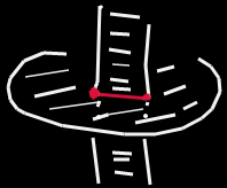
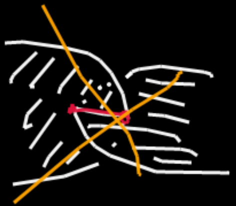
$K \subset S^3$ knot

- K is ribbon if $K = \partial D$ where D is an immersed disk in S^3 with only ribbon singularities:



ribbon



clasp

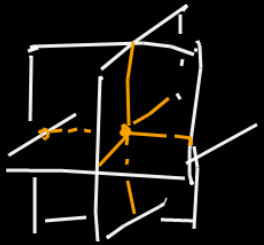
- K is slice if $K = \partial D$ where D is a properly embedded disk in B^4.

Fact: ribbon $\Rightarrow$ slice

Qu: slice $\Rightarrow$ ribbon? (Fox '62)

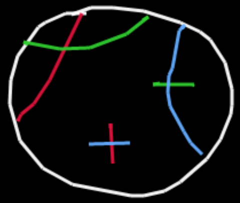
Other singularities

- Triple points



On the preimage disk:

borromean



non borromean

· Branched points

Normal singular disk:

disk immersed in S^3 except at a finite number of branched points.

Marking

ribbon | branched ribbon | circle | branched circle

Theorem (Kawauchi-Shibuya-Suzuki '83)

A knot K in S^3 is slice iff it bounds a marked normal singular disk with no clasp and no boromean triple point.

Sketch of proof

·

$K \mapsto 0$

(b-i) triple point, branched point $\mapsto 1$

other pts on i-line $\mapsto\ > 1$

other pts on b-line $\mapsto\ 0 < \cdot < 1$

- Reciprocally:

Th: The slice genus of a knot K is the minimal genus of a marked normal singular compact surface in S^3 with no clasp and no borromean triple pt.

Kaplan ('79): Any knot bounds a normal singular disk with no clasp.

Murakami–Sugishita ('84):

T-genus $T_s(K)$:

minimal number of borromean triple points on a marked normal singular disk with no clasp.

- T_s is a concordance invariant
- $g_s(K) \leq T_s(K)$
- $T_s(K) \equiv_2 \mathrm{Arf}(K)$ $\rightsquigarrow$ "for any realization"

$\hookrightarrow$ values of $T_s(K)$ for several knots

s.t. $T_s(K) = \begin{cases} g_s(K) \\ g_s(K)+1 \end{cases}$

__Qu°__ : Can $T_s - g_s$ be greater than one?

__Th__ (M.) : $T_s - g_s$ can be arbitrarily large.

__4D clasp number__

$c_4(K) = \min\{\#$ transverse double points on an immersed disk $D \subset B^4$ s.t. $\partial D = K\}$

$c_4^+(K) = \min\{$ — positive double points — $\}$

$c_4^b(K) = \min\{\#$ pairs of double points with opposite signs on a D st. $D \cdot D = 0\}$

$$c_4^+ \leq c_4^b \leq c_4 \leq 2c_4^b$$

__Fact__ : $g_s(K) \leq c_4^b(K)$

__Th__ (Daemi-Scaduto 2020)

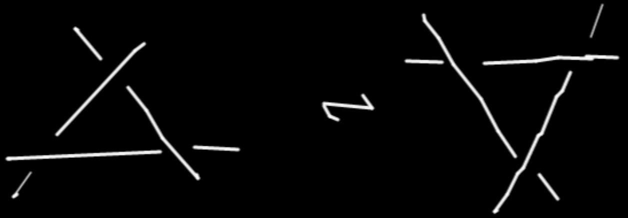
$c_4^+(K) - g_s(K)$ can be arbitrarily large.

__Th__ (M.) $c_4^b(K) \leq T_s(K)$

. T-ribbon disk in S^3 : immersed disk with, no branched pt, no clasp and no non-borromean triple pt.
exist by Kaplan

Ribbon T-genus $T_R(K)$:
minimal number of borromean triple points on a T-ribbon disk with boundary K.

$T_s(K) \leq T_r(K)$

- Δ-move: 

- Δ-slicing number $s_\Delta(K)$:
  minimal number of Δ-moves from K
  to a slice knot.

- Δ-ribboning number $r_\Delta(K)$:
  ____
  to a ribbon knot

Th (Kawauchi – Murakami – Sugishita '83)
$T_s(K) = s_\Delta(K)$

Th (M.) $T_r(K) = r_\Delta(K)$

ribbon knot $K_0 \xrightarrow[\Delta\text{-moves}]{} K$

$K_0 = \partial(\text{ribbon disk})$

$T_r(K) \leq r_\Delta(K)$ $T_s(K) \leq s_\Delta(K)$

$\rightarrow$ at most 1 triple pt on a given ribbon.

Corollary: $g_r(K) \le T_r(K)$

$\uparrow$ ribbon genus

Corollary: The equality $T_s(K) = T_r(K)$ is true for all knots iff all slice knots are ribbon.