Skeins and algebras

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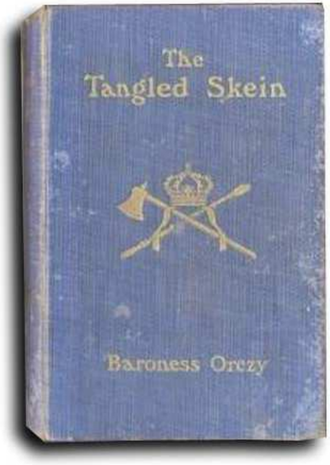
Introduction

Algebras are familiar objects.
What about skeins?

skein (noun): *[skane]* a loosely coiled length of wool

Introduction



John Conway had the knack of finding simple memorable terms for his many innovative ideas.

Among these were the use of the words *Skein* and *Tangle* in knot theory.

Algebras

Algebras that lend themselves to pictorial representations

A common relation in algebras

$$aba = bab$$

brings to mind the Reidemeister III relation for knot diagrams. It lies at the heart of diagrammatic representations of many algebras - sometimes in the guise of the Yang-Baxter equations.

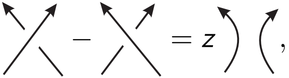
Artin's n-string braid group and its algebraic representation is a prime example of such an interface between algebraic and geometric representations which is familiar to most of us in these seminars.
The Hecke algebra $H_n(z)$ of type A is the group algebra of the braid group B_n over $\mathbb{Z}[z]$ with the relation

$$a^2 = 1 + za$$

for each $a = \sigma_i$, or some rescaled variant of this. It is a deformation of the group algebra of S_n, where $z = 0$.
The quadratic relation, in the form

$$a - a^{-1} = z$$

suggests the diagrammatic picture



which lies at the heart of the Homfly polynomial for knots and links.

Motto

Where there is a quadratic relation in an algebra look for some Homfly-based model.

Algebras with suitable relations include

- Hecke algebras $H_n(z)$
- Affine Hecke algebras $\dot{H}_n(z)$
- Double affine Hecke algebras $\ddot{H}_n(z, q)$

These algebras come with presentations by generators and relations.

We may want to adapt the coordinate ring, for instance to factorise the quadratic relation as

$$(a - s)(a + s^{-1}) = 0,$$

with $z = s - s^{-1}$, or rescale the roots a bit.

The skein side of the picture

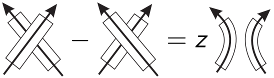
This comes from diagrammatic constructions based on framed oriented curves in 3-manifolds, which give some neat models for the algebras.

These models also highlight certain elements in the algebras that have memorable pictures.
Here is the basic skein framework for these and some further interesting examples of algebras.

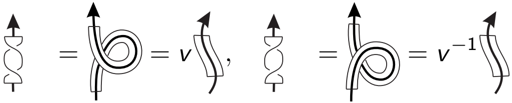
For a 3-manifold M the (HOMFLYPT) *skein* $\mathrm{Sk}(M)$ is based on oriented framed curves in M up to isotopy.

- Take Λ-linear combinations of curves
- Factor by three local relations

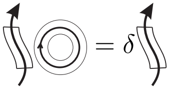
Quadratic



Framing



Unknot



For compatibility we need $\delta z = v^{-1} - v$ in Λ.

Skein algebras

Algebras turn up naturally when $M = F \times [0,1]$ for a surface F.

Stacking copies of $F \times I$ defines a product on $\mathrm{Sk}(F \times I)$, making it an algebra over Λ.

We can fix n points $J \subset F$ and include n arcs from $J \times \{0\}$ to $J \times \{1\}$, to give an algebra for each n.

Write $\mathrm{Sk}_n(F)$ for this algebra - its elements are linear combinations of n-tangles in (F, J).
When $n = 0$ we just have the skein of closed curves in $F \times I$ - write simply as $\mathrm{Sk}(F)$.

The disc

In $\mathrm{Sk}_n(F)$ the n-braids play an important role - these are tangles with just n monotonic arcs. If we stick to these, and just impose the Quadratic relations then the known presentations of surface braid groups give a presentation of the resulting algebra.

The simplest case is $F = D^2$, where this leads directly to the Hecke algebra $H_n(z)$, having a nice basis of $n!$ elements.

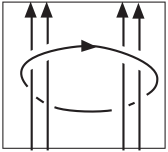
Theorem (M - Traczyk 1986 [2])

$$\mathrm{Sk}_n(D^2) \cong H_n(z) \otimes \Lambda$$

The disc

It is worth noting this element $T^{(n)}$ in $\mathrm{Sk}_n(D^2)$ (and so in $H_n(z)$).

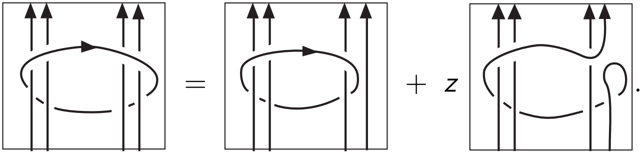
$T^{(n)} =$



It is clearly central - any tangle can slide down from the top to the bottom.

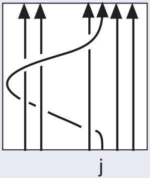
The disc

Use the skein relations to write it in terms of braids.



We finish up with the sum of the Jucys-Murphy elements, T_j.

Theorem (M, 2000 [5])

$$T^{(n)} = \delta + v^{-1} z \sum T_j =$$ 

$\in H_n$.

Symmetric functions of the Jucys-Murphy elements are known algebraically to be central in H_n - diagrammatically we can put more complicated curves around the loop in $T^{(n)}$ to get all the central elements.

The annulus

Look now at the skeins based on the annulus $A = S^1 \times I$.
The case $\mathrm{Sk}(A) := \mathcal{C}$ with closed curves only, is a commutative algebra.

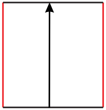
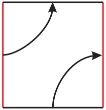
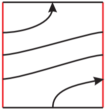
Theorem (Turaev 1987 [7])

$\mathcal{C}$ is a free polynomial algebra on generators $\{A_m, m \in \mathbb{Z} - \{0\}\}$.

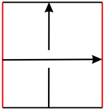
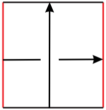
$\mathcal{C}$ is very useful in constructing extra skein invariants of knots K by placing curves in A around a neighbourhood of K to give more complicated elements of a skein.

Interpretations of $\mathcal{C}$ as a ring of symmetric polynomials in commuting variables $\{y_i^{\pm 1}\}$ have proved fruitful, [1]. In particular I will use the elements $P_m \in \mathcal{C}$ which correspond to the power sum $\sum y_i^m$.

In the annulus there are some simple 1-braids, giving elements

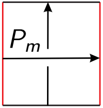
$Id =$ , $x =$ , $x^3 =$ 

in $\mathrm{Sk}_1(A)$. When we include closed curves in the skein the quadratic relation gives a relation

 $-$  $= zx = (s - s^{-1})x$

between 1-tangles.

If we put the element P_m on the closed curve it has the property that



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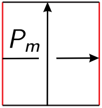
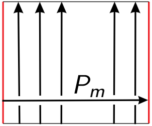


$$= \quad (s^m - s^{-m})x^m.$$

The annulus

This leads to the result



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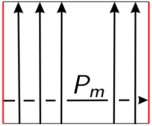
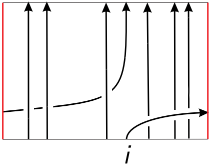


$$= (s^m - s^{-m}) \sum Z_i^m$$

where

$$Z_i =$$



in $\mathrm{Sk}_n(A)$.

The annulus

The skein $\mathrm{Sk}_n(A)$, restricted to using n-braids, is isomorphic to the affine Hecke algebra $\dot{H}_n(z)$. The full skein gives $\dot{H}_n(z)$ also, with extended coefficients.

Theorem

$$\mathrm{Sk}_n(A) \cong \dot{H}_n(z) \otimes \mathcal{C}$$

The torus

Peter Samuelson and I worked out the structure of the algebra $\mathrm{Sk}(T^2)$.

Theorem (M, Samuelson 2015 [3])

$\mathrm{Sk}(T^2)$ *is generated by elements* $P_{\mathbf{x}}$ *for* $\mathbf{x} \in \mathbb{Z}^2 - \{0,0\}$ *represented by simple closed curves in* T^2 *decorated by* P_m.
It is not commutative but

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^k - s^{-k})P_{\mathbf{x}+\mathbf{y}}$$

where $k = \det(\mathbf{xy})$.

The torus

The *double affine Hecke algebra* $\ddot{H}_n$ of Cherednik is an algebra over $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$ generated by

$$\{T_i\}, 1 \leq i \leq n-1, \{X_j\}, \{Y_j\}, 1 \leq j \leq n$$

with relations

$$\begin{aligned}
(T_i + s)(T_i - s^{-1}) &= 0 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
[T_i, T_j] &= 0, |i-j| > 1 \\
X_{i+1} &= T_i X_i T_i, \\
Y_{i+1} &= T_i^{-1} Y_i T_i^{-1} \\
[T_i, X_j] = [T_i, Y_j] &= 0, j \neq i, i+1 \\
[X_i, X_j] = [Y_i, Y_j] &= 0 \\
X_1^{-1} Y_2 &= Y_2 X_1^{-1} T_1^{-2} \\
Y_1 X_1 \cdots X_n &= q X_1 \cdots X_n Y_1
\end{aligned}$$

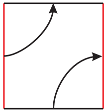
This suggests a comparison with $\mathrm{Sk}_n(T^2)$ based on n-braids in T^2.

The comparison is partially successful, taking $T_i = \sigma_i^{-1}$, $z = s - s^{-1}$ and X_i, Y_i to be pure braids where string i moves once around the x or y direction in T^2.

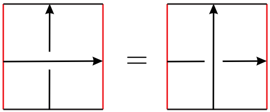
However this only works with braids and with $q = 1$.
If we go beyond braids and include closed curves there is a further immediate problem, even for $n = 1$.

The torus

In T^2 the 1-braids have the form $x^k y^l$, with $yx = xy$.
Our picture of

$x =$ 

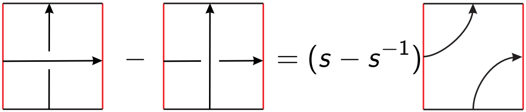
in the annulus gives us a view of x in the torus, while y runs similarly round the y - direction.
If we include a closed curve in the torus we realise that



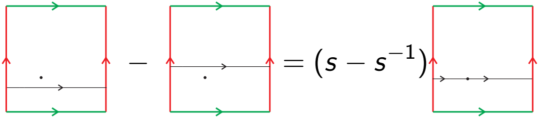
since the closed curve at the back can be moved all the way round the torus in the y-direction to reappear at the front.

The torus

The equation

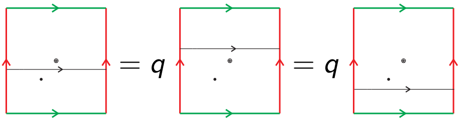


then becomes $0 = (s - s^{-1})x$.
In a plan view of the torus we can see this as

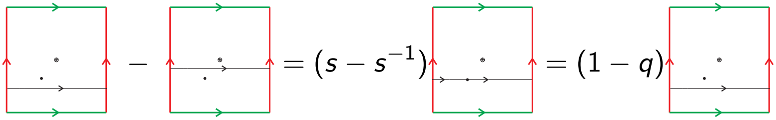


The torus

Samuelson and I have found a fix for both these problems which introduces q and avoids this sort of collapse.
Put in an extra fixed string $* \times I \subset T^2 \times I$. Others can pass through it at the expense of multiplication by q.
Viewed from above



Then



With braids only we get the relation $yx = qxy$ when $n = 1$, and more generally we identify $\ddot{H}_n(z, q)$ with $B\mathrm{Sk}_n(T^2, *)$ using braids only, along with the fixed string $*$. (In a recent arXiv preprint [4]).

We then have a homomorphism

$$\varphi_n : \ddot{H}_n(z, q) \to \mathrm{Sk}_n(T^2, *).$$

So long as $q \neq 1$ we avoid the immediate danger of the full skein $\mathrm{Sk}_n(T^2, *)$ collapsing.
Samuelson and I show that φ_n is surjective. It would be good to be certain that it is also injective when $q \neq 1$.

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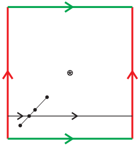
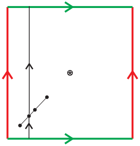
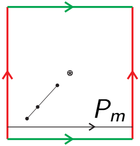
We get nice representations of elements of $\ddot{H}_n$ such as $\sum X_i^m$ and $\sum Y_i^m$ by quite simple elements in the full skein. These are to a large extent independent of n.

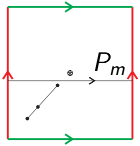
The torus

Using a plan view of the torus, with the braid points and fixed string where indicated, we have diagrams of the braids

$X_i =$  , $Y_i =$ 

Our result from the annulus gives us



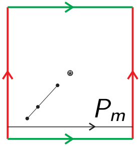
$-$ 

$= (s^m - s^{-m}) \sum_i X_i^m$

The torus

This leads to

$$(s^m - s^{-m})\sum_i X_i^m \;=\; (1-q^m)$$



This closed curve element of the skein fits nicely with the results of Schiffman and Vasserot [6] relating the *elliptic Hall algebra* to a limit of quotients of the algebras $\ddot{H}_n$.

The elliptic Hall algebra

The elliptic Hall algebra has generators $u_{\mathbf{x}}$ for $\mathbf{x} \neq 0 \in \mathbb{Z}^2$, and SV look at images of them in quotients of $\ddot{H}_n$.

Key elements in their model are the power sums

$$X_1^m + \cdots + X_n^m, Y_1^m + \cdots + Y_n^m$$

and their images under automorphisms of $\ddot{H}_n$.

We can use closed curves in the full skein to set up elements $W_{\mathbf{x}}$ that are essentially independent of n, and correspond quite neatly with $u_{\mathbf{x}}$ in the quotients of each $\mathrm{Sk}_n(T^2, *)$.

Construction of elements $W_{\mathbf{x}}$

Fix a disc D in T^2 which includes the braid points and the base point. Any oriented embedded curve in the complement of D^2 is determined up to isotopy by a primitive element $\mathbf{y} \in \mathbb{Z}^2$, representing the homology class of the curve.

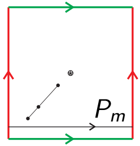
For each primitive $\mathbf{y}$ define an element $W_{\mathbf{y}}$ of the skein $\mathrm{Sk}_n(T^2, *)$ by the oriented curve corresponding to $\mathbf{y}$, along with vertical braid strings and base string.

Construction of elements $W_{\mathbf{x}}$

For any other non-zero $\mathbf{x} \in \mathbb{Z}^2$ write $\mathbf{x} = m\mathbf{y}$ with $m > 0$ and $\mathbf{y}$ primitive, and define $W_{\mathbf{x}}$ to be $W_{\mathbf{y}}$ with the closed curve decorated by the element P_m.

These give elements of $\mathrm{Sk}_n(T^2, *)$ for each n. Our example above is

$W_{m,0} =$



In the correspondence with $\ddot{H}_n$ we think of comparing $W_{\mathbf{x}}$ and $(s^m - s^{-m})u_{\mathbf{x}}$.

The elements $W_{\mathbf{x}}$ really belong in the skein $\mathrm{Sk}(T^2 - D^2)$. If we fill in D^2 they become the elements $P_{\mathbf{x}} \in \mathrm{Sk}(T^2)$. They satisfy some but not all of the commutation relations from $\mathrm{Sk}(T^2)$.

The skein $\mathrm{Sk}(T^2 - D^2)$ is much larger than $\mathrm{Sk}(T^2)$. A part of it can be mapped to a part of the elliptic Hall algebra, using the limit result in [6], with $W_{\mathbf{x}}$ going to a multiple of $u_{\mathbf{x}}$. This depends on the homomorphisms $\varphi_n : \ddot{H}_n \to \mathrm{Sk}_n(T^2, *)$ turning out to be isomorphisms.

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