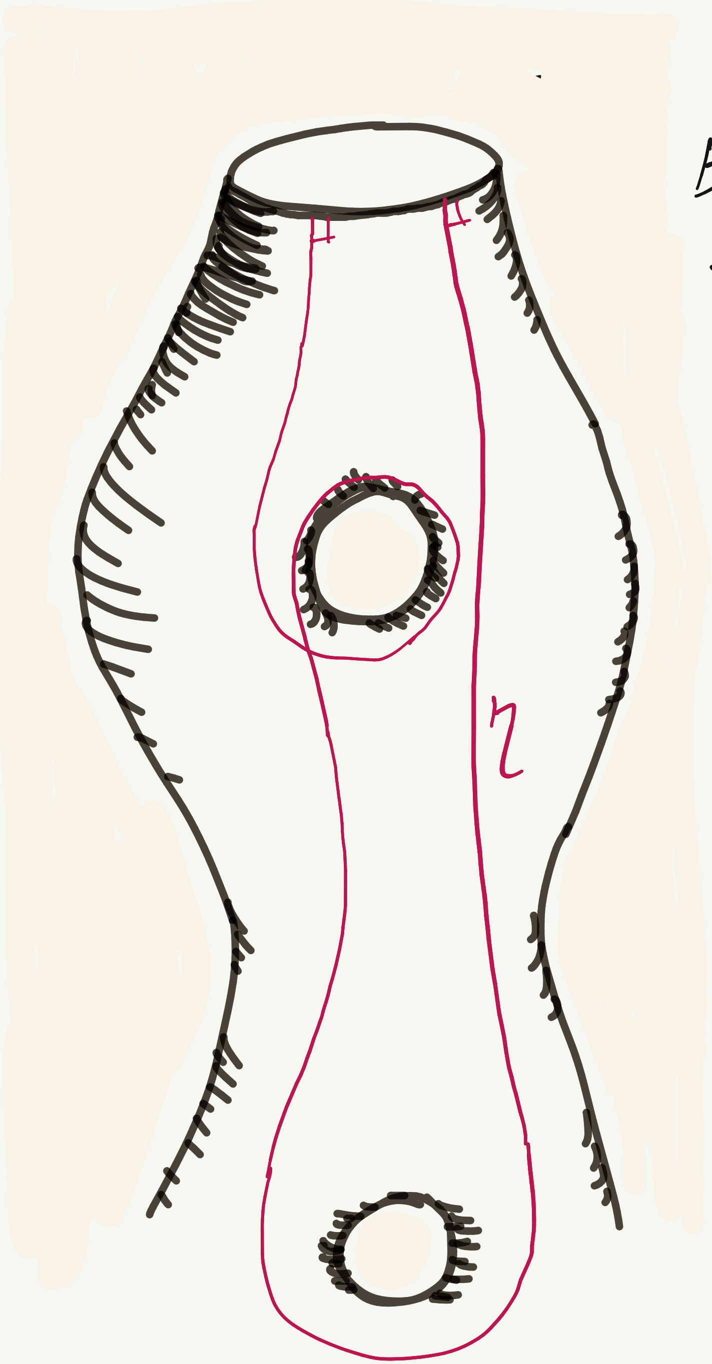
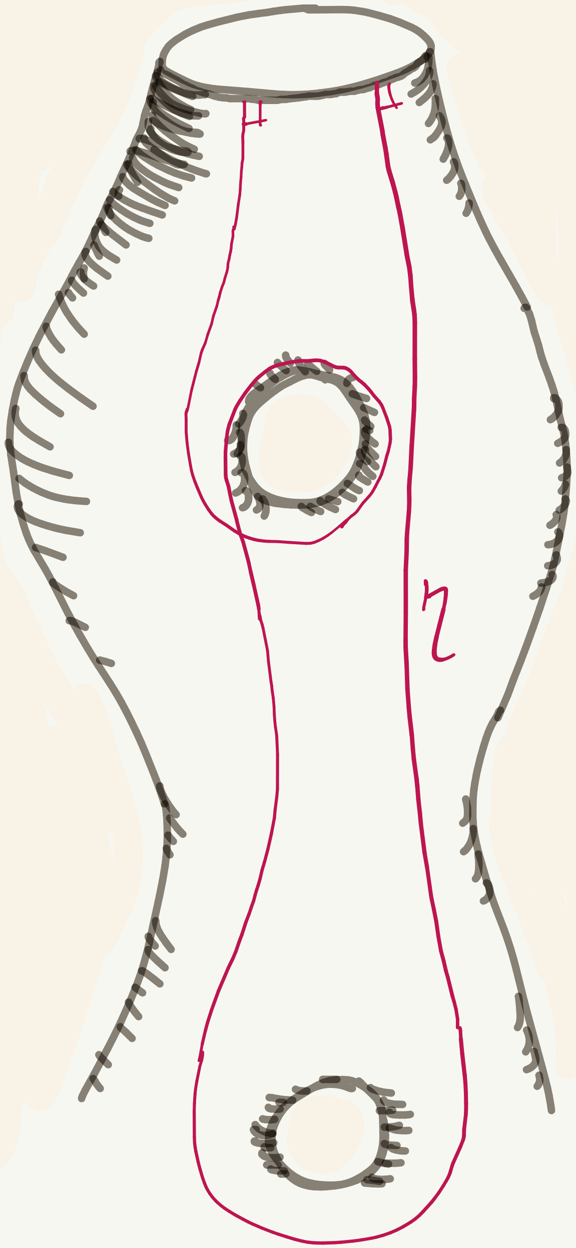
Where orthogeodesics roam

X hyperbolic surface with adjectives



Basmajian (1993)

$$\sum_{\eta} \log\left(\coth^4\left(\ell(\eta)/2\right)\right) = \ell(\beta)$$



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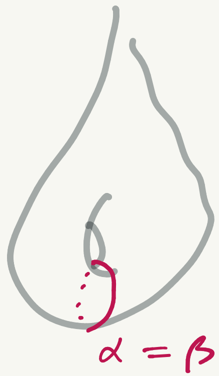
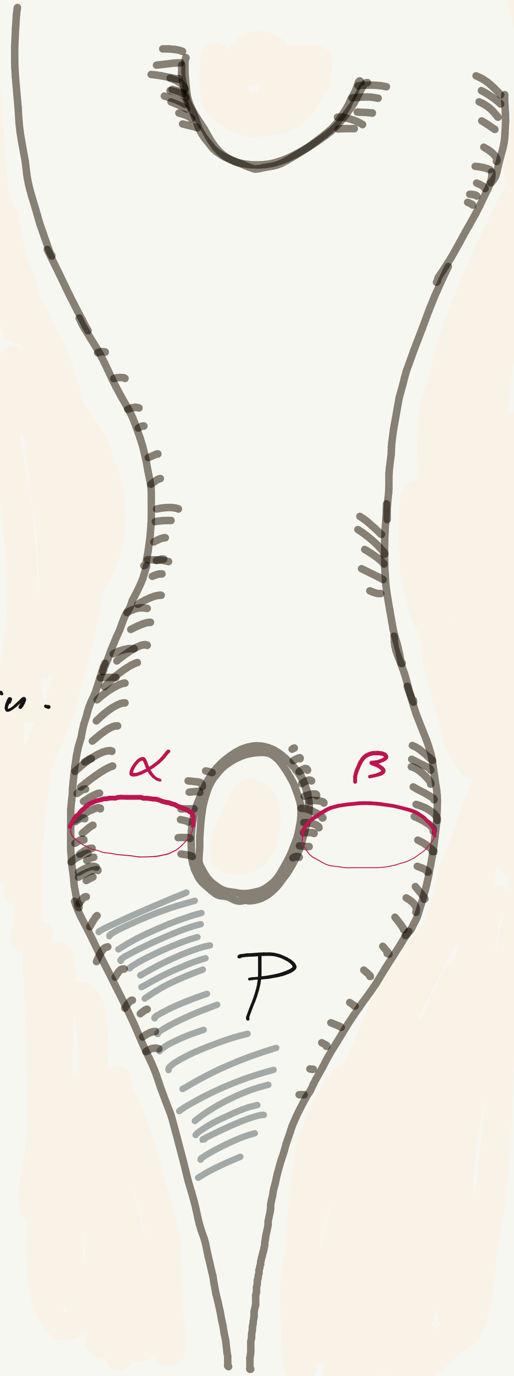


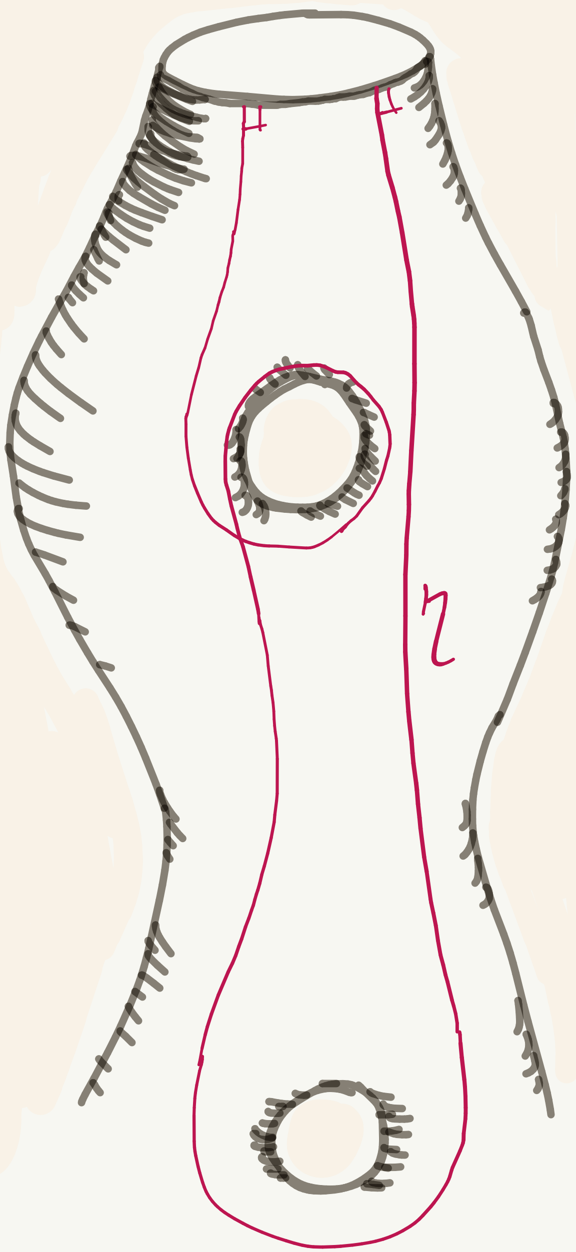
$$\sum_{\alpha} \frac{2}{e^{\ell(\alpha)}+1} = 1$$

↖ simple closed geodesics.

McShane (1991, 1998)

$$\sum_{P} \frac{2}{e^{\frac{\ell(\alpha)+\ell(\beta)}{2}}+1} = 1$$





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Generalized in different contexts (He, Fanoni-Pozzetti, ...)

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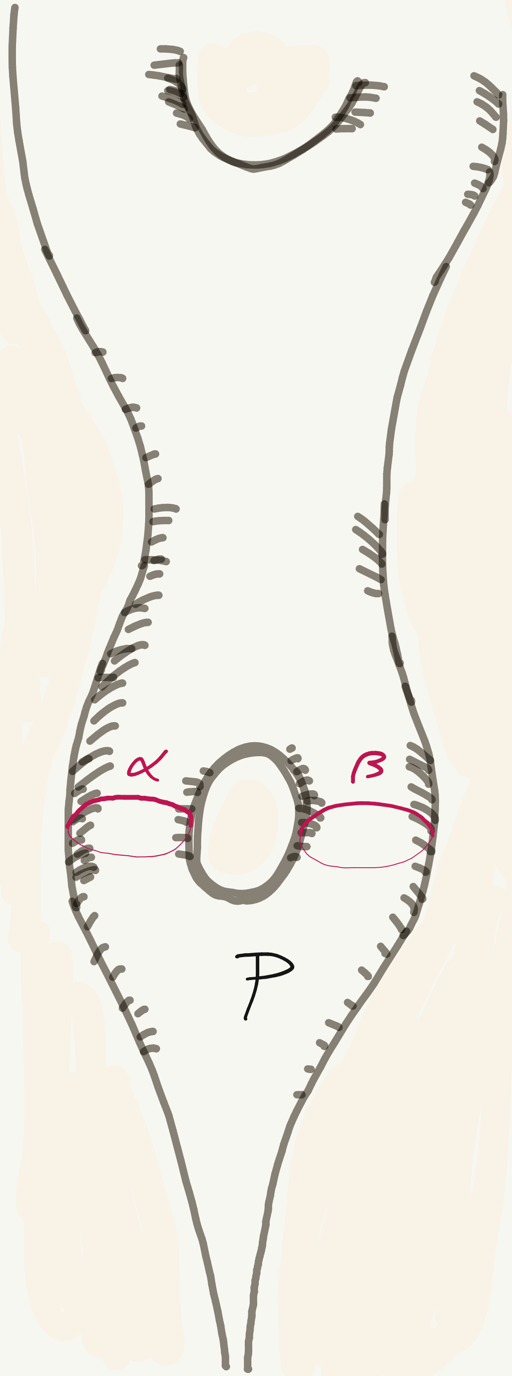

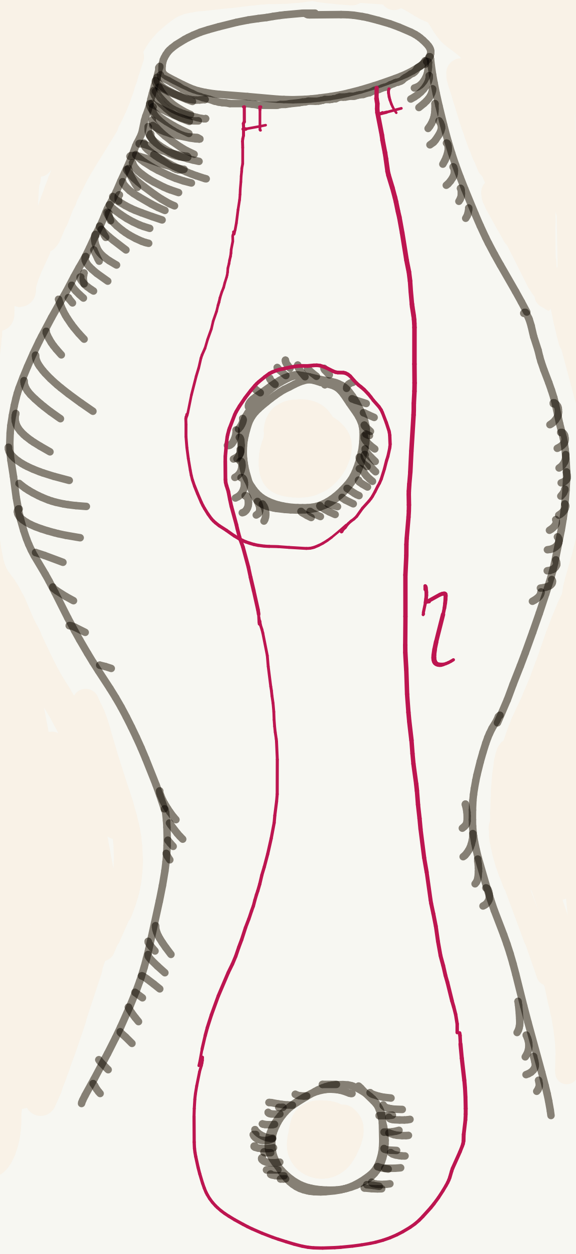

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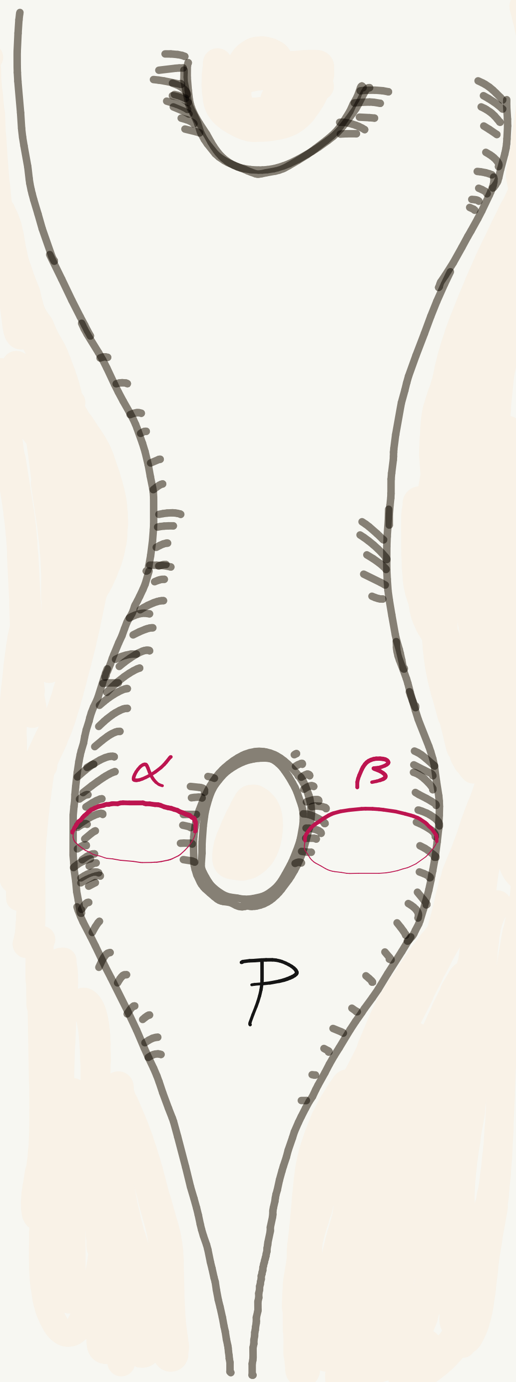
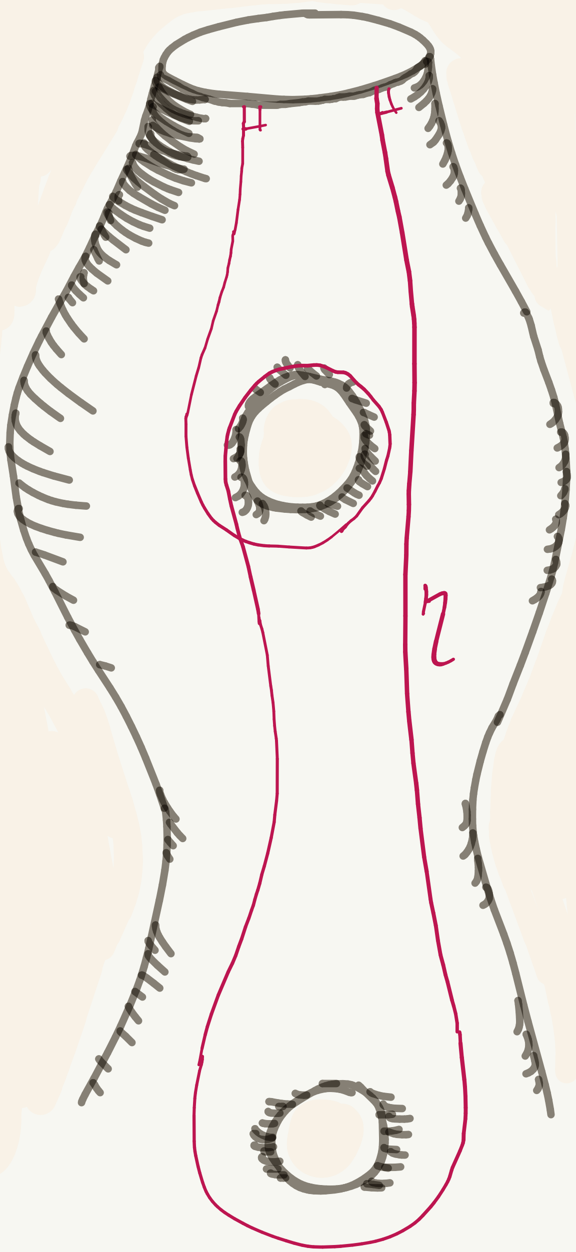
Generalized by Mirzakhani,
Tan-Wong-Zhang ...



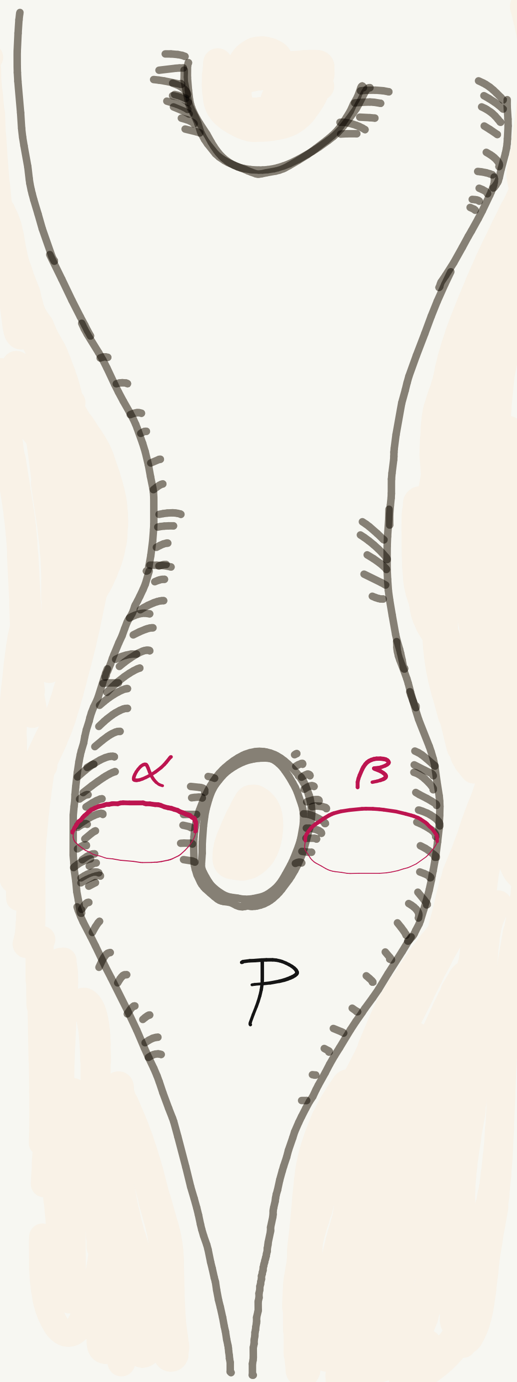


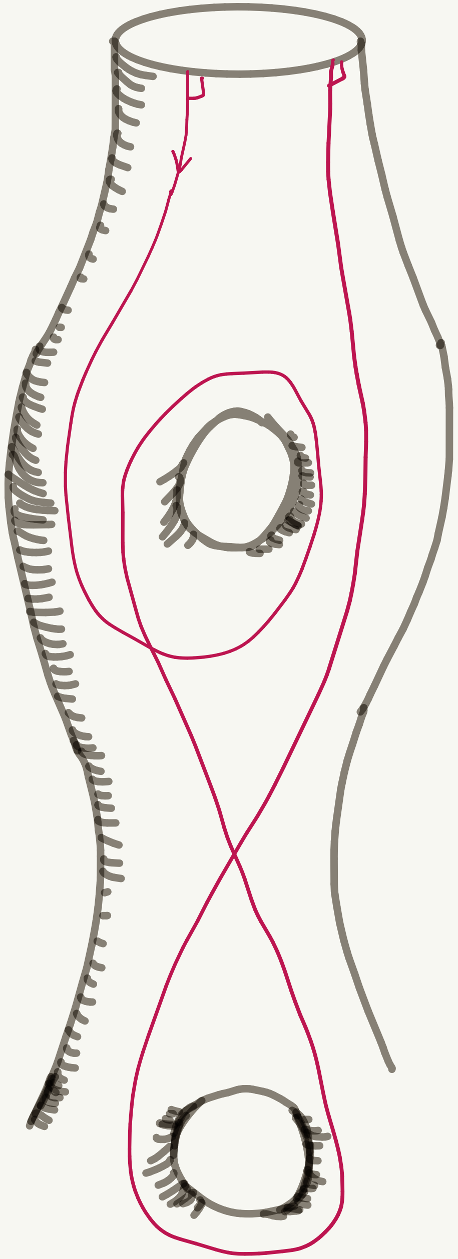
Basmajian Identity

Works for surfaces with at least one boundary geodesic

McShane Identity

Works for surfaces with cusps, boundary geodesics + cone angles...



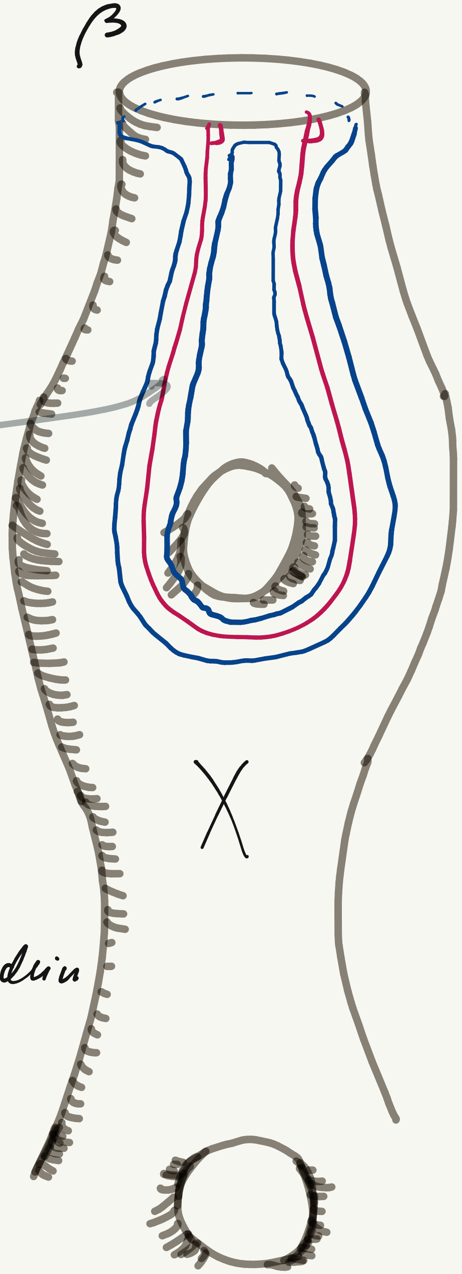


<u>Basmajian</u>

<u>all</u> orthogeodesics

$$\sum_{\eta} (\qquad) = \ell(\beta)$$

embedded pair of pants.

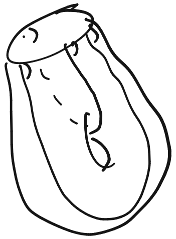


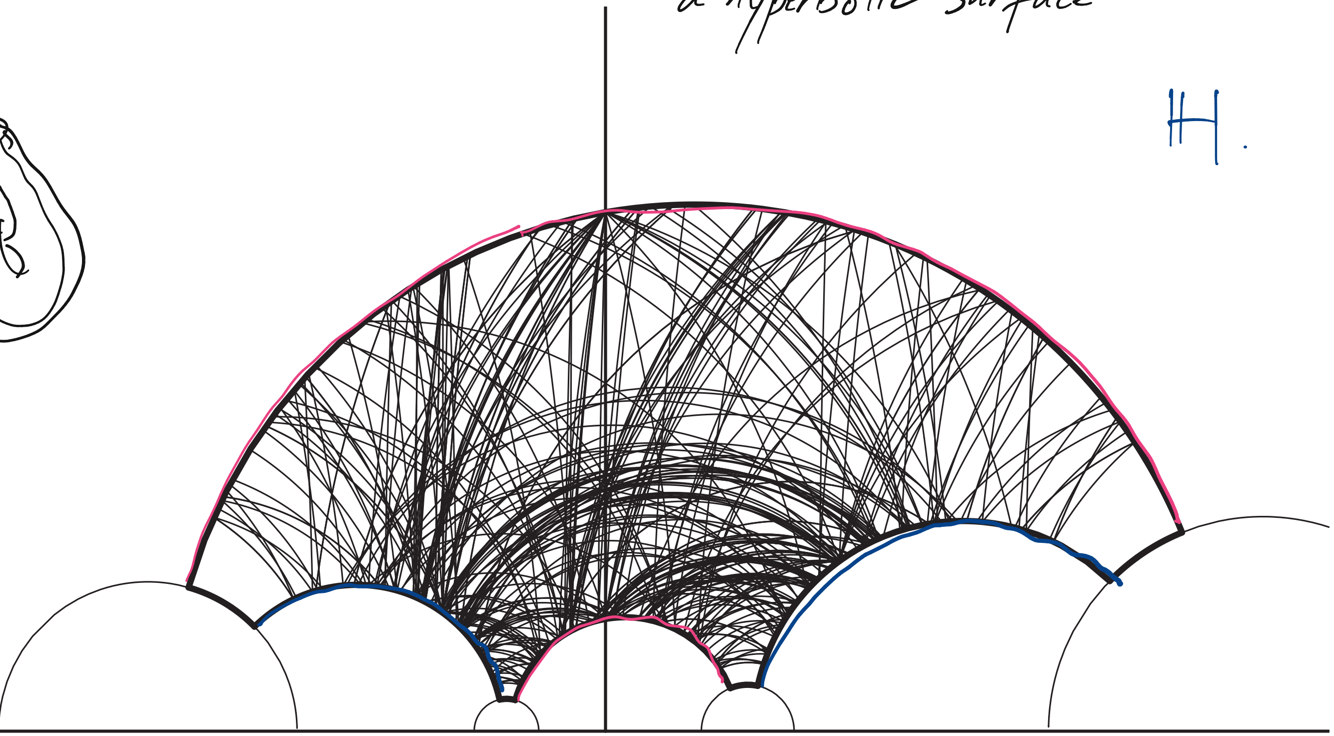
$$\sum_{\substack{\text{simple ortho.} \\ \eta}} f(X, \ell(\eta)) = \ell(\beta)$$

<u>McShane</u>

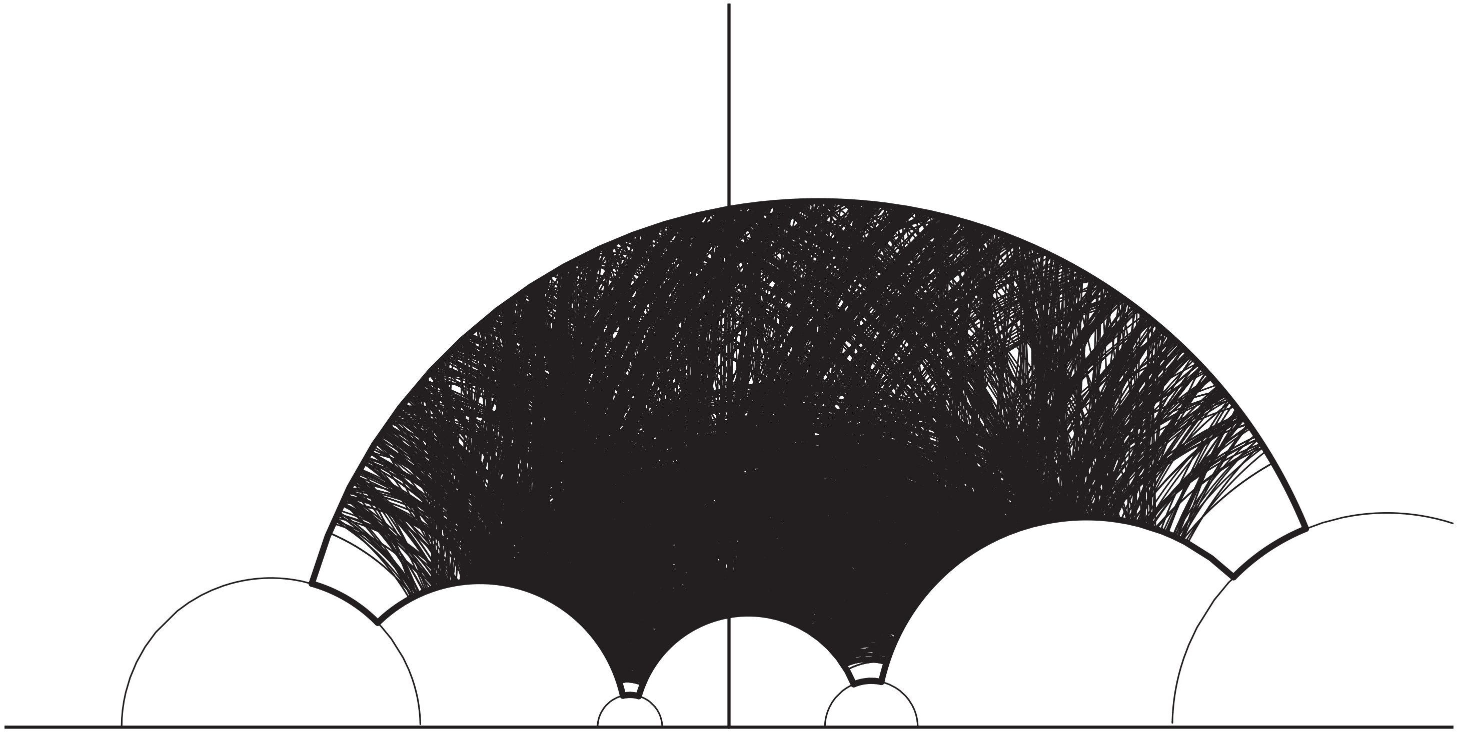
<u>simple</u> orthogeodesics

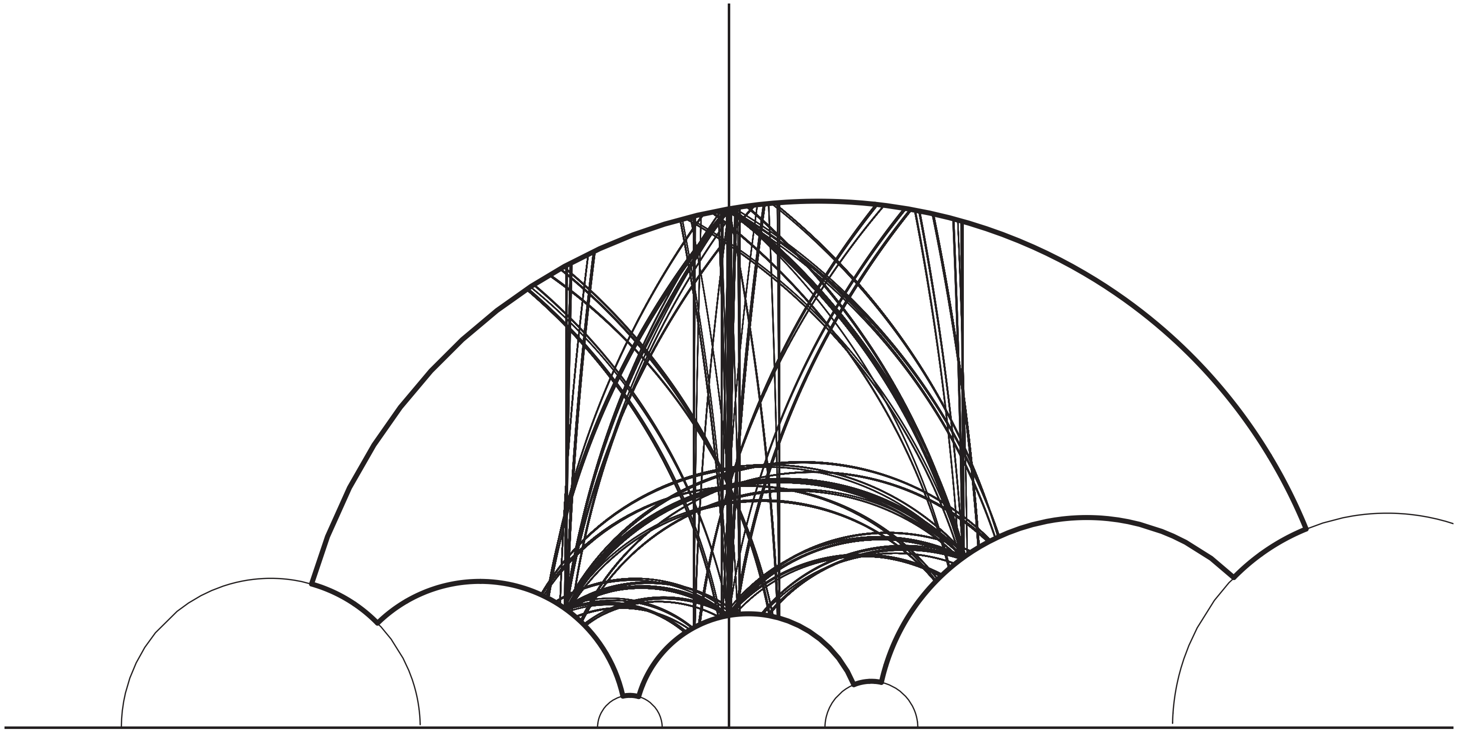
Birman - Series: the set of simple closed geodesics is nowhere dense on a hyperbolic surface

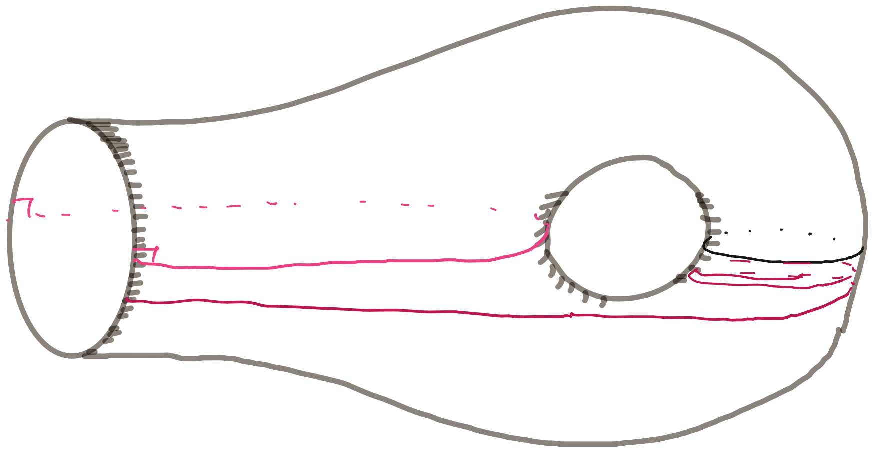


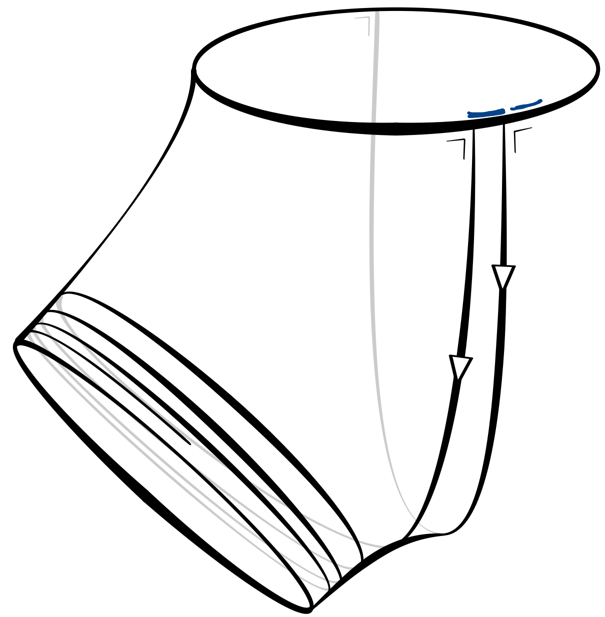


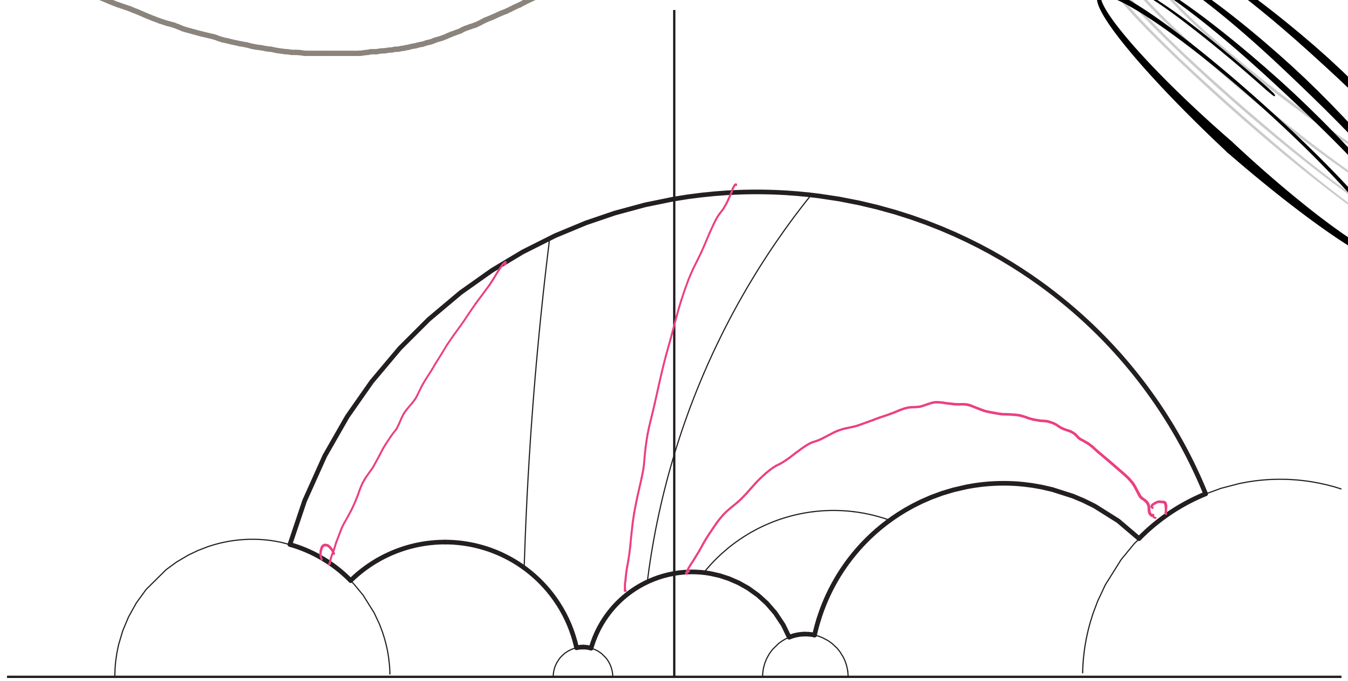
illustrating geodesics, Peter Buser

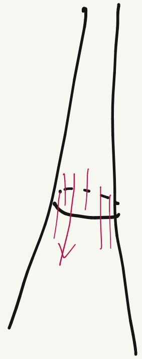




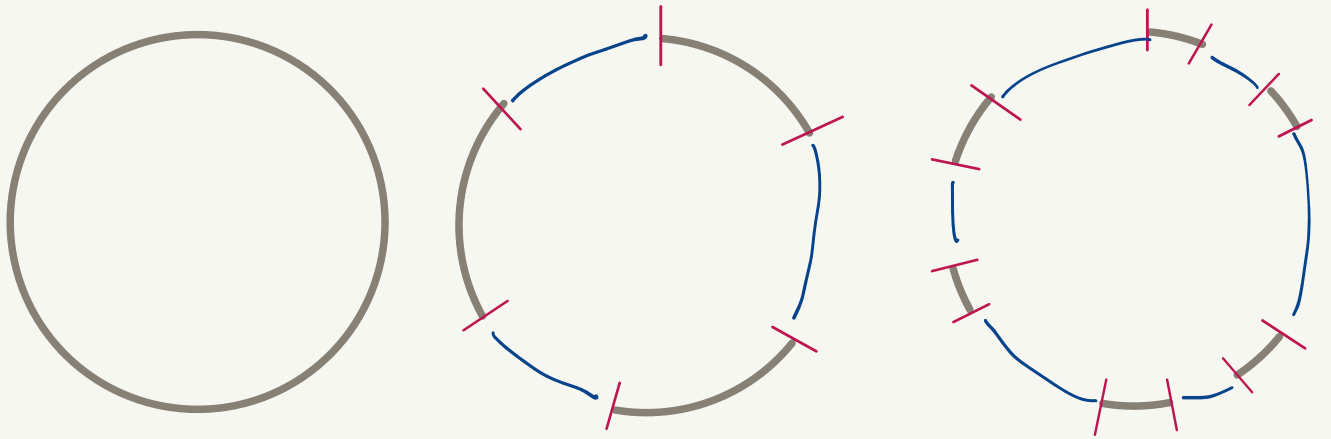






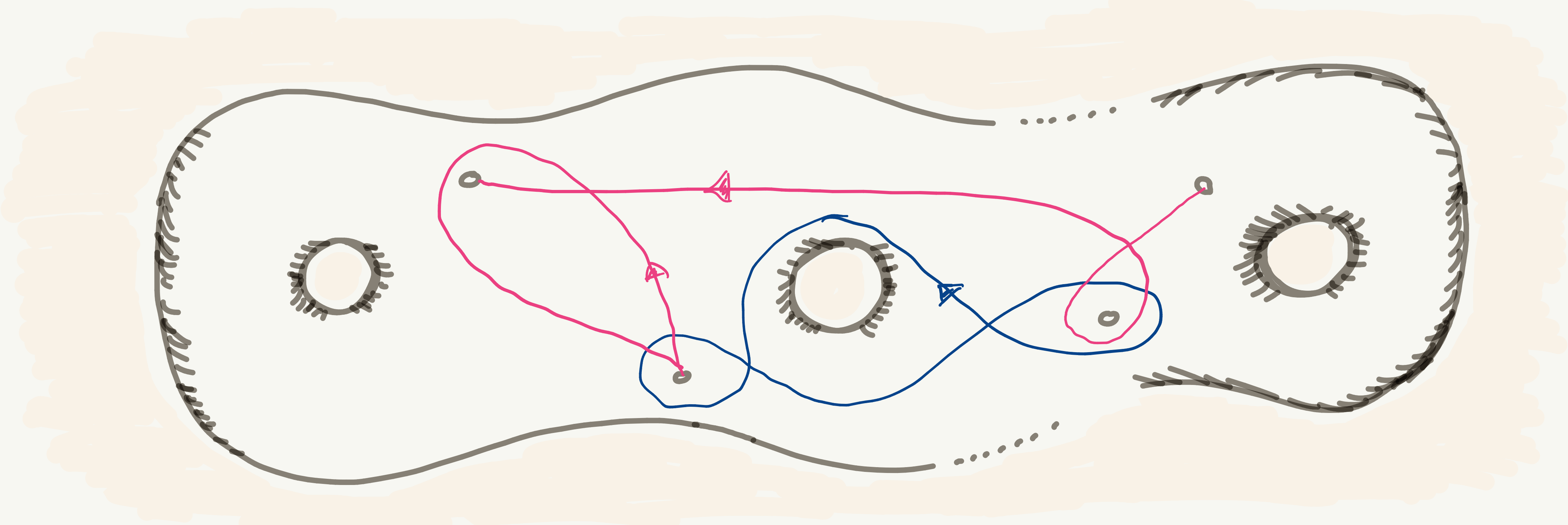


THE FRACTAL NATURE OF SIMPLE ORTHORAYS



The same picture.

CURVES AND ARCS

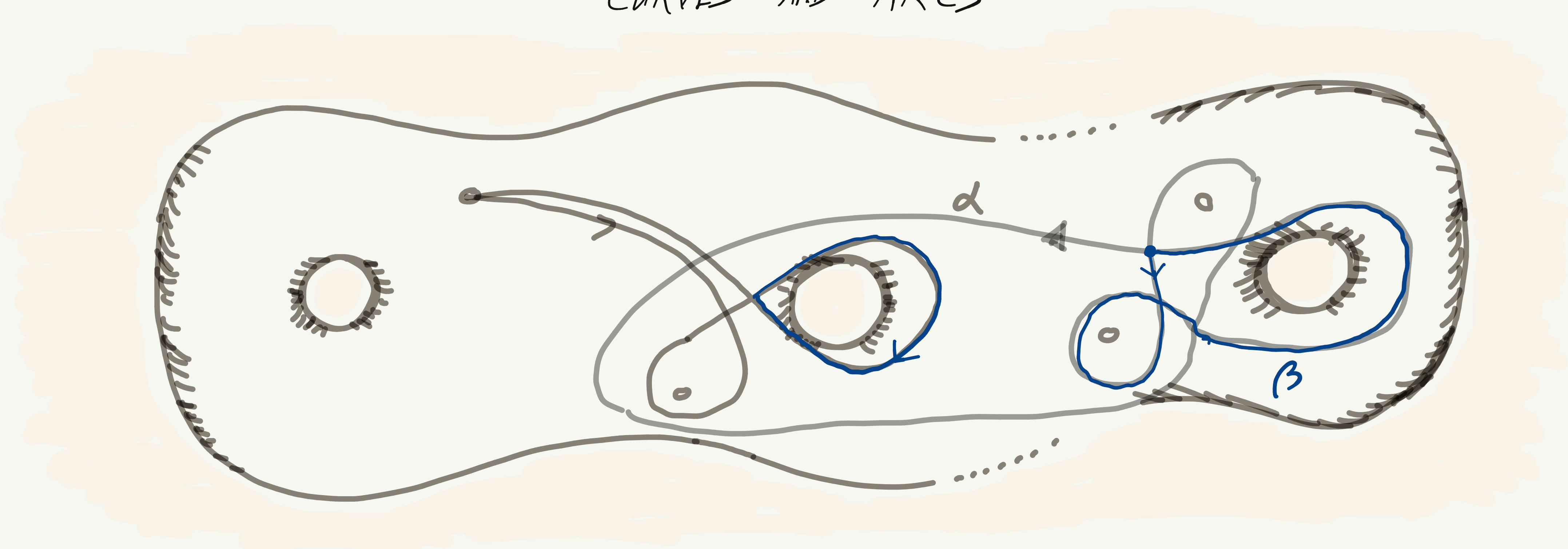


Σ topological surface of genus g w/ n marked points.

$\mathcal{C}(\Sigma)$ = closed curves on Σ / $\sim$ free homotopy [oriented ...]

$\mathcal{A}(\Sigma)$ = arcs with endpoints in the marked point / $\sim$ free homotopy

CURVES AND ARCS



Let $\alpha, \beta \in \mathcal{C}(\Sigma)$: α supports β if β is a proper subloop of α $(\beta \hookrightarrow \alpha)$

If $\eta \in \mathcal{A}(\Sigma)$, $\beta \in \mathcal{C}(\Sigma)$: η supports β if β is a proper subloop of α.

Observation:

- Simple curves and arcs are unsupportive.
- If α is not prime $\left(\alpha = (\alpha')^k \quad k \geq 2 \right)$ then $\alpha' \hookrightarrow \alpha$.
- If $i(\alpha, \alpha) = m > 0$ then it supports at most $2m$ other curves.

COHERENT MARKINGS

A marking M is a subset of $\mathcal{C}(\Sigma)$.

A coherent marking is a marking M with:
if $\alpha \in M$ and $\beta \hookrightarrow \alpha$ then $\beta \notin M$.

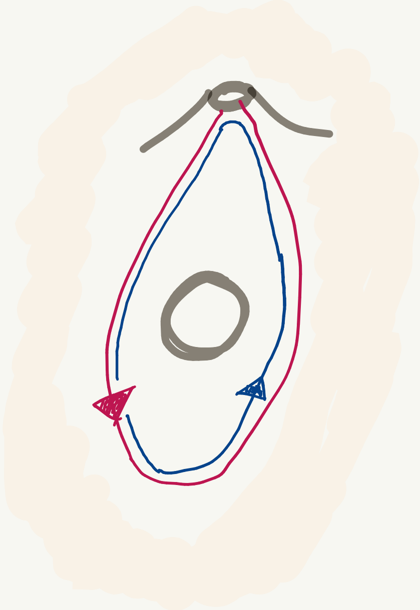
Properties : . If M is coherent, and $M' \subset M$ then M' is coherent.

. There exist uncountably many coherent markings.

Examples : $M = \emptyset$; $M = \{ \alpha \mid$ (primitive) simple closed curves $\}$; $M = \{\alpha\}$

$|M| = +\infty$

BEING PERIPHERAL

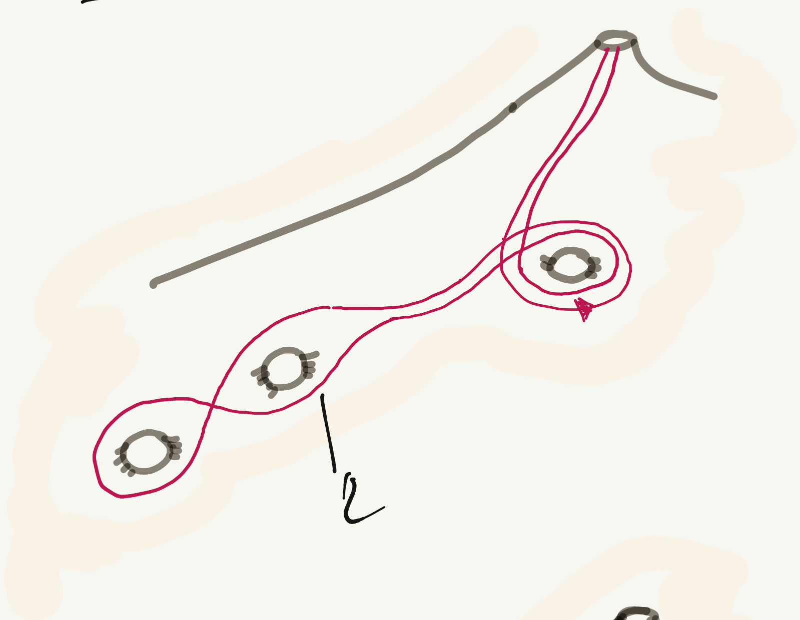


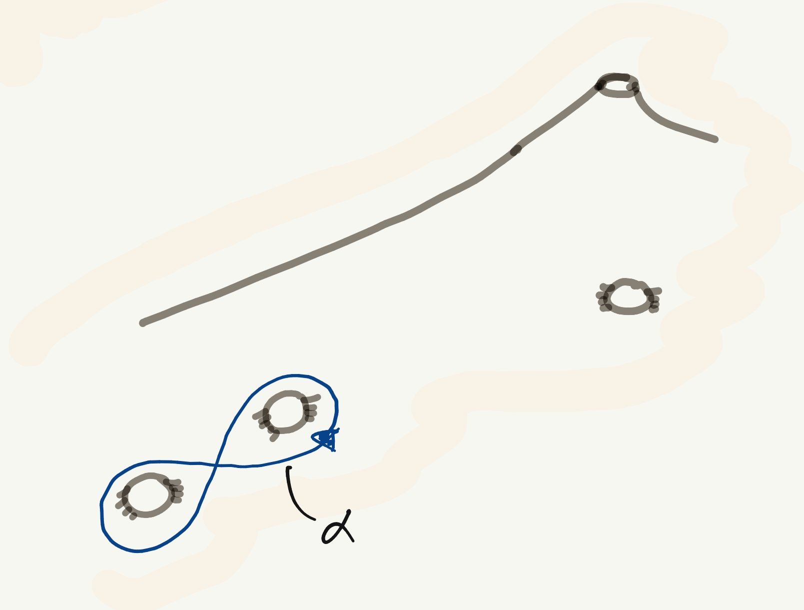
Fix coherent marking M.

Look at unsupportive arcs.

$\eta \in \mathcal{A}(\Sigma)$: $\partial \eta = \{\alpha_1, \alpha_2\}$

If $\alpha \in \partial \eta$
and $\alpha \in M$
then η is
peripheral to α.



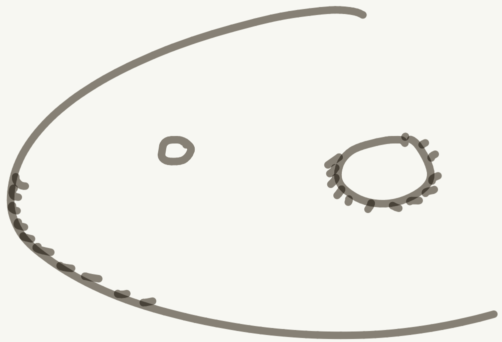


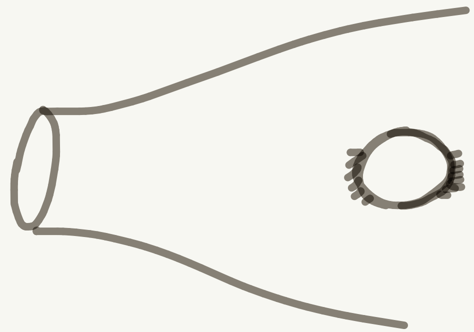
ARCS AND ORTHOGEODESICS

$$\mathcal{C}(\Sigma) \xleftrightarrow{1:1} \mathcal{G}(X) \quad \text{(oriented) closed geodesics}$$

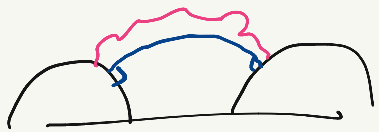
$$\mathcal{A}(\Sigma) \xleftrightarrow{1:1} \mathcal{O}(X) \quad \text{orthogeodesics. (oriented).}$$

Σ topological surface $\cong$ X hyperbolic surface





$\mathbb{H}$



ARCS AND ORTHOGEODESICS

M marking

$$\mathcal{C}(\Sigma) \xleftrightarrow{1:1} \mathcal{G}(X) \quad \text{closed geodesics}$$

$$\mathcal{A}(\Sigma) \xleftrightarrow{1:1} \mathcal{O}(X) \quad \text{orthogeodesics (oriented).}$$

$$\mathcal{O}_M = \mathcal{O}_M(X) \subset \mathcal{O}(X) \quad \text{unsupportive orthogeodesics}$$

$$\mathcal{O}_M^\alpha \subset \mathcal{O}_M \quad \text{peripheral to } \alpha$$

$\alpha \in M$.

<u>Theorem</u> (P.) Let X be a finite type hyperbolic surface with ∂X containing at least one simple closed geodesic. The lengths of geodesics and orthogeodesics of X satisfy

$$\ell(\partial X) = \sum_{\eta \in \mathcal{O}_M} \log\left(\coth^2(\eta/2)\right)$$ ← Basmajian terms.

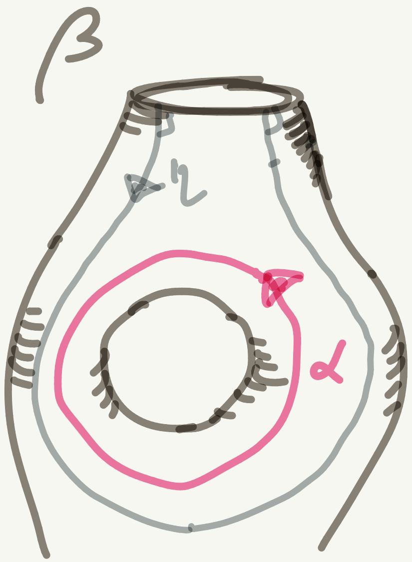
$$+ \sum_{\alpha \in M} \left(\sum_{\eta \in \mathcal{O}_M^\alpha} \log\left(\frac{\cosh(\alpha/2) + \sqrt{\cosh^2(\alpha/2) + \cosh^2(\eta/2) - 1}}{\sinh(\alpha/2) + \sqrt{\cosh^2(\alpha/2) + \cosh^2(\eta/2) - 1}} \right) \right)$$

McShane term.

$$\ell(\beta) = \sum_{\eta \in \mathcal{O}\beta_M} \log\left(\coth^2(\eta/2)\right)$$

$$+$$

$$\sum_{\alpha \in M} \sum_{\eta \in \mathcal{O}\beta_M^\alpha} \log\left(\frac{\cosh(\alpha/2) + \sqrt{\cosh^2(\alpha/2) + \cosh^2(\eta/2) - 1}}{\sinh(\alpha/2) + \sqrt{\cosh^2(\alpha/2) + \cosh^2(\eta/2) - 1}}\right)$$



$\mathcal{O}\beta_M \subset \mathcal{O}_M$ set of those that leave from β

(HALF) TRACE FORMULATION

$$b+\sqrt{b^2-1} = \left(\prod_{\eta}\frac{t^2}{t^2-1}\right)\prod_{\alpha\in M}\left(\prod_{\substack{\eta\\ \partial\eta\ni\alpha}}\left(\frac{a+\sqrt{a^2+t^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+t^2-1}}\right)\right)$$

where $b = \cosh(\beta/2)$, $a = \cosh(\alpha/2)$ and

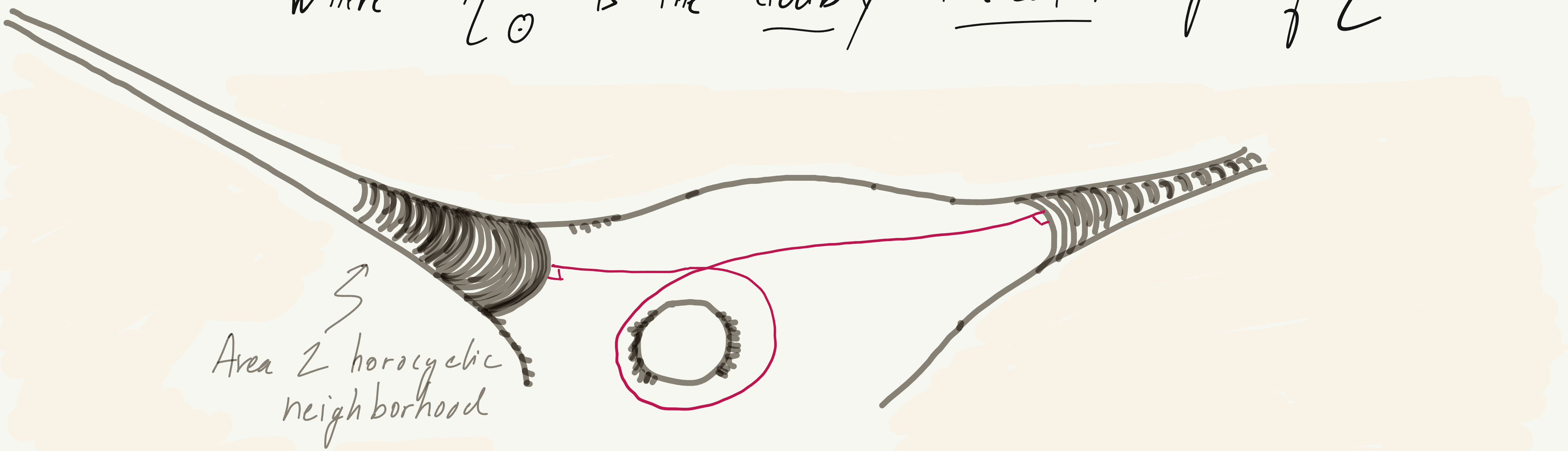
$t = \cosh(\eta/2)$

Theorem (cusps)

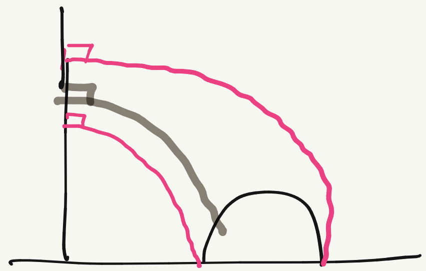
Let X be a finite type hyperbolic surface with $n > 0$ cusps as boundary and M a coherent marking. Then

$$n = \frac{1}{2} \sum_{\alpha \in M} e^{-\alpha/2} \sum_{\eta \in \mathcal{O}_M^\alpha} e^{-\eta_\odot/2}$$

where $\eta_\odot$ is the doubly truncated length of η.



WHY ?



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