Genus bounds for non-semisimple quantum invariants

Daniel López Neumann

(joint work with Roland van der Veen)

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Quantum invariants of knots:

- Come from the representation theory of quantum groups $U_q(\mathfrak{g})$ (Jones, Witten, Reshetikhin-Turaev, 80-90s).

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Question: Topological content of quantum invariants?

Many conjectures about (colored) Jones. But what about ADO?

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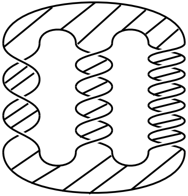

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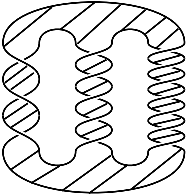

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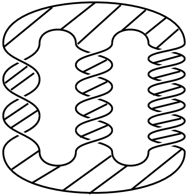

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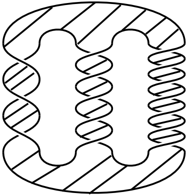

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- Alexander 1928: $\deg \Delta_K(t) \leq 2g(K)$.
- No such bound for $J_K(q)$.
- Bound for 2-loop expansion of colored Jones (Ohtsuki '06).

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Theorem (LN-van der Veen, '22)

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(LN-vdV, '22) If $H =$ Borel of $\mathfrak{u}_{\zeta_p}(\mathfrak{sl}_2)$ then $P_H = ADO_p$ invariant so

$$\deg ADO_p(K, t) \leq 2g(K)(p - 1).$$

Outline

- Drinfeld doubles $D(H)$ of Hopf algebras.
- Universal invariant $Z_H(K) \in D(H)$ of knots.
- Twist the above story (by $\mathbb{N}$-degree) to get knot polynomials.
- Proof of thm.

Hopf algebra:

- Algebra $(H, m, 1)$ over $\mathbb{K}$.
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1. $H_q = \langle K^{\pm 1}, E \mid KE = q^2 EK \rangle$ with

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, & \Delta(K) &= K \otimes K, \\ \epsilon(E) &= 0, & \epsilon(K) &= 1, \\ S(E) &= -EK^{-1}, & S(K) &= K^{-1}. \end{aligned}$$

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2. If $q = \zeta_p = e^{\pi i/p}$, the quotient $B_{\zeta_p} = H_q/(E^p = 0, K^{2p} = 1)$ is a Hopf algebra.

The **Drinfeld double** of a Hopf algebra H is $D(H) := H^* \otimes H$ as a coalgebra with multiplication

$$(p \otimes a) \cdot (q \otimes b) := \langle q_{(1)}, S^{-1}(a_{(3)}) \rangle \langle q_{(3)}, a_{(1)} \rangle p \cdot q_{(2)} \otimes a_{(2)} b$$

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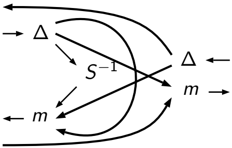

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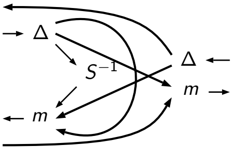

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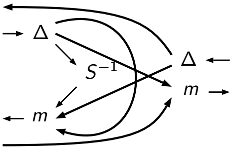

Why is $D(H)$ interesting? Because $R \in D(H)^{\otimes 2}$ defined by

$$R = \sum (\epsilon \otimes h_i) \otimes (h^i \otimes 1) = \sum h_i \otimes h^i \in D(H)^{\otimes 2}$$

where h_i is a basis of H and $h^i \in H^*$ is the dual basis, satisfies the **Yang-Baxter eqtn**.

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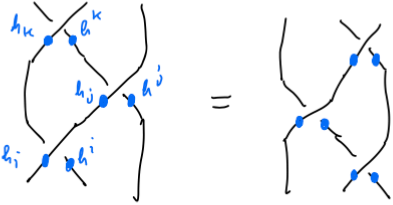

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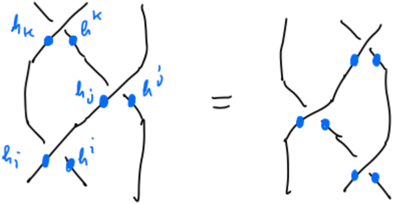

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We will also suppose $D(H)$ is “ribbon”. Thus, there is $g \in D(H)$ such that $S^2(x) = gxg^{-1}$ and $\Delta(g) = g \otimes g$

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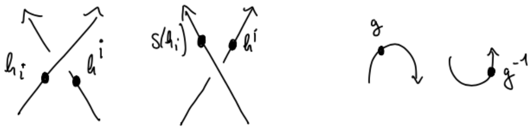

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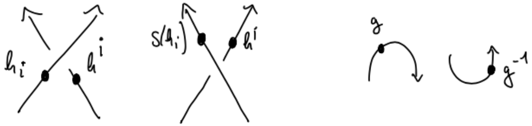

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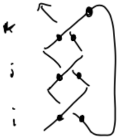

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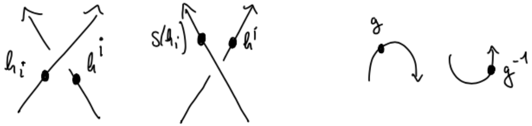

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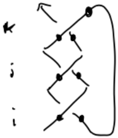

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The result is $Z_H(T) \in D(H)^{\otimes m}$ (m = number of cmpnts. of T).

If you prefer categories:

$X \in \text{Rep } D(H)$ iff $X \in \text{Rep}(H)$ together with a collection of **half-braidings**

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The invariant $Z_H(T)$ is "universal" (for Rep $D(H)$ quantum invariants) in that

$$\text{tr}_V(Z_H(T)) = RT_V(\widehat{T})$$

for $V \in \text{Rep } D(H)$.

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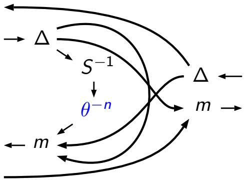

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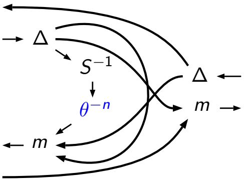

This is NOT a Hopf algebra (if $n \neq 0$) but there is a "coproduct"

$$\Delta_{n,m} : D_{n+m} \to D_n \otimes D_m$$

and antipode $S_n : D_n \to D_{-n}$, satisfying graded versions of Hopf axioms.

Theorem

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Rmk: We could have used the full Aut(H) to define an Aut(H)-crossed braided category. But note:

- Aut(H) is finite for f.d. **semisimple** Hopf algebras (Radford).
- More generally: finite semisimple tensor categories can only carry finite group actions! (Etingof-Gelaki-Nikshych '05).

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$$\{\sigma_{Y,X} : Y \otimes X \to X \otimes \theta^n(Y)\}_{Y \in \mathrm{Rep}(H)}.$$

This is the **relative Drinfeld center** of $\mathrm{Rep}(H) \ltimes \mathbb{Z}$.

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But here $K^{2p} = 1$. If we set $K' = t^{n/2}K, E' = t^{n/2}E$ then get

$$[E', F] = \frac{K' - k'}{q - q^{-1}}$$

and $(K')^{2p} = t^{pn}$. So the twisted Drinfeld double is related to semi-restricted unrolled $U_{\zeta_p}(\mathfrak{sl}_2)$.

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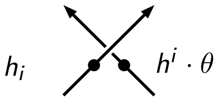

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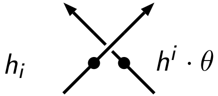

Note: the above lives in a semidirect product $D_1(H) \ltimes \mathbb{Z}$. Also: don't forget $g^{\pm 1}$ on right caps/cups.

Step 2: Multiply in $D_1(H) \ltimes \mathbb{Z}$ the elements encountered while following the orientation of T. This results in an element $Z_H^\theta(T) \in H^* \otimes H[t^{\pm 1}]$.

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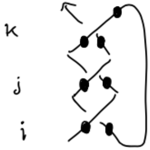

$$\Rightarrow Z_H^{\theta}(T) = h^k \cdot \theta \cdot h_j \cdot h^i \cdot \theta \cdot g \cdot h_k \cdot h^j \cdot \theta \cdot h_i$$
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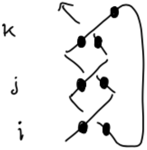

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Step 3: If T is a $(1,1)$-tangle whose closure is K then

$$P_H(K,t) = \epsilon_{D(H)}(Z_H^{\theta}(T)) \in \mathbb{K}[t^{\pm 1}].$$

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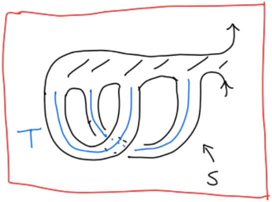

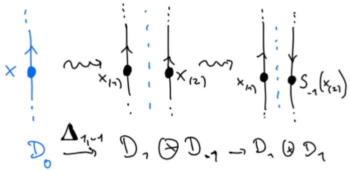

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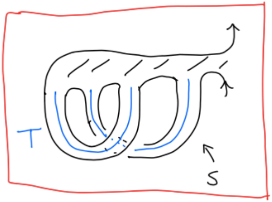

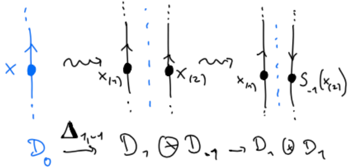

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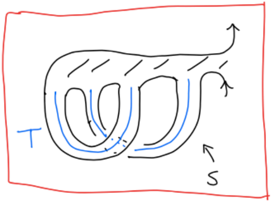

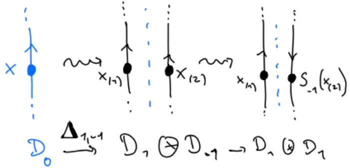

Then the invariant of each band is of the form $\Delta_{1,-1}(x), x \in D_0 = D(H)$! Thus,

$$Z_H^{\theta}(K) = m_1^{(4)} P(\mathrm{id} \otimes S_{-1} \circ \Delta_{1,-1})^{\otimes 2}(Z_H(T)), \ Z_H(T) \in D(H)^{\otimes 2}$$

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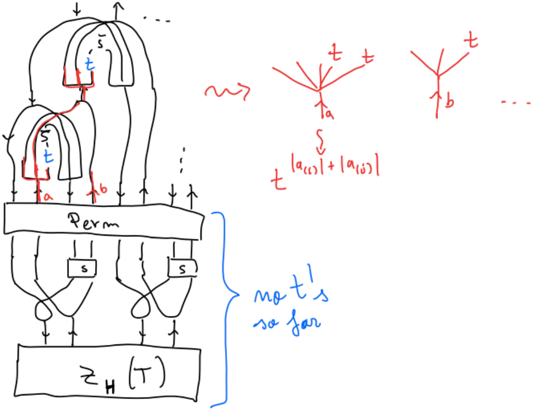

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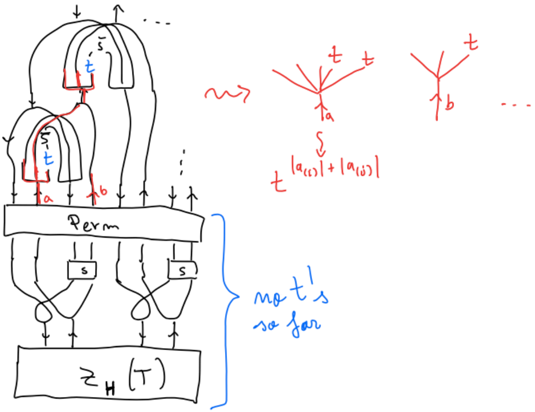

But $|a_{(i_1)}| + |a_{(i_2)}| + \cdots \leq |a| \leq d(H)$ for any $a \in H$. Thus each H-leg of T contributes a t-power $\leq d(H)$.

Thanks!