Some applications of the volume conjecture

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Plan

1. Volume conjecture

2. Applications

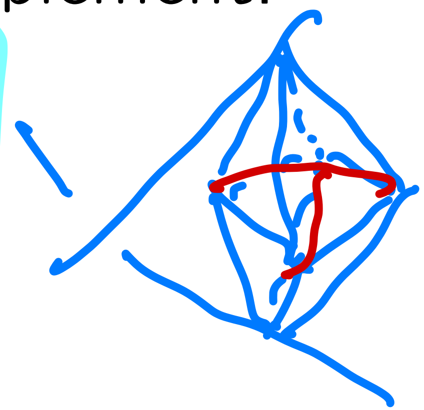
3. Symmetry of quantum $6j$ symbols

1. Volume Conjecture

R. Kashaev:

Construct a new invariant $\langle K \rangle_N$ of a knot K from the quantum dilogarithm function whose asymtotics seems to relate to the hyperbolic volume of the complement.

$$\boxed{\lim_{N\to\infty} \frac{2\,\pi}{N} \log |\langle K \rangle_N| \underset{?}{=} \mathrm{Vol}(S^3 \setminus K)}$$



H. Murakami-J.M.:

$\langle K \rangle_N$ is equal to $(N-1)$-th Colored Jones polynomial $V_K^{(N-1)}(t)$ at N-th root of unity $t = \exp(2\pi i/N)$.

Y. Yokota:

Correspondence with a octahedral decomposition of the knot complement.

Strategy to prove the V. C.

(by **T. Ohtsuki**)

$$\boxed{\sum_{i_1\cdots i_k} R\cdots R}$$

quantum sl_2 Knot invariant

↓

Volume potential function

↓

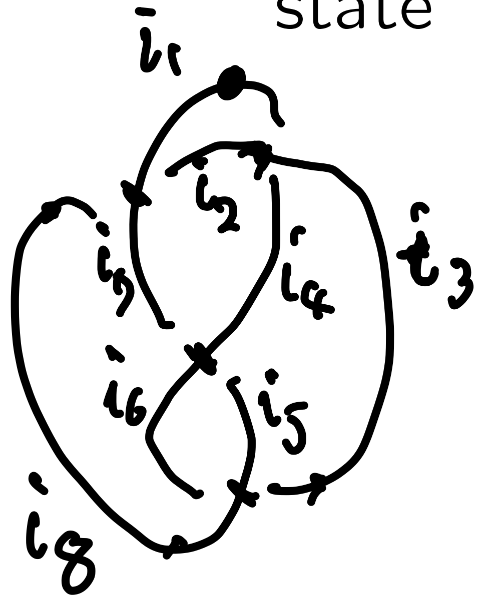
Poisson summation formula

↓

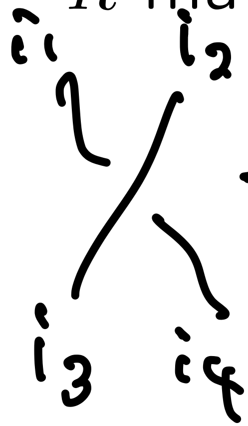
apply the saddle point method

State sum

state



R-matrix



$$\longrightarrow R^{i_1 i_2}_{i_3 i_4}$$

$$\uparrow$$

$$\{n\}!$$

$$\{n\} = t^{n/2} - t^{-n/2}$$

state parameter

$$\{n\}! = \{n\}\{n-1\}\cdots\{1\}$$

Strategy to prove the V. C.

(by **T. Ohtsuki**)

$\sum_{i_1 \cdots i_n} \underbrace{R \cdots R}$

quantum sl_2 Knot invariant

$\downarrow$

Volume potential function

$\downarrow$

Poisson summation formula

$\downarrow$

apply the saddle point method

quantum factorial $\longrightarrow$ $\underbrace{\{n\}!\{\cdots\}!\cdots}$

dilogarithm function

Replace $\{x\}!$ by $Li_2(x)$

$x = t^n$

$Li_2(x_1) + Li_2(x_2) + \cdots$

$P(t^{i_1}, t^{i_2}, \cdots, t^{i_n})$

$\text{inv.} = \sum_{i_1, \cdots, i_n} e^{\frac{N}{2\pi} P(t^{i_1}, t^{i_2}, \cdots, t^{i_n})}$

Strategy to prove the V. C.

(by **T. Ohtsuki**)

quantum sl_2
Knot invariant

$\downarrow$

Volume potential
function

$\downarrow$

Poisson summation
formula

$\downarrow$

apply the saddle
point method

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

$\hat{f}$: the Fourier transform of f.

$$\hat{f}(x) = \int_{\mathbb{R}} e^{2\pi i x y} \underline{f(y)}\, dy$$

$$\uparrow$$

$$e^{\frac{N}{2\pi} P(x_1, \ldots, x_n)}$$

Strategy to prove the V. C.

(by **T. Ohtsuki**)

quantum sl_2
Knot invariant

$\downarrow$

Volume potential
function

$\downarrow$

Poisson summation
formula

$\downarrow$

apply the saddle
point method

large

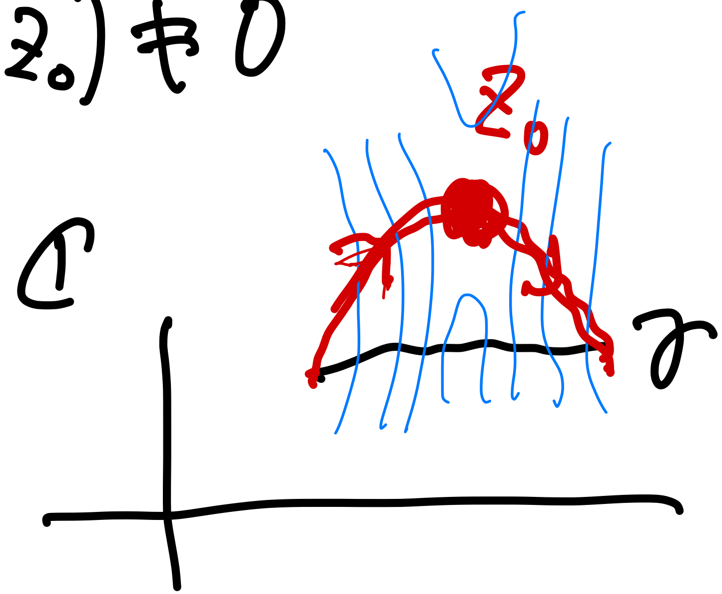
$$\int_\gamma f(z) e^{Ng(z)}\, dz \underset{N\to\infty}{\sim} \sqrt{\frac{2\pi}{-Ng''(z_0)}}\, f(z_0) e^{Ng(z_0)}$$

where $g'(z_0) = 0$.

Conditions:

1. $f(z_0) \neq 0$

2,



Example: Whitehead link

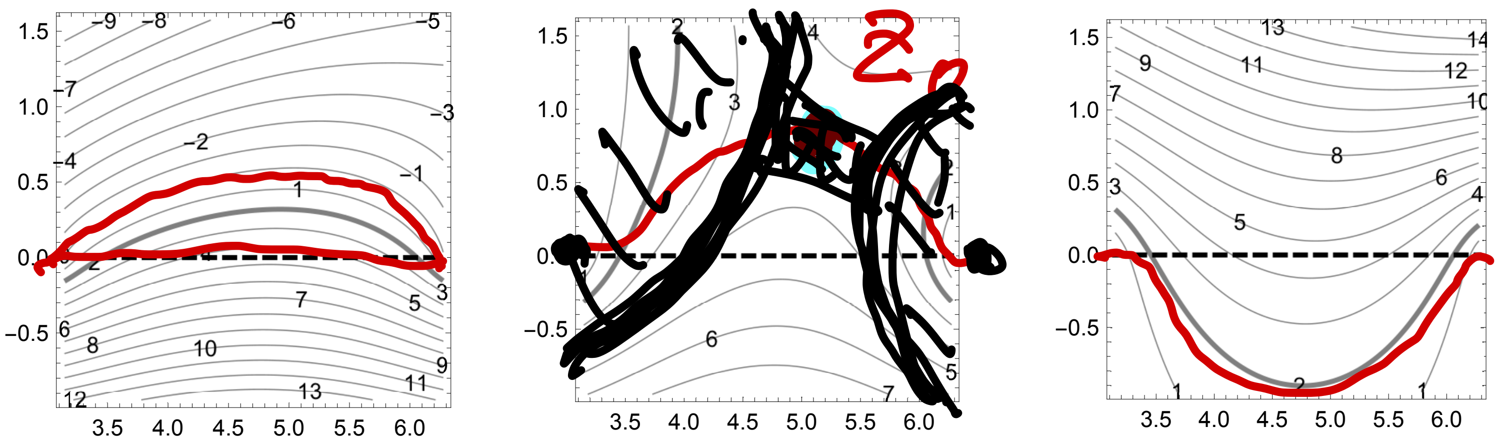
$$V_W^{(N-1,N-1)}(t) = (-1)^{N-1} \sum_i t^{\frac{1}{4}(-i^2-3i)} \frac{\{N+i\}!^2\,\{i\}!}{\{N-1-i\}!^2\,\{2i+1\}!\{N\}}$$

$$P(x) = \frac{1}{i}\left(\pi\sqrt{-1}\log x - \frac{1}{2}(\log x)^2 - 5\,\mathrm{Li}_2(x) + \mathrm{Li}_2(x^2) + \frac{2\pi^2}{3}\right)$$

Poisson sum

$$\frac{2\pi}{N}\log|V_W^{(N-1,N-1)}(e^{2\pi i/N})| \fallingdotseq \frac{2\pi}{N}\log\left|\frac{N}{2\pi}\sum_k \int_0^\pi \exp\left(\frac{N}{2\pi}\boxed{\left(P(e^{i\theta}) + 2\pi k\theta i\right)}\right) d\theta\right|$$

Let z_0 be a solution of $\frac{d}{dz}\left(P(e^{iz}) + 2\pi kzi\right) = \frac{d}{dz}P(e^{iz}) + 2\pi ki = 0$.

$\cdots$  $\cdots$

$k=0$ $\quad$ $k=1$ $\quad$ $k=-2$

$\cdots$ $\quad$ $\mathrm{Re}\,P(e^{iz})$ $\quad$ $\mathrm{Re}\left(P(e^{iz}) - 2\pi iz\right)$ $\quad$ $\mathrm{Re}\left(P(e^{iz}) - 4\pi iz\right)$ $\quad$ $\cdots$

Chen-Yang's Volume Conjecture

N : odd, $\quad t=\exp(2\pi i/N) \longrightarrow t=\exp(4\pi i/N)$.

$$\lim_{N\to\infty}\frac{2\pi}{N}\log TV_N(M) \underset{?}{=} \mathrm{Vol}(M)$$

Variations of Chen-Yang's volume conjecture:

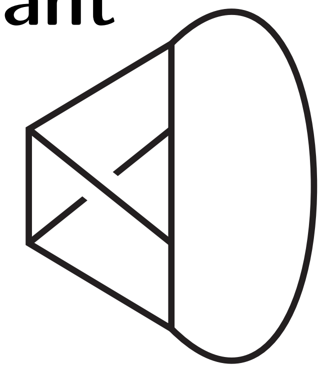
Turaev-Viro invariant

Witten-Reshetikhin-Turaev invariant

Kirillov-Reshetikhin invariant

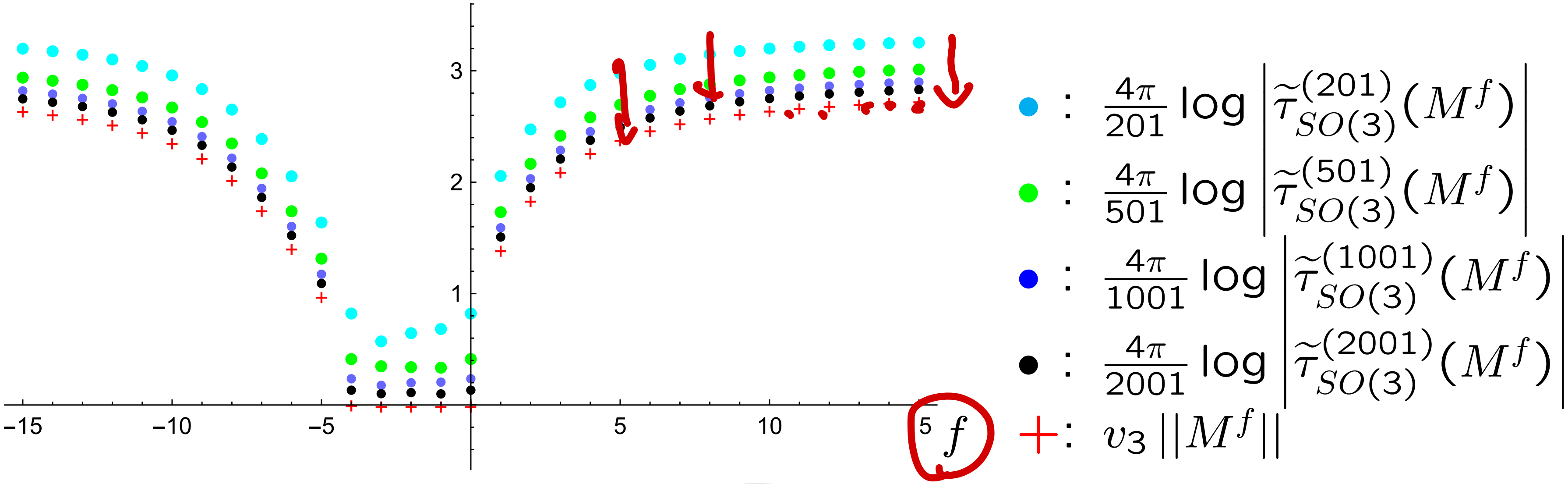
for knotted graphs including

- Colored Jones polynomial
- Quantum $6j$ symbol
- Polyhedral Graph

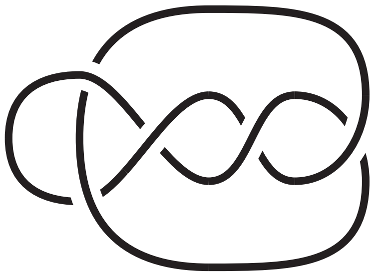


Chen-Yang's Volume Conjecture

Witten-Reshetikhin-Turaev invariant of 5_2 surgery



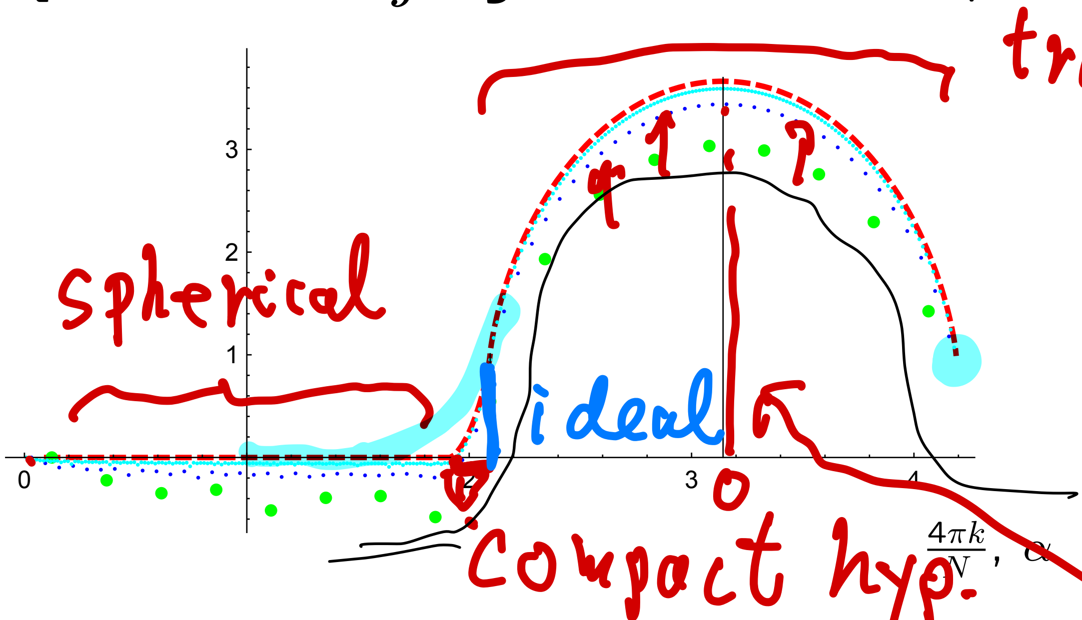
f : framing



It is proved for integral surgeries of 4_1 by **T. Ohtsuki**.

V. C. for Kirillov-Reshetikhin inv.

Quantum $6j$ **symbol** corresponding to regular tetrahedra



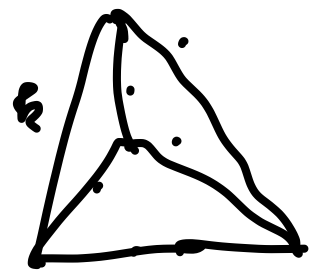
● : $\frac{2\pi}{51}\log\left|\begin{vmatrix} k & k & k \\ k & k & k \end{vmatrix}_{t=\exp\frac{4\pi i}{51}}\right|$

● : $\frac{2\pi}{201}\log\left|\begin{vmatrix} k & k & k \\ k & k & k \end{vmatrix}_{t=\exp\frac{4\pi i}{201}}\right|$

● : $\frac{2\pi}{801}\log\left|\begin{vmatrix} k & k & k \\ k & k & k \end{vmatrix}_{t=\exp\frac{4\pi i}{801}}\right|$

- - - : $\mathrm{Vol}T_{|\pi-\alpha|})$

For truncated case, it is proved by **Q. Chen** and **J. M.**.

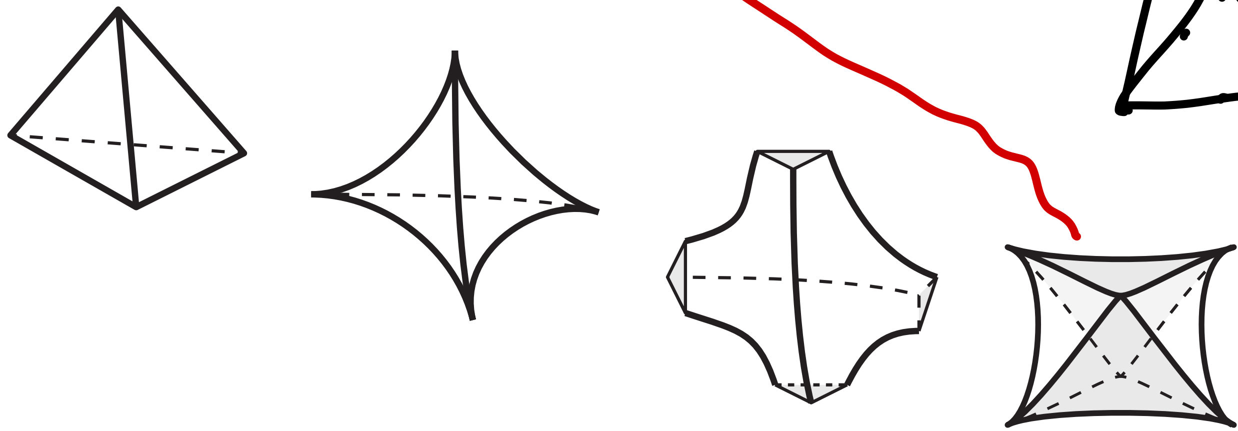


compact

ideal

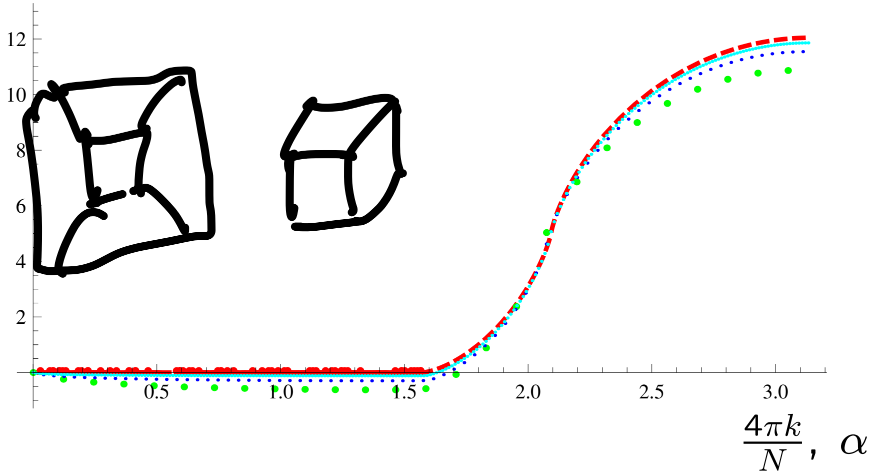
truncated

ideal octahedron



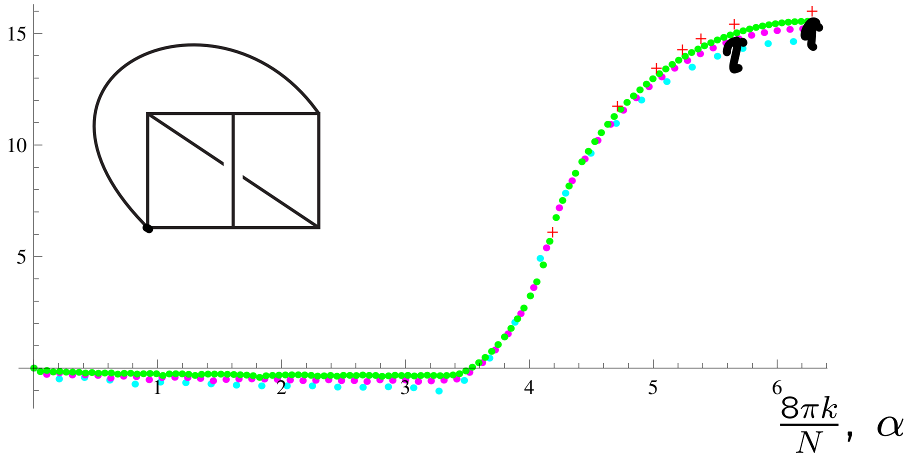
V. C. for Kirillov-Reshetikhin inv.

Polyhedral Graph corresponding to regular cubes



- $\color{green}\bullet$: $\frac{2\pi}{103}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{103}}$
- $\color{blue}\bullet$: $\frac{2\pi}{303}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{303}}$
- $\color{cyan}\bullet$: $\frac{2\pi}{1003}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{1003}}$
- $\color{red}- - -$: $v_3\,||C_{\pi-\alpha}||$

Knotted graph (quantum spin network) $K_{3,3}$.



- $\color{cyan}\bullet$: $\frac{4\pi}{123}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{123}}$
- $\color{magenta}\bullet$: $\frac{4\pi}{243}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{243}}$
- $\color{green}\bullet$: $\frac{4\pi}{483}\mathrm{Re}\log\langle\Gamma, c(k)\rangle^U\Big|_{q=\exp\frac{4\pi i}{483}}$
- $\color{red}+$: $v_3\,||M_{2\pi-\alpha}||$

2. Applications

- Volume ← potential function t
  saddle point method
- A-polynomial (A-ideal)
- Twisted Reidemeister torsion
- Complexified tetrahedron
- Symmetry of quantum $6j$-symbols

! many saddle points.
must choose correct one.

2. Applications

- Volume
- A-polynomial (A-ideal)
- Twisted Reidemeister torsion
- Complexified tetrahedron
- Symmetry of quantum $6j$-symbols

links $K = K_1 \cup \cdots \cup K_\ell$

Colored Jones λ ← meridian $\lambda_1, \lambda_2, \cdots, \lambda_\ell$

$P(\lambda, x_1, \cdots, x_n)$

$$\begin{cases} \dfrac{\partial P}{\partial x_1} = 0 \\ \vdots \\ \dfrac{\partial P}{\partial x_n} = 0 \end{cases}$$

saddle pt. eq.

$$\dfrac{\partial P}{\partial \lambda_1} = \mu_1 \quad \vdots \quad \dfrac{\partial P}{\partial \lambda_\ell} = \mu_\ell$$

longitude

Eliminate $x_1, \cdots, x_n$

$\Rightarrow A(\lambda, \mu) = 0$

2. Applications

- Volume
- A-polynomial (A-ideal)
- Twisted Reidemeister torsion
- Complexified tetrahedron
- Symmetry of quantum $6j$-symbols

$$V_K^{(N-1)}(t) \underset{N\to\infty}{\sim} \text{///}\, e^{\frac{N}{2\pi}\mathrm{Vol}(K)}$$

conjecture

Ohtsuki - Takata proved several cases.

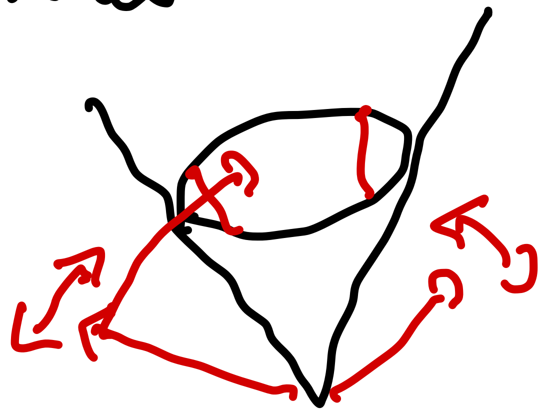
2. Applications

- Volume
- A-polynomial (A-ideal)
- Twisted Reidemeister torsion
- Complexified tetrahedron
- Symmetry of quantum $6j$-symbols

Generalized tetrahedron

proj. model

by 4 planes



compact

ideal

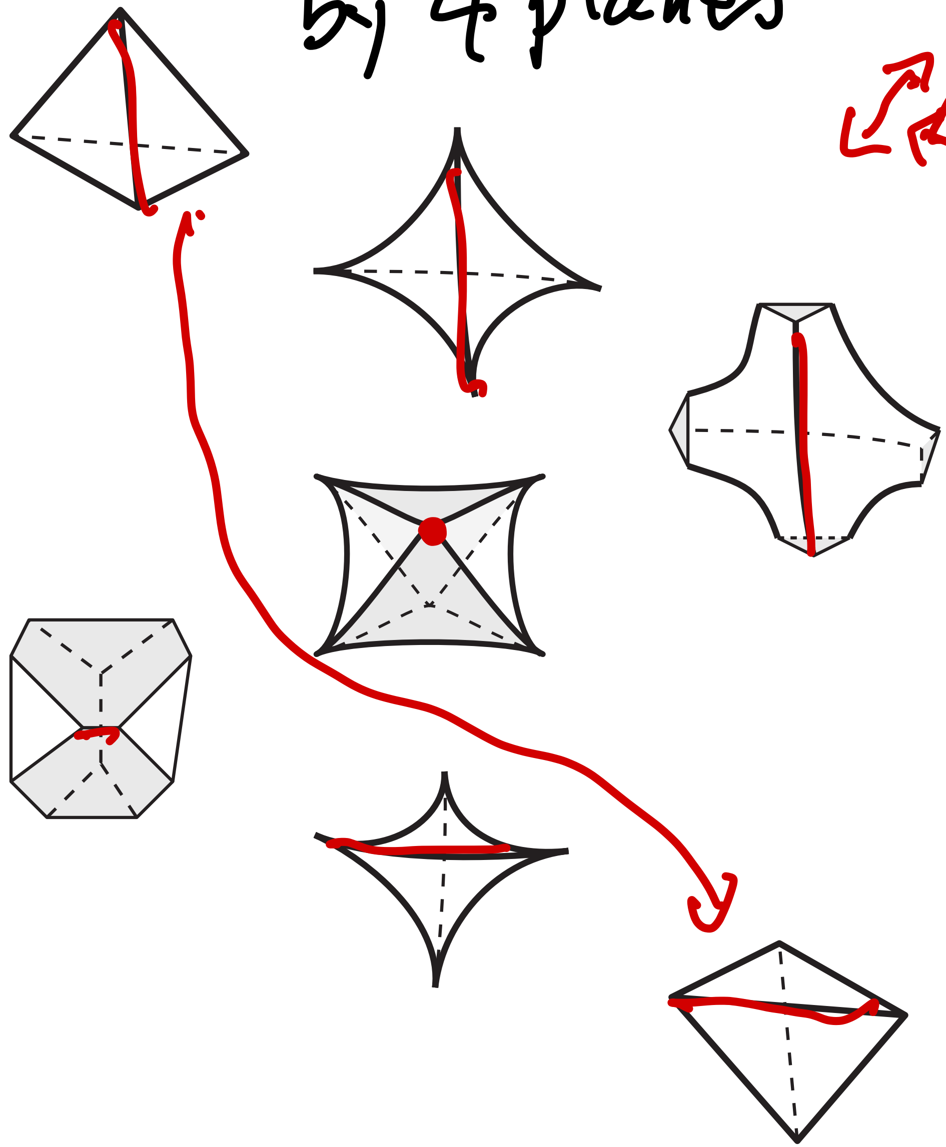
truncated

ideal octahedron

doubly truncated

ideal (dual)

compact (dual)



$\mathbb{R}^4$

$\rightsquigarrow$

$\mathbb{C}^4$

2. Applications

- Volume
- A-polynomial (A-ideal)
- Twisted Reidemeister torsion
- Complexified tetrahedron
- Symmetry of quantum $6j$-symbols

3. Symmetry of quantum $6j$-symbols

Volume Conjecture $$2\pi \lim_{N\to\infty} \frac{\log |TV_N(M)|}{N} \overset{?}{=} \mathrm{Vol}(M).$$

Recall the strategy to prove V. C.

quantum sl_2 invariant

$\downarrow$

Volume potential function

$\downarrow$

Poisson summation formula

$\downarrow$

apply saddle point method

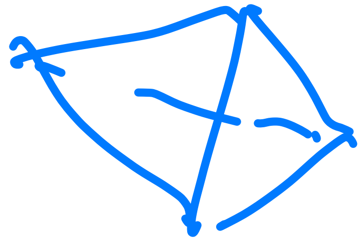
$TV_N(M)$ $\quad$ M : Fujii's manifold

$P(e^{i\theta})$ $\quad$ $e^{i\theta} = t^{a+\frac{1}{2}}$

$$\frac{N}{4\pi} \frac{1}{\sin \frac{2\pi}{N}} \sum_{k} \int_\gamma e^{\frac{iNk}{2}\theta} \sin\theta \, \exp\left(\frac{N}{4\pi} P(e^{i\theta})\right) d\theta$$

Turaev-Viro invariant of 3-manifolds

M : 3-manifold



$\triangle$: a tetrahedral decomposition of M
$V_\triangle$: vertices, $E_\triangle$: edges, $F_\triangle$: faces,
$T_\triangle$: tetrahedra, $N \geq 2$
$I = \{0, \frac{1}{2}, 1, \cdots, \frac{N-2}{2}\}$: the set of spins

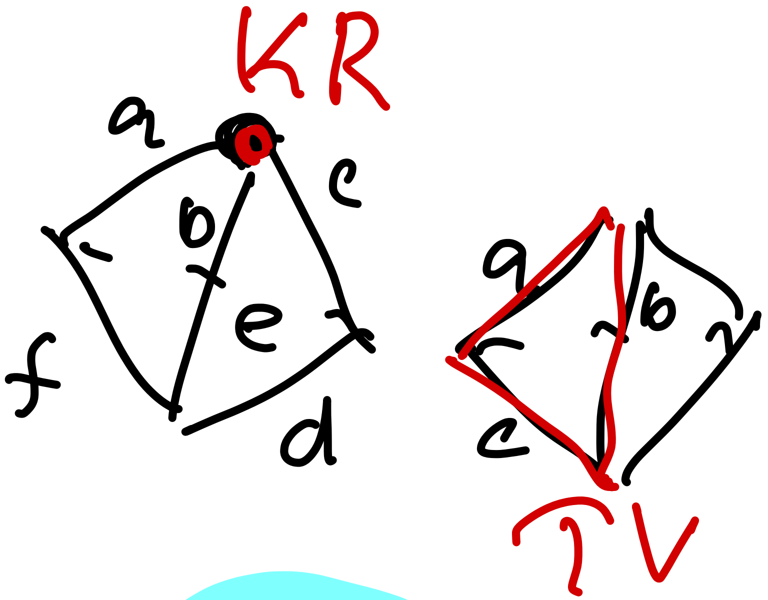
$a, b, c \in I$ is N-*admissible* iff

$$|a-b| \leq c \leq a+b, \quad a+b+c \leq N-2 \quad \text{and} \quad a+b+c \in \mathbb{Z}$$

$|\sigma| = \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$: quantum $6j$-simbol (by **A. Kirillov** and **N. Reshetikhin**)

Turaev-Viro invariant of 3-manifolds

M : 3-manifold



$\triangle$: a tetrahedral decomposition of M
$V_\triangle$: vertices, $E_\triangle$: edges, $F_\triangle$: faces,
$T_\triangle$: tetrahedra, $N \geq 2$
$I = \{0, \frac{1}{2}, 1, \cdots, \frac{N-2}{2}\}$: the set of spins

$|\sigma| = \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}$: quantum $6j$-simbol

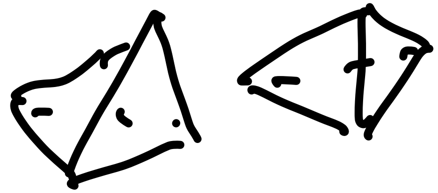
$$[n] = \frac{\{n\}}{\{1\}}$$

$$TV_N(M) = \sum_{\substack{\varphi : E_\triangle \to I \\ N\text{-admissible}}} \prod_{e \in E_\triangle} [2\varphi(e)+1] \prod_{\sigma \in T_\triangle} |\sigma_\varphi^*|$$

Chen-Yang's V. C.	$\displaystyle\lim_{N\to\infty} \frac{2\pi \log |TV_N(M)|}{N} \underset{?}{=} \mathrm{Vol}(M).$

Turaev-Viro invariant of 3-manifolds

M : **cusped 3-mfd or 3-mfd w. boundary**



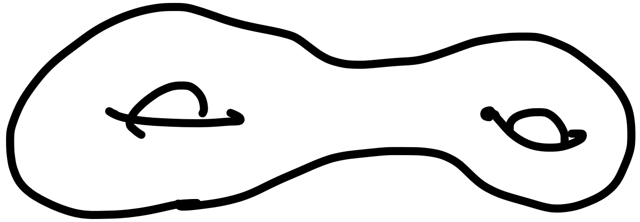
$$TV_N(M) = \sum_{\substack{\varphi : E_\triangle \to I \\ N\text{-admissible}}} \prod_{e \in E_\triangle} [2\varphi(e)+1] \prod_{\sigma \in T_\triangle} \left|\sigma_\varphi^*\right|$$

Chen-Yang's V. C. $\displaystyle \lim_{N\to\infty} \frac{2\pi \log |TV_N(M)|}{N} \underset{?}{=} \mathrm{Vol}(M).$

For 4_1 and Borromean rings, this conjecture is proved by **R. Detcherry, E. Kalfagianni** and **T. Yang**.

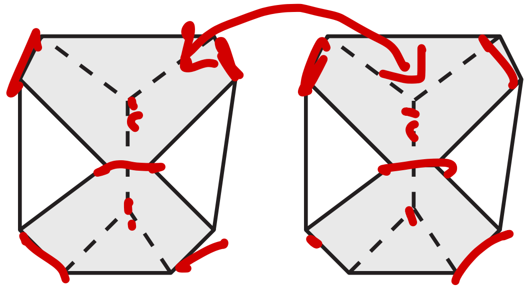
For a knot K, $TV_N(S^3 \setminus K) = 2^{N+1} \frac{\sin^2 \frac{2\pi}{N}}{N} \sum_j |V_j(K)|^2$.

Fujii's manifolds

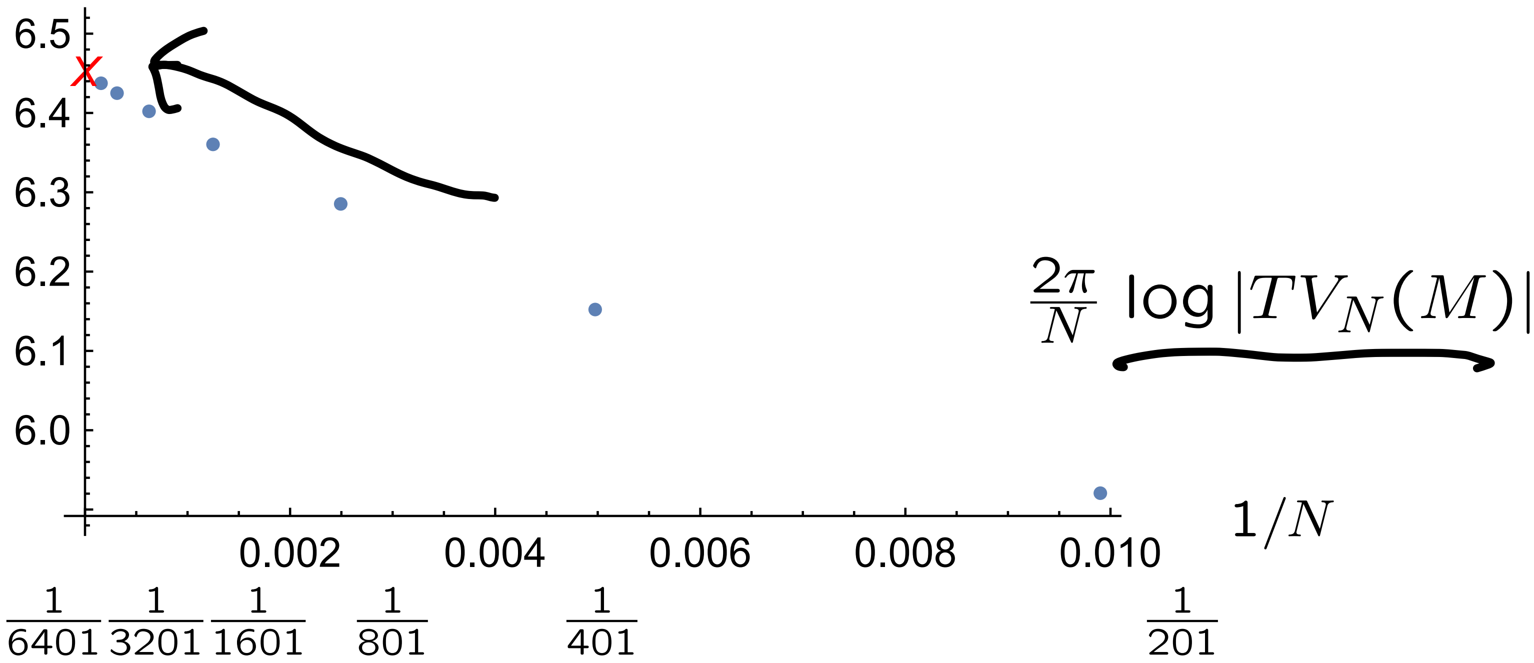


Construction:

Three-manifolds with genus 2 geodesic boundary consists of two truncated regular tetrahedron with dihedral angles $\frac{\pi}{6}$.

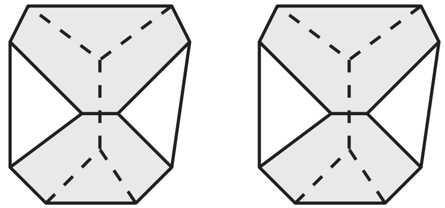


There are 8 different manifolds having the same volume.



TV invariant of Fujii's manifolds

M : Three-manifolds with genus 2 geodesic boundary consists of two truncated tetrahedron. There are 8 different manifolds having the same volume.

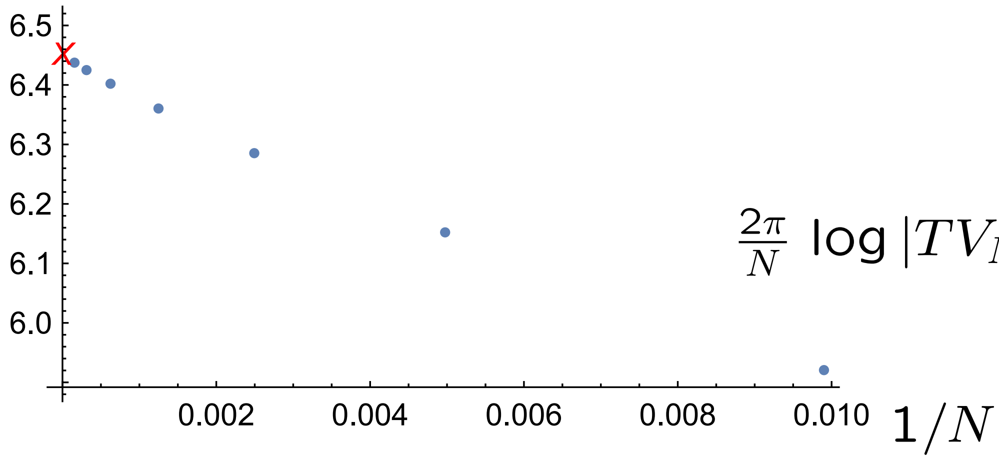


$$TV_N(M) = \sum_a [2a+1] \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix}^2$$

$\{n\}!$

$P_N(e^{i\theta})$: $\boxed{\text{continuous version of } \frac{4\pi}{N} \log \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix}^2}$

$$TV_N(M) = \frac{N}{4\pi} \frac{1}{\sin \frac{2\pi}{N}} \sum_k \int_0^{2\pi} \sin\theta \exp\left(\frac{N}{4\pi} \left(P_N(e^{i\theta}) + 2\pi k i \theta \right) \right) d\theta$$



Apply the saddle point method

$$\boxed{\int_\gamma f(z)e^{Ng(z)}\,dz \underset{N\to\infty}{\sim} \sqrt{\frac{2\pi}{-Ng''(z_0)}}\,f(z_0)e^{Ng(z_0)} \quad \text{where } g'(z_0)=0.}$$

$$\underbrace{TV_N(M) = \frac{N}{4\pi}\frac{1}{\sin\frac{2\pi}{N}}\sum_k \int_0^{2\pi} \boxed{\sin\theta}\,\exp\left(\frac{N}{4\pi}\left(P_N(e^{i\theta}) + 2\pi\, k\, i\,\theta\right)\right)^{\varphi} d\theta}$$

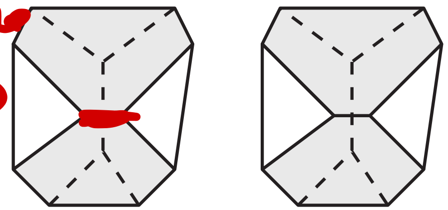
Let $f(\theta)=\sin\theta$, $\quad g_k(\theta)=P(e^{i\theta})+2\pi k\theta$

For $k=0$, $\theta_0=\pi$, $P(-1)=7.32772$.

For $k=\pm1$, $\theta_1=\boxed{\pm 0.596134}\,i+\pi$,

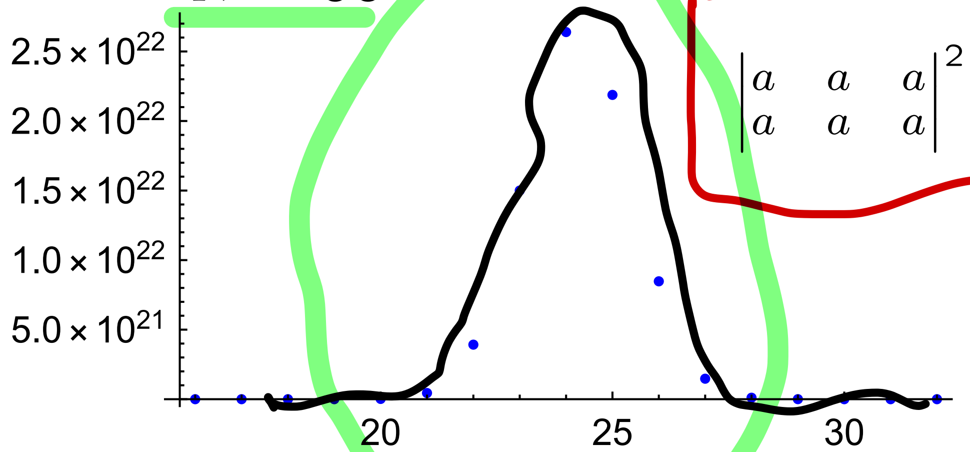
Re $\left(P(e^{i\theta_1})+(\pm 2\pi i\theta_1)\right) = \pm\boxed{6.45199} = \pm\mathrm{Vol}(M)$.

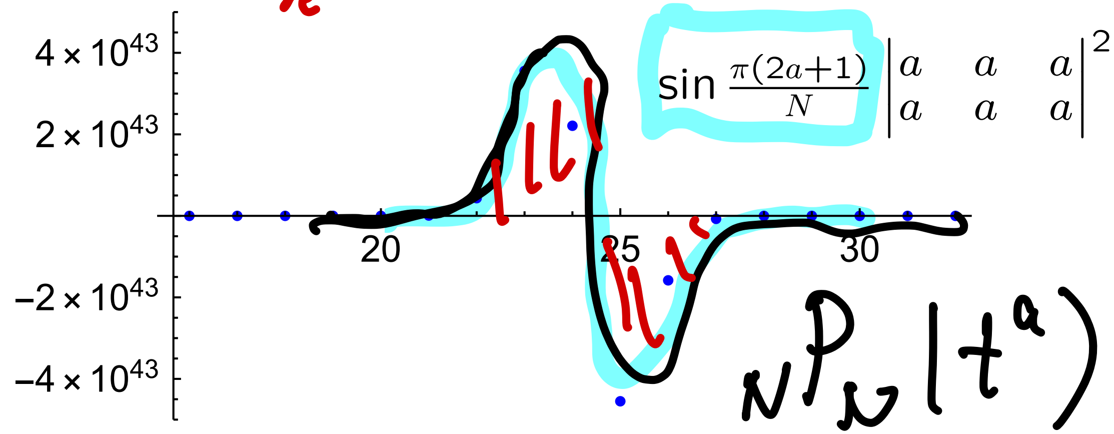
lengths



$k=0$

$N=99$





$_N P_N(t^a)$

$\sin\theta\; e$

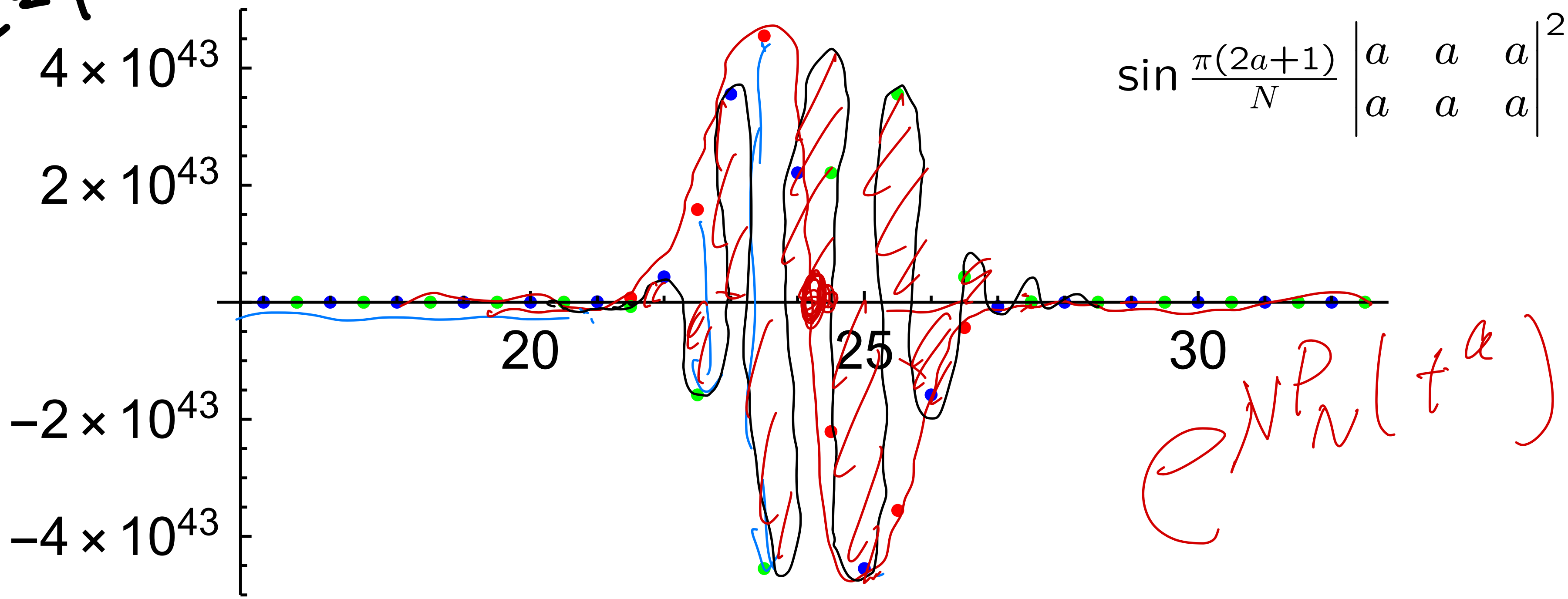
Symmetry of quantum $6j$-symbols

Let $h_k^{(N)} = \int_0^{2\pi} \sin\theta \exp\left(\frac{N}{4\pi}\left(P_N(e^{i\theta}) + \boxed{2\pi} k\, i\,\theta\right)\right) d\theta.$

$|h_k^{(N)}| < |h_{\pm 1}^{(N)}|, \quad |h_0^{(N)}| \leq |h_{\pm 1}^{(N)}|.$

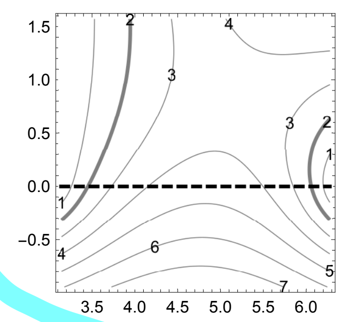
?

$k = 1$



Question: Is $P_N(e^{i\theta})$ symmetric at $\theta = \pi$?

Towards a proof of V. C.



Obstruction for proving V. C.

- **Colored Jones and WRT invariant**
  The condition for the saddle point method.
- **Turaev-Viro invariant**
  Symmetry of the quantum $6j$ symbol.

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix}^{\text{cont.}} \underset{?}{=} \begin{vmatrix} a^* & b & c \\ d & e & f \end{vmatrix}^{\text{cont.}} \quad \text{where } a^* = N - 2 - a.$$

Cf. $\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ d^* & e^* & f^* \end{vmatrix}$ (by **R. Detcherry**, **E. Kalfagianni**, **T. Yang**)

- **Kirillov-Reshetikhin invariant**
  Complexified tetrahedron

- **Hennings invariant**
  The quantum $6j$-symbol has natural symmetry.
  Well-definedness for knotted graph is not obvious.

  non-semisimple repr., modified trrace, ADO inv., CGP inv.
  Diagramatic interpretation is given by **M. de Renzi**