Canonical Forms of Automorphisms of Free Groups

Vogtmann's Dictionary:

compact

① $GL_n\mathbb{Z} \longleftrightarrow$ ② $MCG(S) \longleftrightarrow$ ③ $Out(F_n)$

1) $GL_n\mathbb{Z}$: Jordan Canonical Form.

Given $A \in GL_n\mathbb{Z}$,

$$\begin{array}{ccccccc} \mathbb{Z}^n & \leq & \mathbb{C}^n & = & V_1 & \oplus \cdots \oplus & V_k \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ A & & A & & A_1 & & A_k \end{array}$$

$A \rightsquigarrow$ eigenvalues $\lambda_1, \dots, \lambda_k$

: $\quad i = 1, \dots, k$

$V_i = \ker (A - \lambda_i I)^n$

$A_i = \lambda_i I + N$

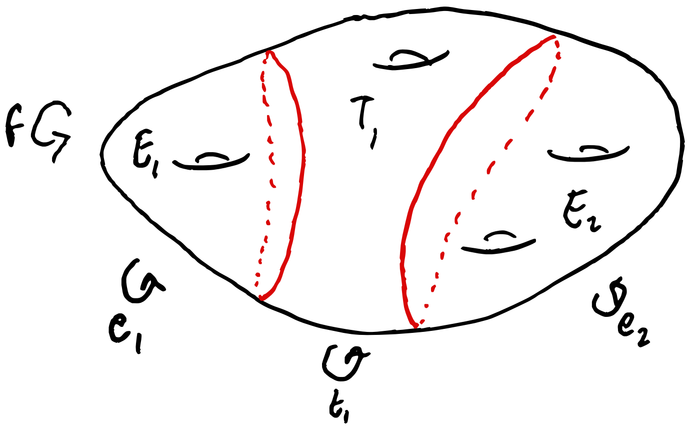
$|\lambda_i| > 1$

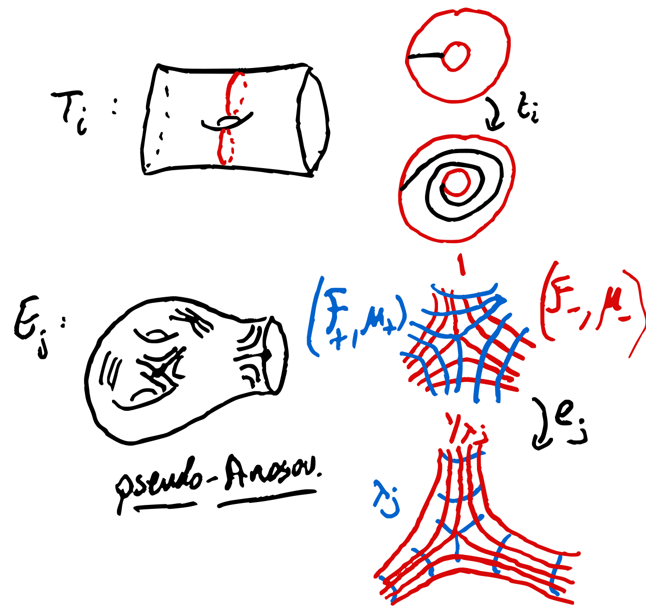
2) $MCG(S) = \mathrm{Homeo}(S)/\text{isotopy} = \pi_0(\mathrm{Homeo}(S))$:

<u>Nielsen-Thurston Decomposition</u>

Assume $\chi(S) < 0$.

Given $[f] \in MCG(S)$,





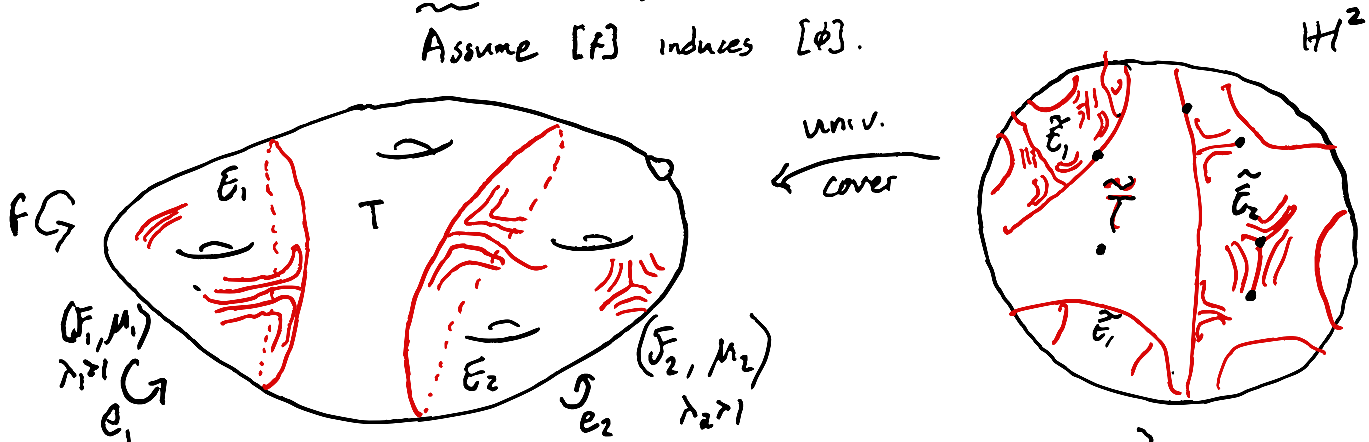
3) $\mathrm{Out}(F) = \mathrm{Aut}(F)/\mathrm{Inn}(F)$: ???

Given $[\phi] \in \mathrm{Out}(F)$,

<u>Geometric Case</u>: Assume $\partial S \neq \emptyset$ and let $F = \pi_1(S)$.

<u>Fact</u>: $MCG(S) \leq \mathrm{Out}(F)$ (Nielsen)

Assume $[f]$ induces $[\phi]$.



$\mathbb{H}^2/\sim$

$(Y, d) \circlearrowleft g$ homeo.

properties

<u>Facts</u>:

1) $F \curvearrowright (Y, d)$ by isometries.
   - trivial arc stabilizers

2) $g: (Y,d) \to (Y,d)$ is a <u>dilation</u>

3) x is loxodromic $\Leftrightarrow$ x^{t^F} has <u>exp. growth</u>

$\mathbb{R}$-tree
- geodesic
- triangle

$\cong$

Original Goal Given $[\phi] \in Out(F)$,

Find an $\mathbb{R}$-tree (T, d) with an isometric F-action

1) trivial arc stabilizers.

2) $f : (T,d) \to (T,d)$ that is

- ϕ-equiv. : $f(x \cdot p) = \phi(x) \cdot f(p)$
- dilation: $(T,d) \sim (T_1, d_1)$ "$\oplus$" ... "$\oplus$" (T_k, d_k)

  + $f|_{T_i}$ is a λ_i-homothety w/ $\lambda_i > 1$.

3) $x \in F$ is loxodromic $\Leftrightarrow$ x has exp. growth:

$$\lim_{n \to \infty} \|\phi^n(x)\|_B^{1/n} > 1.$$

This fails because "exponential dynamics aren't ~~ss~~ semi-simple."

$F = F(a,b,c,d)$

1) ϕ: $a \mapsto ab$, $b \mapsto bab$ ← "semi-simple"

$c \mapsto d\underline{[a,b]}$

$d \mapsto dc$

← not diagonalizable.

2) ψ: $a \mapsto ab$, $b \mapsto bab$

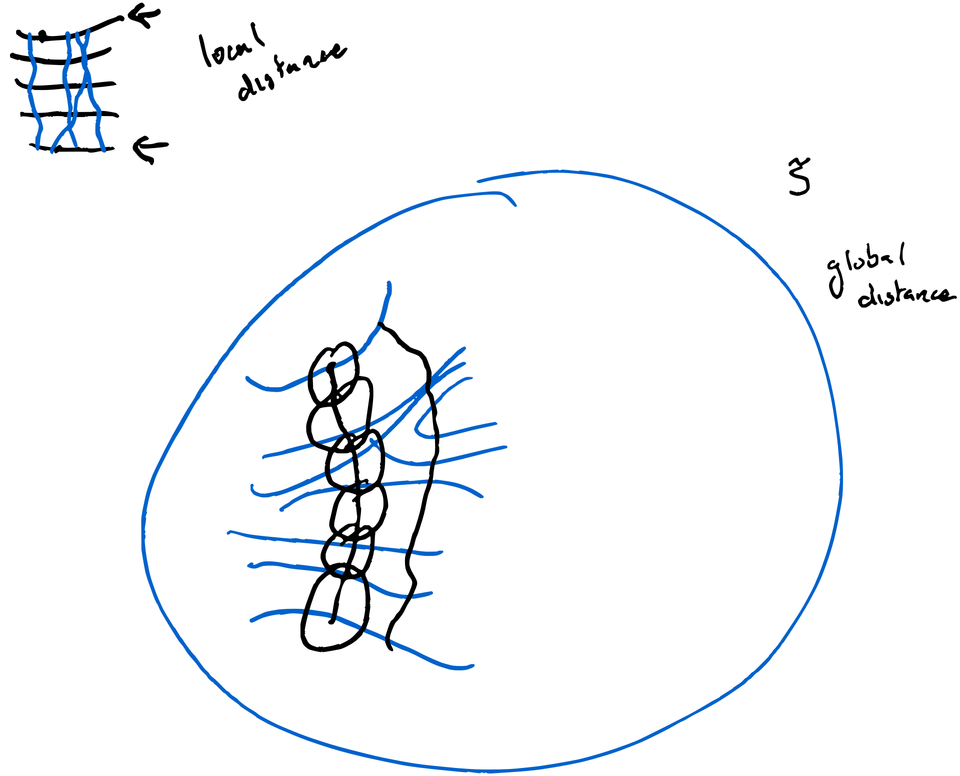
$c \mapsto d\underline{a}$

$d \mapsto dc$

Iterating $a \overset{\phi}{\underset{\psi}{\mapsto}} \cancel{b}\ ab \mapsto \cancel{dcab}\ \underline{\underline{abbab}} \mapsto \cdots$

$c \overset{\phi}{\mapsto} d[a,b] \mapsto dc[a,b] \mapsto dcd\underline{[a,b]}[a,b].$

$c \overset{\psi}{\mapsto} da \mapsto dcab \mapsto dcda\,\underline{\underline{abbab}} \mapsto \cdots$


local distance
ξ
global distance.