Abelian quotients of the Y-filtration on the homology cylinders via the LMO functor

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joint work with
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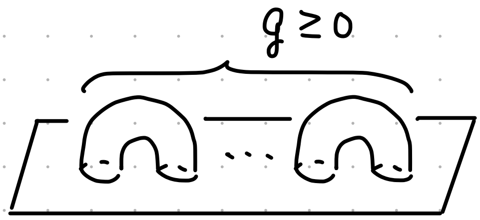
[K-OS]

2021/6/3

☆ [NSS] arXiv: 2001.09825 to appear in Geom. Topol.
[NSS2] arXiv: 2103.07086

§1 Introduction

$\Sigma_{g,1} :=$ 

$\mathcal{M}_{g,1} := \mathrm{Homeo}(\Sigma_{g,1} \text{ rel } \partial)/\text{isotopy}$ the mapping class group

$\mathcal{I}_{g,1} := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}\, H_1(\Sigma_{g,1};\mathbb{Z}))$ the Torelli group

$$\begin{array}{ccccccc} \mathcal{I}_{g,1}(1) & \supset & \cdots & \supset & \mathcal{I}_{g,1}(n) & \supset & \cdots \\ \| & & & & \| & & \\ \mathcal{I}_{g,1} & & & & [\mathcal{I}_{g,1}(n-1), \mathcal{I}_{g,1}] & & \end{array}$$

the lower central series

$\rightsquigarrow \mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1) \cong ??$

$\uparrow$ as abelian groups

$\mathcal{I}_{g,1}(1)/\mathcal{I}_{g,1}(2) = \mathcal{I}_{g,1}^{ab}$ Johnson '85

(Johnson hom. & Birman-Craggs hom.
<u>Fact</u> $\mathrm{tor}\, \mathcal{I}_{g,1}^{ab} \neq 0$ for $g \geq 3$)

$(\mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1)) \otimes \mathbb{Q}$ Hain '97 Morita '99
Morita - Sakasai - Suzuki '20

<u>Prob 1</u> For $n \geq 2$, $\exists ?$ torsion in $\mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1)$

<u>Thm 1</u> (N.-Sato-Suzuki) $\exists$ torsion for $n = 3, 5$ & $g \gg 1$

$$\mathcal{K}_{g,1} := \mathrm{Ker}\left(\mathcal{I}_{g,1} \to \mathrm{Aut}\left({}^{\pi_1\Sigma_{g,1}}/_{\pi_1\Sigma_{g,1}(3)}\right)\right)$$ the Johnson kernel

$$\cup \qquad = \mathrm{Ker}\left(\tau_1 : \mathcal{I}_{g,1} \to \Lambda^3 H_1(\Sigma_{g,1})\right)$$

$$\mathcal{I}_{g,1}(2) \qquad = \langle t_c\text{'s} \rangle$$ C is a separating curve

<u>Rem</u> $\mathcal{M}_g \supset \mathcal{I}_g \supset \mathcal{K}_g$ are defined similarly.

- $\dim H_1(\mathcal{K}_g; \mathbb{Q}) < \infty$ for $g \geq 4$ Dimca-Papadima '13
- $\mathcal{K}_{g(,1)}$ is finitely generated for $g \geq 4$ Church-Ershov-Putman '17 (arXiv)

By using Dimca-Hain-Papadima's result ('14),

<u>Thm</u> (MSS '20) For $g \geq 6$

$H_1(\mathcal{K}_g; \mathbb{Q}) \cong$ "Casson inv." $\oplus$ "Morita's refinement $\tilde{\tau}_2$ of Johnson hom."

<u>Prob 2</u> $\exists$? torsion in $H_1(\mathcal{K}_g; \mathbb{Z})$

<u>Thm 2</u> (NSS) $\exists$ torsion in $H_1(\mathcal{K}_g; \mathbb{Z})$ for $g \geq 6$.

Key tool : the LMO functor in 3-dim (quantum) topology
(Le-Murakami-Ohtsuki)

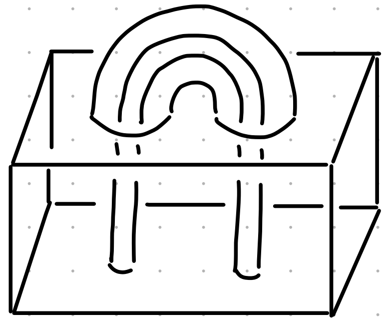
$$\tilde{Z} : \mathcal{L}\mathcal{C}ob_q \longrightarrow {}^{ts}\!\mathcal{A}$$

$\mathcal{L}\mathcal{C}ob_q$: the category of Lagrangian q-cobordisms

${}^{ts}\!\mathcal{A}$: the category of top-substantial Jacobi diagrams

M: connected oriented compact 3-mfd

$m : \partial(\Sigma_{g,1}\times[-1,1]) \xrightarrow{\cong} \partial M \quad (m \rightsquigarrow m_+, m_-)$



$g=1$

$(M,m)\sim(M',m') \overset{\text{def}}{\Longleftrightarrow}$
$$\begin{tikzcd} M \arrow[rr, "\exists\cong"] & \circlearrowleft & M' \\ & \partial(\Sigma_{g,1}\times[-1,1]) \arrow[ul, "m"] \arrow[ur, "m'"'] & \end{tikzcd}$$

↑ cobordism

Def (M,m) is a homology cylinder over $\Sigma_{g,1}$

$\Longleftrightarrow (m_+)_* = (m_-)_* : H_*(\Sigma_{g,1}) \xrightarrow{\cong} H_*(M)$.

$\mathcal{C}_{g,1} := \{\text{homology cylinders over } \Sigma_{g,1}\}$

is a monoid by $M\circ M' = \begin{array}{|c|}\hline M' \\ \hline M \\ \hline\end{array}$

$c : \mathcal{I}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$

$f \longmapsto (\Sigma_{g,1}\times[-1,1],\ f\times 1 \cup \mathrm{id}\times(-1))$

$Y_1\mathcal{C}_{g,1} \supset \cdots \supset Y_n\mathcal{C}_{g,1} \supset \cdots$: the "Y-filtration"

$Y_1\mathcal{C}_{g,1} = \mathcal{C}_{g,1}$ submonoid satisfying $c(\mathcal{I}_{g,1}(n)) \subset Y_n\mathcal{C}_{g,1}$

c induces a group hom $\mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1) \longrightarrow Y_n\mathcal{C}_{g,1}/\sim_{Y_{n+1}}$

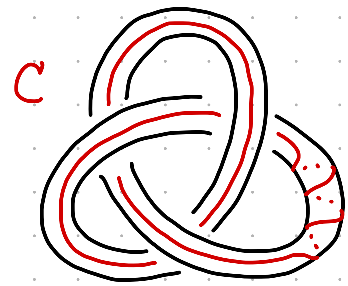
$Y_n\mathcal{C}_{g,1}/\sim_{Y_{n+1}}$ $\qquad Y_n\mathcal{C}_{g,1}$

Abelian quotients of the Y-filtration

on the homology cylinders via the LMO functor

$\mathcal{C}_{g,1} \qquad \tilde{Z}$

§2. Preliminaries



◎ Dehn surgery

M : a 3-mfd
$\cup$
K : a framed knot, i.e., $C \subset \partial N(K)$ is specified

$M \rightsquigarrow M_K := (M \setminus \mathring{N}(K)) \cup D^2 \times S^1$

◎ Clasper surgery $\subset$ Dehn surgery $\left(\begin{array}{l}\text{Goussarov '99} \\ \text{Habiro '00}\end{array}\right)$

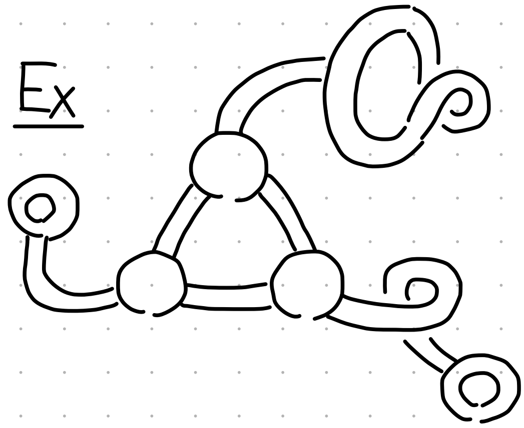
$M = (M, m) \in \mathcal{C}_{g,1}$
$\cup$
G : a graph clasper

$\Big\downarrow$ $\left(\begin{array}{l}\text{a surface consisting of} \\ \text{disks, bands and annuli}\end{array}\right)$

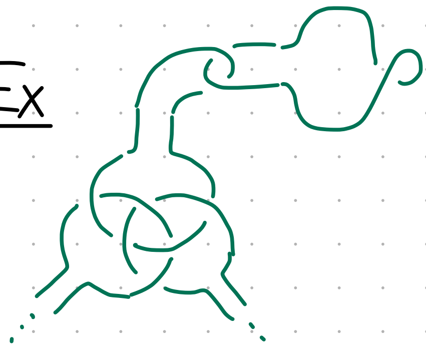
L_G : framed link in M

$M_G := M_{L_G} \in \mathcal{C}_{g,1}$

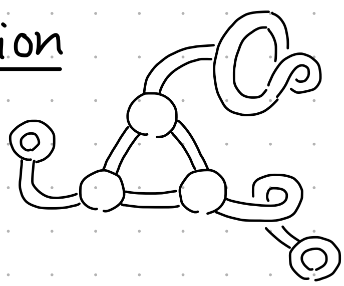
Ex



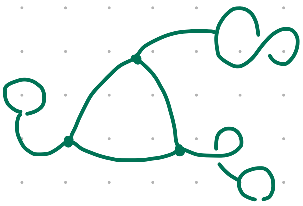
Ex



Convention



$=:$



$\underline{\text{Def}}$ M is Y_n-equivalent to M'

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\, G_1, \cdots, G_r \subset M$ s.t. $M_{\coprod_{j=1}^{r} G_j} = M'$ & $\deg G_j = n$ (# of disks)

$Y_n \mathcal{C}_{g,1} := \{ M \in \mathcal{C}_{g,1} \mid M \underset{Y_n}{\sim} \Sigma_{g,1} \times [-1,1] \}$ submonoid

$\mathcal{C}_{g,1} = Y_1 \mathcal{C}_{g,1} \supset \cdots \supset Y_n \mathcal{C}_{g,1} \supset \cdots$: the Y-filtration

$\underline{\text{Fact}}$ · $c(\mathcal{I}_{g,1}(n)) \subset Y_n \mathcal{C}_{g,1}$, where $c : \mathcal{I}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$

· $M \underset{Y_n}{\sim} M' \Rightarrow \pi_1 M / \pi_1 M(n+1) \cong \pi_1 M' / \pi_1 M'(n+1)$

· $Y_n \mathcal{C}_{g,1} / Y_{n+1}$ is a finitely generated abelian group

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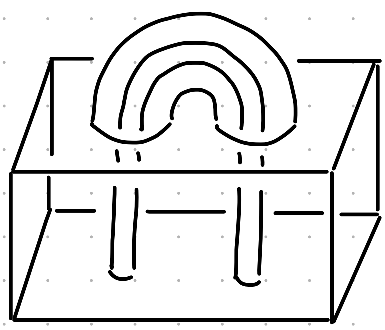
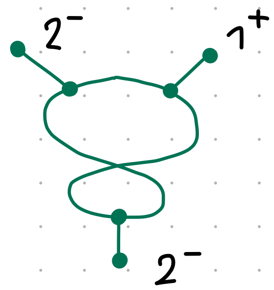
$$\underbrace{(\log \tilde{Z})_n}_{\text{the LMO functor}} : Y_n \mathcal{C}_{g,1} / Y_{n+1} \xrightarrow{\text{hom}} \mathcal{A}_n^c \otimes \mathbb{Q}$$

3-manifold $\longmapsto$ graph / diagram

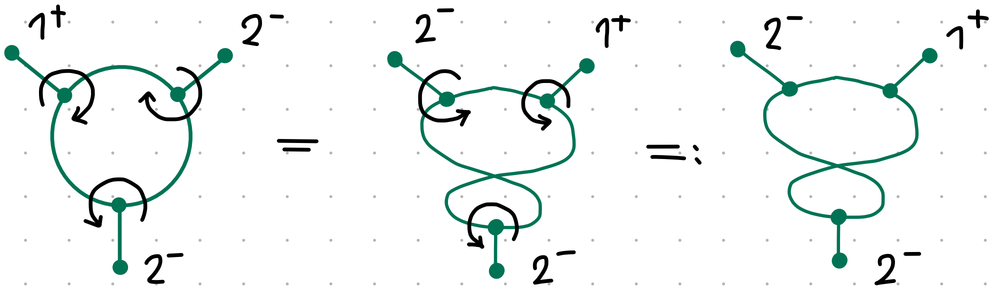




◎ Jacobi diagrams

Def A Jacobi diagram colored with $\{1^+, \cdots, g^+, 1^-, \cdots, g^-\}$ is
a uni-trivalent graph
s.t. each $\begin{cases} \text{univalent vertex is colored} \\ \text{trivalent vertex has a cyclic order} \end{cases}$

Ex

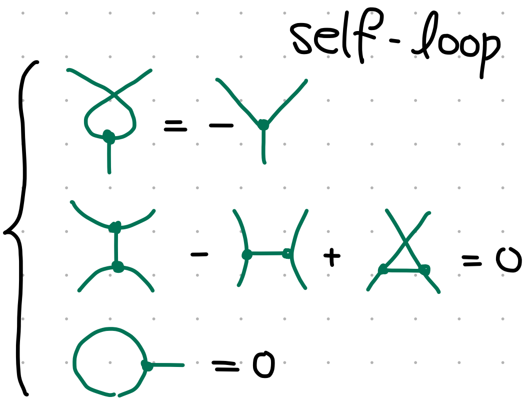


$\in \mathcal{A}^c_{3,1}$

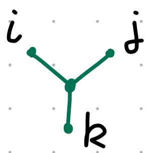
$\mathcal{A}^c_n := \mathbb{Z}\{\text{conn. Jacobi diagrams of i-deg} = n\}/\text{AS, IHX}$

i-deg ~ # of trivalent vertices (internal)

$\| \quad \bigoplus_{\ell \geq 0} \mathcal{A}^c_{n,\ell}$ ← ℓ ~ first Betti number

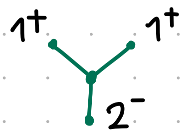


Ex $\mathcal{A}^c_1 = \langle$



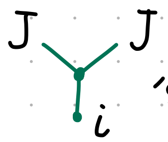
's $\rangle$

$\cong H_1(\Sigma_{g,1})^{\otimes 3} / x_1 \otimes x_2 \otimes x_3 \sim \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$



$(\neq 0)$ is a 2-torsion

Thm (Conant-Schneiderman-Teichner '12)
$\mathrm{tor}\, \mathcal{A}^c_{2k+1,0}$ is generated by



's

J ~ tree

→ We use $\mathrm{tor}\, \mathcal{A}^c_{2k+1,0}$ and $\mathrm{tor}\, \mathcal{A}^c_{2k+1,1}$

◎ LMO functor & surgery map
(Cheptea - Habiro - Massuyeau '08)

$$\log \tilde{Z}(-) : \mathcal{C}_{g,1} \longrightarrow \hat{\mathcal{A}}^c \otimes \mathbb{Q}, \quad M \mapsto J_0 + J_1 + \cdots + \underset{\mathcal{A}_n^c \otimes \mathbb{Q}\ \ni}{J_n} + \cdots$$

$$\cup$$

$$Y_n \mathcal{C}_{g,1} \qquad\qquad M \mapsto J_0 + 0 + \cdots + 0 + J_n + \cdots$$

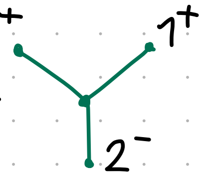
$$Y_n \mathcal{C}_{g,1}/Y_{n+1} \longrightarrow \mathcal{A}_n^c \otimes \mathbb{Q}, \quad [M] \mapsto J_n$$

Q. How to compute J_n from $[M]$?

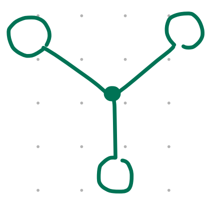
A. When $[M] = s(J)$ for $J \in \mathcal{A}_n^c$, we have $J_n = \pm J$.

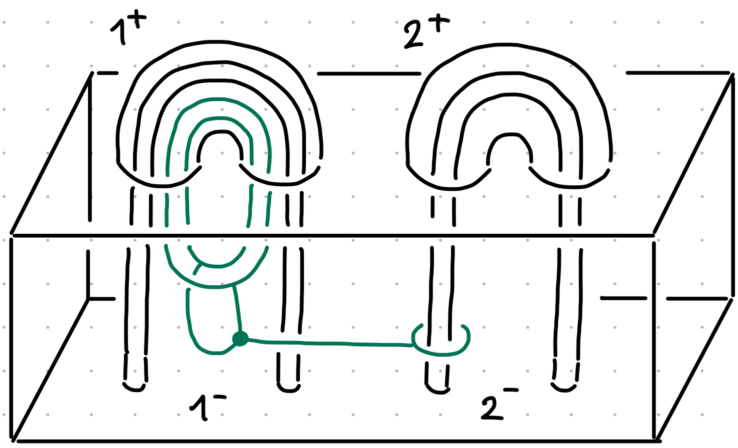
The surgery map $s : \mathcal{A}_n^c \xrightarrow{\text{hom}} Y_n \mathcal{C}_{g,1}/Y_{n+1}$ is defined by...

Ex ($n=1$, $g=2$)

$J =$ 

$\rightsquigarrow$ $\Sigma_{2,1} \times [-1,1] \supset G_J$





$s(J) = (\Sigma_{2,1} \times [-1,1])_{G_J}$ is well-defined up to $\sim_{Y_2}$

Fact $\mathcal{A}_n^c \xrightarrow{s} Y_n \mathcal{C}_{g,1}/Y_{n+1}$ is surjective if $n \geq 2$

$$\begin{tikzcd} \mathcal{A}_n^c \arrow[r, "s"] \arrow[d] & Y_n \mathcal{C}_{g,1}/Y_{n+1} \arrow[d] \\ \mathcal{A}_n^c \otimes \mathbb{Q} \arrow[r, "\cong"] & Y_n \mathcal{C}_{g,1}/Y_{n+1} \otimes \mathbb{Q} \arrow[l, bend left, "(\log \tilde{Z}(-))_n"] \end{tikzcd}$$

- $\mathrm{Ker}\, s \subset \mathrm{tor}\, \mathcal{A}_n^c$
- Goussarov - Habiro Conjecture

§3. Main theorem

We expect that $(\log \tilde{Z}(-))_{n+1} \overset{?}{\longleftrightarrow} \mathrm{tor}\,(Y_n\mathcal{C}_{g,1}/Y_{n+1})$

$$\begin{tikzcd}
{[M]} \in Y_n\mathcal{C}_{g,1}/Y_{n+1} \arrow[r, "\times"] \arrow[dr, "\bar{z}_{n+1}"'] & \mathcal{A}^c_{n+1} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d] \\
& \mathcal{A}^c_{n+1} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \ni (\log \tilde{Z}(M))_{n+1} = J_{n+1}
\end{tikzcd}$$

well-defined

Rem $\bar{z}_2$ = Birman–Craggs homomorphism

Q. How to compute $\bar{z}_{n+1}([M]) = J_{n+1} \bmod \mathbb{Z}$?

A. When $[M] = \mathfrak{s}(J)$ for $J \in \mathcal{A}^c_n$, we have $\bar{z}_{n+1}([M]) = \frac{1}{2}\delta(J)$:

Thm 3 (N.-Sato-Suzuki) For $n \geq 1$, $g \geq 0$,

$$\begin{tikzcd}
\mathcal{A}^c_n \arrow[r, "\mathfrak{s}"] \arrow[d, "``\delta\text{''}"] \arrow[dr, phantom, "\circlearrowleft"] & Y_n\mathcal{C}_{g,1}/Y_{n+1} \arrow[r, two heads] \arrow[d, "\bar{z}_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & Y_n\mathcal{C}_g/Y_{n+1} \arrow[d, "``\hat{z}_{n+1}\text{''}"] \\
\mathcal{A}^c_{n+1} \otimes \mathbb{Z}/2 \arrow[r, "\mathrm{id} \otimes \frac{1}{2}"'] & \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, two heads] & (\mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z})/\sim
\end{tikzcd}$$

combinatorial operation	our invariant in the case of $\Sigma_{g,1}$	our invariant in the case of Σ_g
easy to compute	difficult to compute from the definition	

Thm 3 $\rightsquigarrow$ Thm 1 & 2 in §4

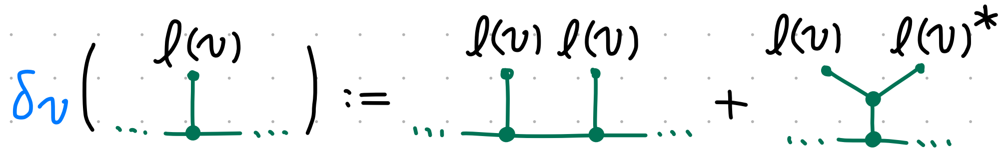
§4. The map δ and Proof of theorems

Def & Prop $U(J) = \{\text{univalent vertices of } J\}$

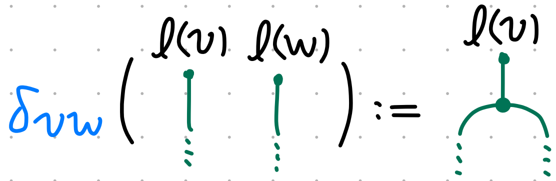
$\quad\quad\quad\quad\quad v \in U(J) \qquad \ell(v) \in \{1^+, \cdots, g^+, 1^-, \cdots, g^-\}$

The map $\delta : \mathcal{A}_n^c \longrightarrow \mathcal{A}_{n+1}^c \otimes \mathbb{Z}/2$ is well-defined.

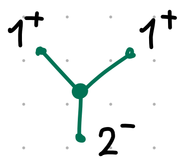
$$J \longmapsto \sum_{v \in U(J)} \delta_v(J) + \sum_{\substack{v \neq w \in U(J) \\ \ell(v) = \ell(w)}} \delta_{vw}(J)$$



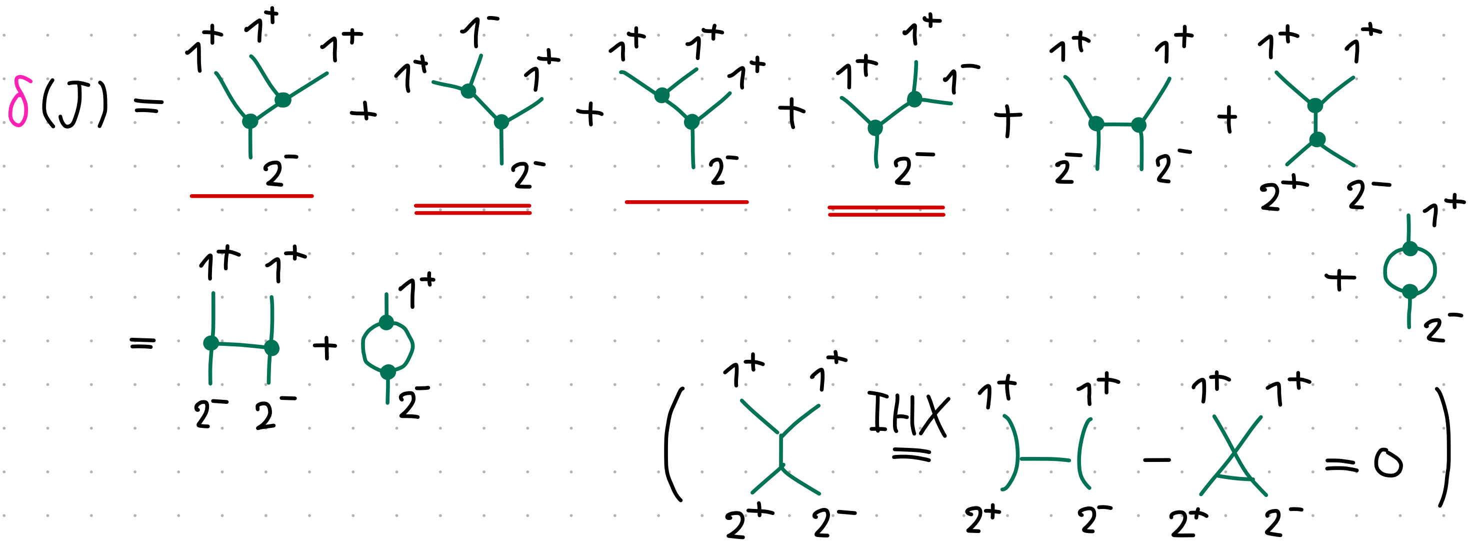
$(j^\pm)^* := j^\mp$



Ex $(n=1)$ $J =$



$(J \neq 0,\ 2J = 0)$



Note We gave some examples of $\bar{Z}_{n+1}$ by the computer and found the map δ.

Thm 3 (N.-Sato-Suzuki)

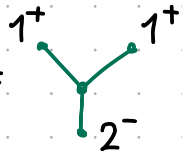
$$\begin{tikzcd}
\mathcal{A}_n^c \arrow[r, "\mathfrak{s}"] \arrow[d, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & Y_n\mathcal{C}_{g,1}/Y_{n+1} \arrow[r, two heads] \arrow[d, "\bar{z}_{n+1}"] \arrow[dr, phantom, "\circlearrowleft"] & Y_n\mathcal{C}_g/Y_{n+1} \arrow[d, "\hat{z}_{n+1}"] \\
\mathcal{A}_{n+1}^c \otimes \mathbb{Z}/2 \arrow[r, "\mathrm{id}\otimes\frac{1}{2}"'] & \mathcal{A}_{n+1}^c \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, two heads] & (\mathcal{A}_{n+1}^c \otimes \mathbb{Q}/\mathbb{Z})/\sim
\end{tikzcd}$$

Proof of $\circlearrowleft$ on the left

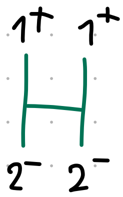
Step 1: Decompose $\mathfrak{s}(J)$ into elementary pieces.

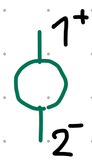
Step 2: Compute $\tilde{Z}_{\leq 2}$ for these pieces.

Step 3: Compute $\tilde{Z}_{n+1}(\mathfrak{s}(J))$ by the functoriality of $\tilde{Z}$.
Then $\frac{1}{2}\delta(J)$ appears. $\square$

Ex $(n=1)$ $J =$ 

$\left(2\mathfrak{s}(J) = 0,\ \mathfrak{s}(J) \overset{?}{\neq} 0\right)$

$\bar{z}_2(\mathfrak{s}(J)) = \frac{1}{2}\delta(J) = \frac{1}{2}$ 

$+ \frac{1}{2}$ 

$\neq 0$

In particular, $\mathfrak{s}(J) \neq 0$.

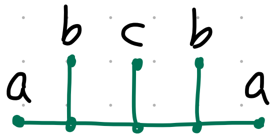
Thm (Morita-Sakasai-Suzuki '20) For $g \geq 6$

$H_1(\mathcal{K}_g; \mathbb{Q}) \cong$ "Casson inv." $\oplus$ "Morita's refinement $\tilde{\tau}_2$ of Johnson hom."

Thm 2 (NSS) $\mathrm{tor}\, H_1(\mathcal{K}_g;\mathbb{Z}) \neq 0$ for $g \geq 6$

Proof Step 1: Check that ζ is a homomorphism.

$$\begin{tikzcd}
\mathcal{K}_g \arrow[rrr, "(\log\tilde{Z}\circ c(-))_4 =: \zeta"] & & & (\mathcal{A}_4^c\otimes\mathbb{Q}/\mathbb{Z})/\sim \\
\cup & & & \\
\mathcal{I}_g(2) \arrow[r, two heads] & \mathcal{I}_g(2)/\mathcal{I}_g(4) \arrow[r, "c"] & Y_2\mathcal{C}_g/Y_4 \arrow[ddr, dashed] \arrow[ur, phantom, "\circlearrowleft"] & \\
\rotatebox{90}{$\in$} & & \cup & \\
f \arrow[ddrr, mapsto, bend right] & & Y_3\mathcal{C}_g/Y_4 \arrow[r, "\hat{Z}_4"'] \arrow[ur, phantom, "\circlearrowleft"] & (\mathcal{A}_4^c\otimes\mathbb{Q}/\mathbb{Z})/\sim \arrow[uuuu, two heads] \\
& & \rotatebox{90}{$\in$} & \\
& & S(J) &
\end{tikzcd}$$

Step 2: For $J =$  , find $f \in \mathcal{I}_g(2)$ s.t. $f \mapsto S(J)$

Step 3: Show $\zeta(f) \neq 0$ by Thm 3 $(\therefore [f] \neq 0 \in H_1(\mathcal{K}_g))$

Step 4: Prove that Casson inv. $\oplus\, \tilde{\tau}_2$ sends f to 0

(Key: $2J = 0$)

Therefore, [MSS'20] implies $[f] \neq 0 \in \mathrm{tor}\, H_1(\mathcal{K}_g)$ ☒

Similarly, we can prove the following:

Thm 1 (NSS)

$\mathrm{tor}\left(\mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1)\right) \neq 0$ for $n = 3, 5$ & $g \gg 1$

Future perspective

We would like to

- determine the structures of $\mathcal{I}_{g,1}(n)/\mathcal{I}_{g,1}(n+1)$ and $H_1(\mathcal{K}_g;\mathbb{Z})$.

- clarify $\begin{cases} \text{the topological/geometric meaning of } \bar{z}_{n+1}. \\ \text{the relations to other invariants.} \end{cases}$

$\bar{z}_{n+1,0} \leftrightarrow$ higher Sato-Levine invariant [NSS]

$\bar{z}_{n+1,1}$ [NSS 2]

- study "$\bar{z}_{n+2}$" which is defined and partially computed
  ($Y_n\mathcal{C}_{g,1}/Y_{n+2}$ is also an abelian group for $n \geq 2$)

- determine $Y_n\mathcal{C}_{g,1}/Y_{n+1}$ and solve Goussarov-Habiro Conjecture

  $n=1,2$ Massuyeau-Meilhan '03 '13

  $n=3$ [NSS] $\quad$ $n=4$ [NSS 2]

Goussarov-Habiro Conjecture

<u>Conj</u>

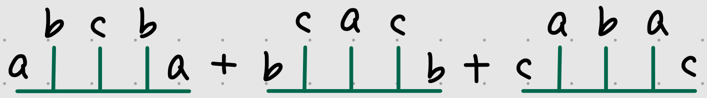
$M \underset{Y_{n+1}}{\sim} M' \iff f(M)=f(M')$ for $\forall f$ "finite type inv. of deg $\leq n$"

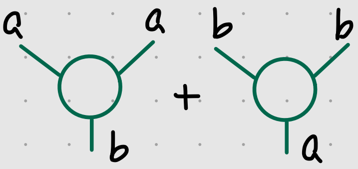
<u>Ex</u> $\tilde{Z}^Y_n$ and $\bar{z}_{n+1}$ are finite type inv's of deg $\leq n$

$\underline{\text{Key}}$ $0 \to \operatorname{Ker} s \to \mathcal{A}_n^c \xrightarrow{s} Y_n\mathcal{C}/Y_{n+1} \to 0 \quad (n \geq 2)$

Thm 4 (NSS)

$$0 \to (\overset{3}{\wedge} H \oplus \overset{2}{\wedge} H) \otimes \mathbb{Z}/2 \to \mathcal{A}_3^c \xrightarrow{s} Y_3\mathcal{C}_{g,1}/Y_4 \to 0 \quad \text{(exact)}$$

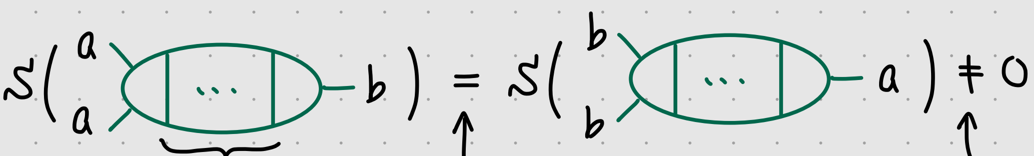
$a \wedge b \wedge c \longmapsto$  ①

$a \wedge b \longmapsto$  ②

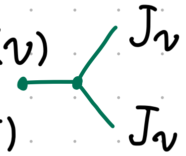
Cor (NSS & 2) Conj is true for $n = 3, 4$

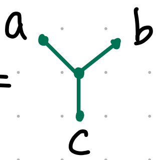
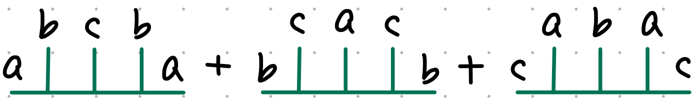
Recently, we generalized ② to

Thm 5 (NSS 2)



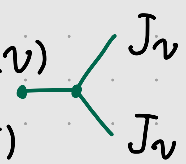
clasper calculus $\qquad$ $\bar{z}_{2\ell+2}$ & weight system

Also, ① is generalized as $\sum_{v \in U(J)} \ell(v)$  for $J \in \mathcal{A}_{n,0}^c$

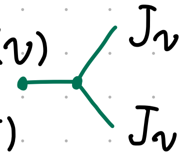
$n=1$ $J =$  $\rightsquigarrow$ 

$n=2$ OK $\qquad n \geq 3$??

Thm 6 (NSS)

$$\sum_{v \in U(J)} \ell(v)$$  $\in \operatorname{Ker}\left(\mathcal{A}_{2n+1,0}^c \xrightarrow{s} Y_{2n+1}\mathcal{C}_{g,1}/Y_{2n+2} \xrightarrow{\bar{z}_{2n+2}} \mathcal{A}_{2n+2}^c \otimes \mathbb{Q}/\mathbb{Z}\right)$

cf. Conant - Schneiderman - Teichner '16

$\sum_{v\in U(J)} \ell(v)$ 

$\in \mathrm{Ker}\left(\mathcal{A}^c_{2n+1,0} \xrightarrow{\mathfrak{s}} Y_{2n+1}\mathcal{C}_{g,1}/Y_{2n+2} \twoheadrightarrow Y_{2n+1}\text{``}\mathcal{H}_{g,1}\text{''}/Y_{2n+2}\right)$

◎ $\mathcal{H}_{g,1} = \mathcal{C}_{g,1}/\sim_H$ the homology cobordism group of homology cylinders

$(M,m)\sim_H(N,n) \overset{\text{def}}{\Longleftrightarrow} \exists W^4$: cpt ori smooth s.t.

$\partial W = M \underset{m\circ n^{-1}}{\cup} (-N)$ & $H_*(M) \xrightarrow{\cong} H_*(W) \xleftarrow{\cong} H_*(N)$

$m : \partial(\Sigma_{g,1}\times[-1,1]) \xrightarrow{\cong} \partial M$

<u>Fact</u>

- $\mathcal{H}_{0,1} \cong \Theta^3$ the homology cobordism group of $\mathbb{Z}HS^3$'s
- $\mathcal{I}_{g,1} \hookrightarrow \mathcal{C}_{g,1} \twoheadrightarrow \mathcal{H}_{g,1}$ is injective. Garoufalidis - Levine '01

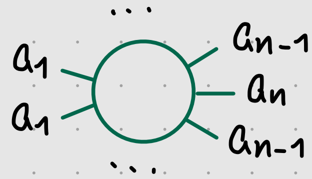
$$\begin{array}{ccc} \mathcal{C}_{g,1} & \twoheadrightarrow & \mathcal{H}_{g,1} \\ \cup & & \cup \\ Y_n\mathcal{C}_{g,1} & \twoheadrightarrow & Y_n\mathcal{H}_{g,1} \end{array}$$

- Johnson hom & Birman - Craggs hom factor through $Y_n\mathcal{H}_{g,1}$

<u>Prop</u> (NSS) $\bar{z}_{n+1}$ ($n>1$ odd) does NOT factor through $Y_n\mathcal{H}_{g,1}$

Prop (NSS) · $\mathcal{A}^c_{2n,1}$ is free abelian

· $\mathrm{tor}\,\mathcal{A}^c_{2n-1,1} = \mathcal{A}^{c,\mathrm{sym}}_{2n-1,1} \cong H_1(\Sigma_{g,1})^{\otimes n} \otimes \mathbb{Z}/2$



Thm (NSS) $\bar{z}_{2n,\ell}|_{\mathrm{tor}\cap \mathrm{Im}\, c}$ is non-trivial for $\ell = 0,1$ $(g \geq n+2)$.

$$\mathcal{I}_{g,1}(2n-1)/\mathcal{I}_{g,1}(2n) \xrightarrow{c} \underset{\underset{\mathrm{tor}}{\cup}}{Y_{2n-1}\mathcal{C}_{g,1}/Y_{2n}} \xrightarrow{\bar{z}_{2n,\ell}} \mathcal{A}^c_{2n,\ell} \otimes \mathbb{Q}/\mathbb{Z}$$

Prob Is $\mathcal{I}_{g,1}(k)/\mathcal{I}_{g,1}(k+1) \xrightarrow{\mathrm{incl}_*} J_k\mathcal{M}_{g,1}/J_{k+1}\mathcal{M}_{g,1}$ injective?

$(k \neq 2)$

$$J_k\mathcal{M}_{g,1} := \mathrm{Ker}\left(\mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}\left(\pi_1\Sigma_{g,1}/\pi_1\Sigma_{g,1}(k+1)\right)\right)$$

Prob $\otimes\mathbb{Q}$ was proposed by Morita '99

Habiro-Massuyeau '09

Cor For $k = 2n-1$ $(g \geq n+2)$, incl_* is NOT injective.

Rem $n=1$ is due to Johnson '85 $(\mathcal{I}^{ab}_{g,1} \longrightarrow \mathcal{I}_{g,1}/\mathcal{K}_{g,1})$