immediate danger of the full skein $\text{Sk}_n(T^2, *)$ collapsing.

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We get nice representations of elements of \ddot{H}_n such as $\sum X_i^m$ and $\sum Y_i^m$ by quite simple elements in the full skein. These are to a large extent independent of n .

The torus

Using a plan view of the torus, with the braid points and fixed string where indicated, we have diagrams of the braids

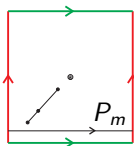
$$X_i = \begin{array}{|c|} \hline \xrightarrow{\hspace{2cm}} \\ \hline \end{array}, \quad Y_i = \begin{array}{|c|} \hline \xrightarrow{\hspace{2cm}} \\ \hline \end{array}$$

Our result from the annulus gives us

$$\begin{array}{|c|} \hline \xrightarrow{\hspace{2cm}} \\ \hline \end{array} - \begin{array}{|c|} \hline \xrightarrow{\hspace{2cm}} \\ \hline \end{array} = (s^m - s^{-m}) \sum_i X_i^m$$

This leads to

$$(s^m - s^{-m}) \sum_i \chi_i^m = (1 - q^m)$$



This closed curve element of the skein fits nicely with the results of Schiffman and Vasserot [6] relating the *elliptic Hall algebra* to a limit of quotients of the algebras \check{H}_n .

The elliptic Hall algebra

The elliptic Hall algebra has generators u_x for $x \neq 0 \in \mathbb{Z}^2$, and SV look at images of them in quotients of \check{H}_n .

Key elements in their model are the power sums

$$X_1^m + \cdots + X_n^m, Y_1^m + \cdots + Y_n^m$$

and their images under automorphisms of \check{H}_n .

We can use closed curves in the full skein to set up elements W_x that are essentially independent of n , and correspond quite neatly with u_x in the quotients of each $\text{Sk}_n(T^2, *)$.

Construction of elements W_x

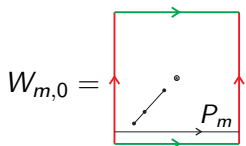
Fix a disc D in T^2 which includes the braid points and the base point. Any oriented embedded curve in the complement of D^2 is determined up to isotopy by a primitive element $\mathbf{y} \in \mathbb{Z}^2$, representing the homology class of the curve.

For each primitive \mathbf{y} define an element $W_{\mathbf{y}}$ of the skein $\text{Sk}_n(T^2, *)$ by the oriented curve corresponding to \mathbf{y} , along with vertical braid strings and base string.

Construction of elements W_x

For any other non-zero $\mathbf{x} \in \mathbb{Z}^2$ write $\mathbf{x} = m\mathbf{y}$ with $m > 0$ and \mathbf{y} primitive, and define W_x to be W_y with the closed curve decorated by the element P_m .

These give elements of $\text{Sk}_n(T^2, *)$ for each n . Our example above is



In the correspondence with \ddot{H}_n we think of comparing W_x and $(s^m - s^{-m})u_x$.

The elements W_x really belong in the skein $\text{Sk}(T^2 - D^2)$. If we fill in D^2 they become the elements $P_x \in \text{Sk}(T^2)$. They satisfy some but not all of the commutation relations from $\text{Sk}(T^2)$.

The skein $\text{Sk}(T^2 - D^2)$ is much larger than $\text{Sk}(T^2)$. A part of it can be mapped to a part of the elliptic Hall algebra, using the limit result in [6], with W_x going to a multiple of u_x . This depends on the homomorphisms $\varphi_n : \dot{H}_n \rightarrow \text{Sk}_n(T^2, *)$ turning out to be isomorphisms.

Liverpool Knot Theory publications link.

<https://www.liverpool.ac.uk/~su14/knotprints.html>

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