alg. top
handle structures
proofs

A users guide to simple

(compact, w/ boundary) exotica

Def^n: A smooth 4-mfld W is <u>exotic</u> if $\exists$ smooth 4-mfld W' s.t. W is homeomorphic to W' ($W \underset{TOP}{\cong} W'$) but W is not diffeomorphic to W' ($W \not\cong_{sm} W'$).

Today, W compact with ∂.

Def^n: A pair of sm 4-mflds W, W' w/ homeo $f: \partial W \to \partial W'$ are <u>exotic relative to f</u> if $\exists$ homeo $F: W \to W'$ s.t. $F|_\partial = f$ but <u>no</u> diffeo $\mathcal{F}: W \to W'$ s.t. $\mathcal{F}|_\partial = f$.

(easier, <u>not</u> what we're after today)

* (Akbulut ($n \neq 0$) '92, <u>Yasui ($n=0$) '15</u>): $\exists$ exotic W^4 homotopy equiv. ($\simeq$) S^2, (w/ $Q_W = [n]$)

(<u>Hayden-P. '19</u>): $\exists$ exotic $W \simeq S^2$ s.t. W' v. v. distinct from W

in terms of handle decomp.

(Akbulut-Ruberman '15): $\exists$ exotic contractible W^4

* (<u>Hayden-Mark-P '19</u>): $\exists$ exotic contractible W^4 which are as simple as possible ($W \not\cong_{sm} B^4$)

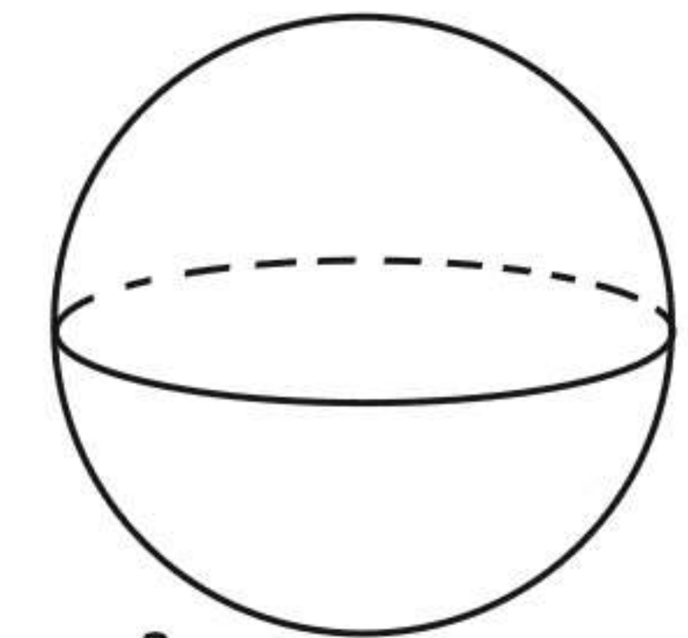
(<u>Hayden '20</u>): $\exists$ exotic $W^4 \simeq S^1$

V. similar v. simple techniques. We'll try to give a users guide today.

Handle calculus main ideas

- Always start w/ B^n (0-handle)

- 1-handles: $B^1 \times B^{n-1}$ attached along $(\partial B^1) \times B^{n-1} \cong S^0 \times B^{n-1}$



B^3

Handle calculus main ideas

- Always start w/ B^n (0-handle)

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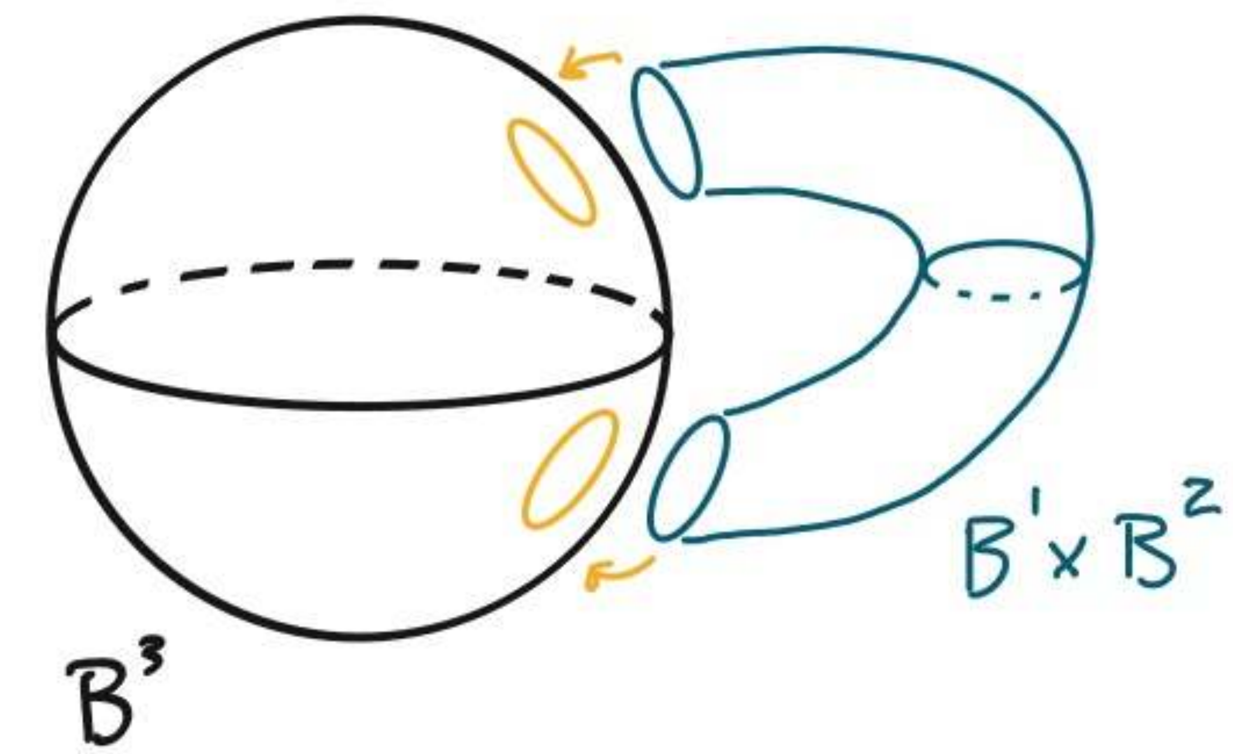
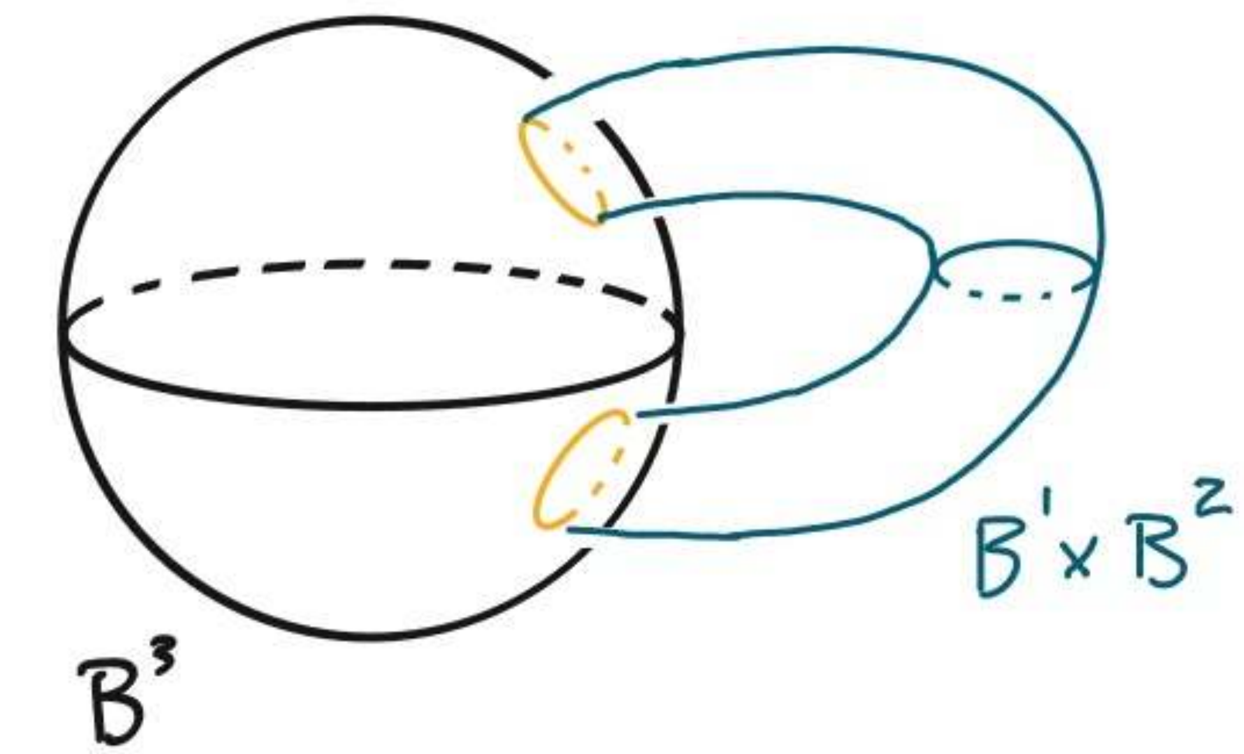


Handle calculus main ideas



- Always start w/ B^n (0-handle)
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- 2-handles: $B^2 \times B^{n-2}$ attached along $(\partial B^2) \times B^{n-2} \cong S^1 \times B^{n-2}$

Handle calculus main ideas

- Always start w/ B^n (0-handle)
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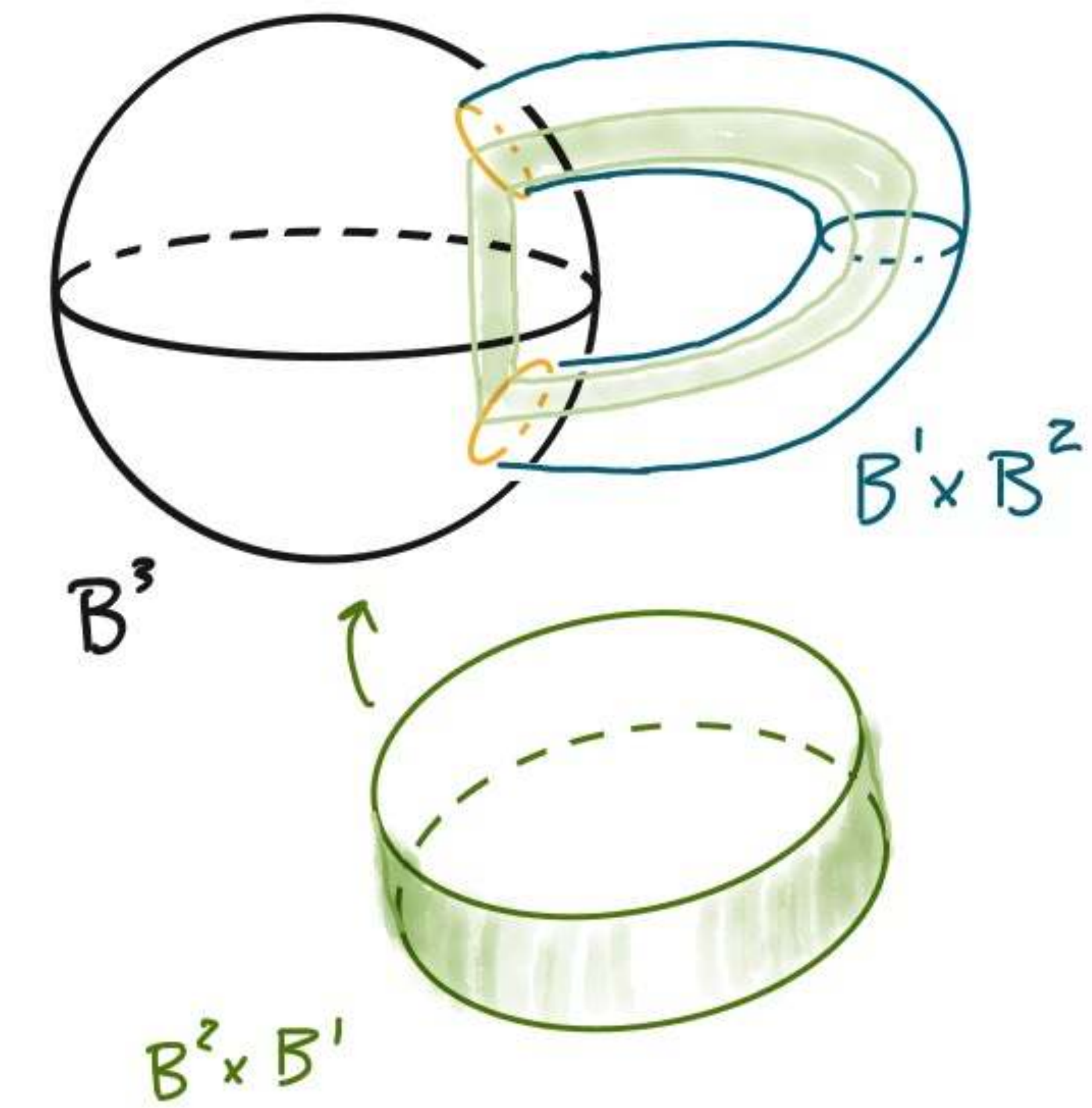
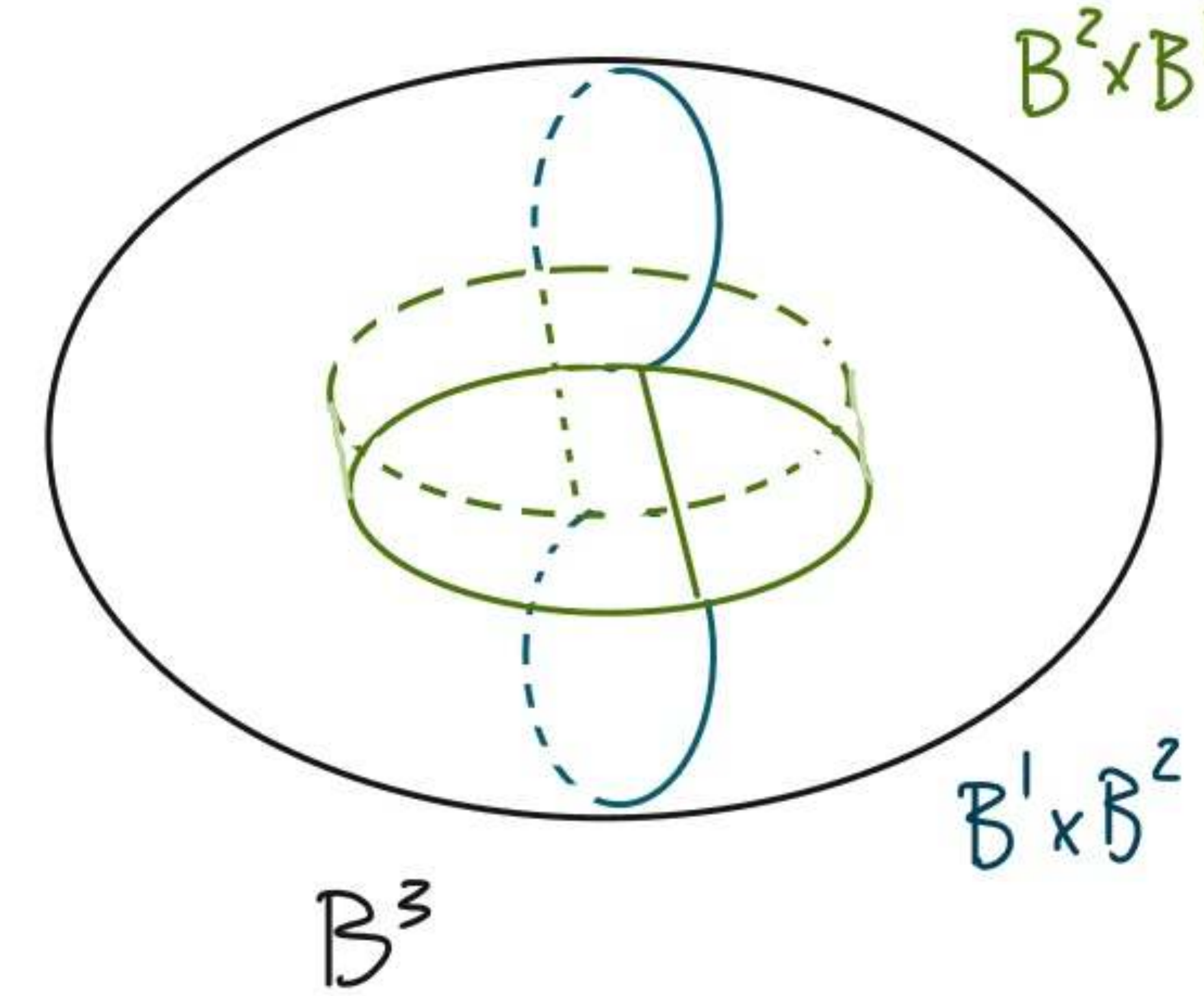


Handle calculus main ideas

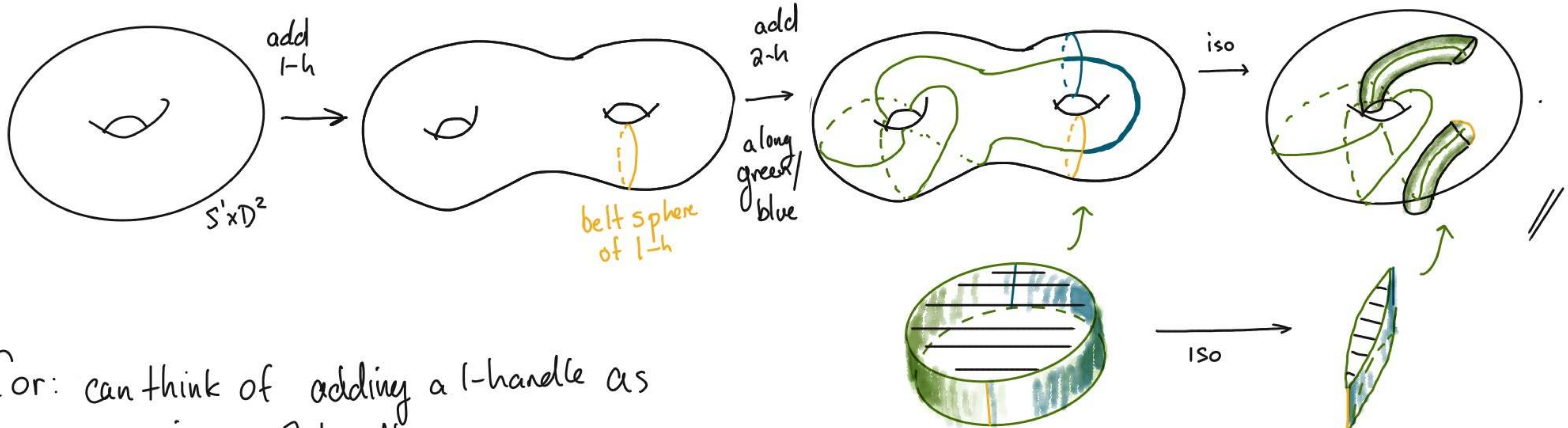


- Always start w/ B^n (0-handle)
- 1-handles: $B^1 \times B^{n-1}$ attached along $(\partial B^1) \times B^{n-1} \cong \underline{S^0} \times B^{n-1}$
- 2-handles: $B^2 \times B^{n-2}$ attached along $(\partial B^2) \times B^{n-2} \cong \underline{S^1} \times B^{n-2}$

attaching sphere

Lemma: If a 2-handle runs 1x geometrically over a 1-handle $\left(\underbrace{(\partial B^2 \times \{pt\})}_{\text{attaching sphere of 2-h}} \cap \underbrace{(\{pt\} \times \partial(B^{n-1}))}_{\text{belt sphere of 1-h}} = \{pt\} \right)$
then the pair of handles is cancelling

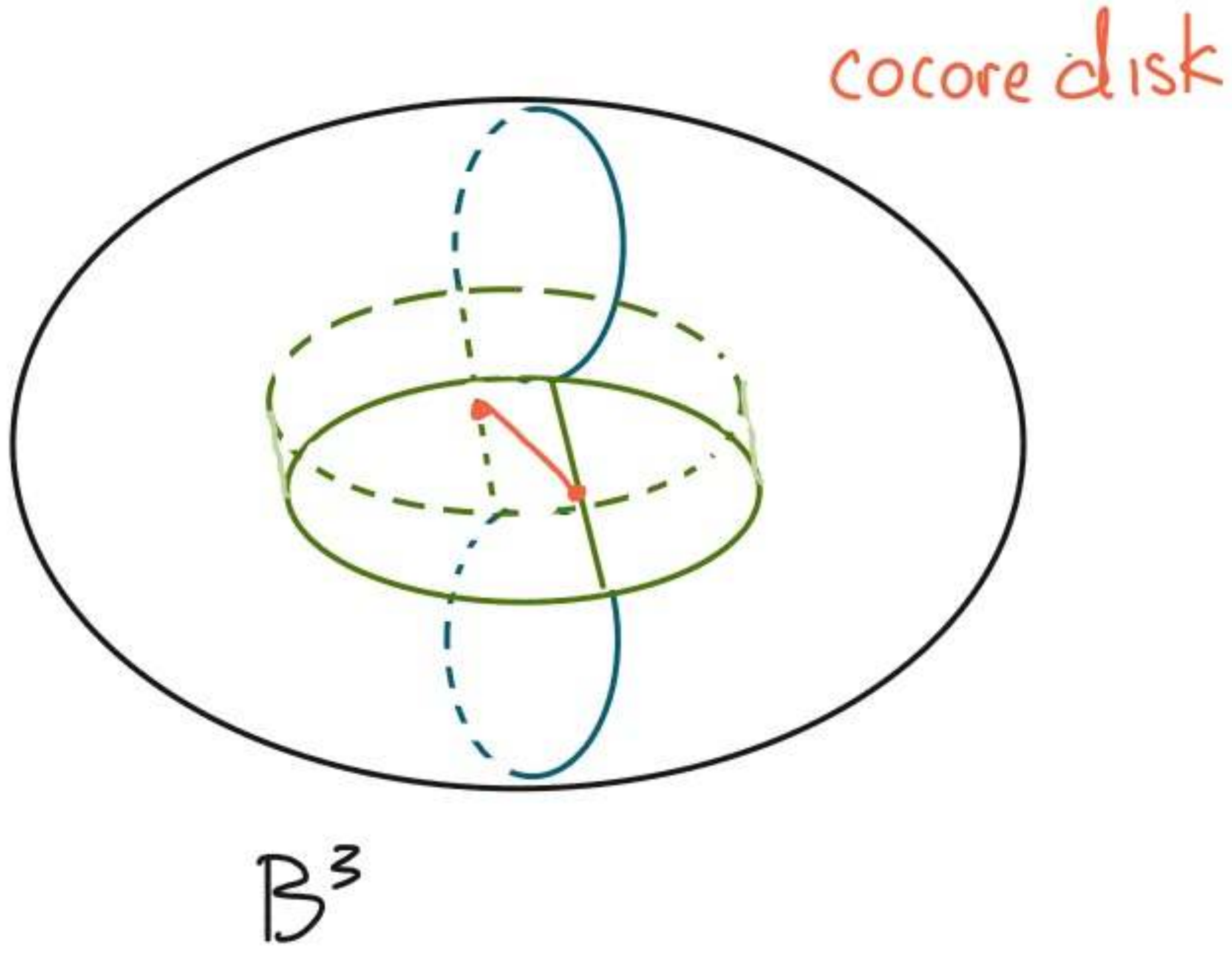
pf:



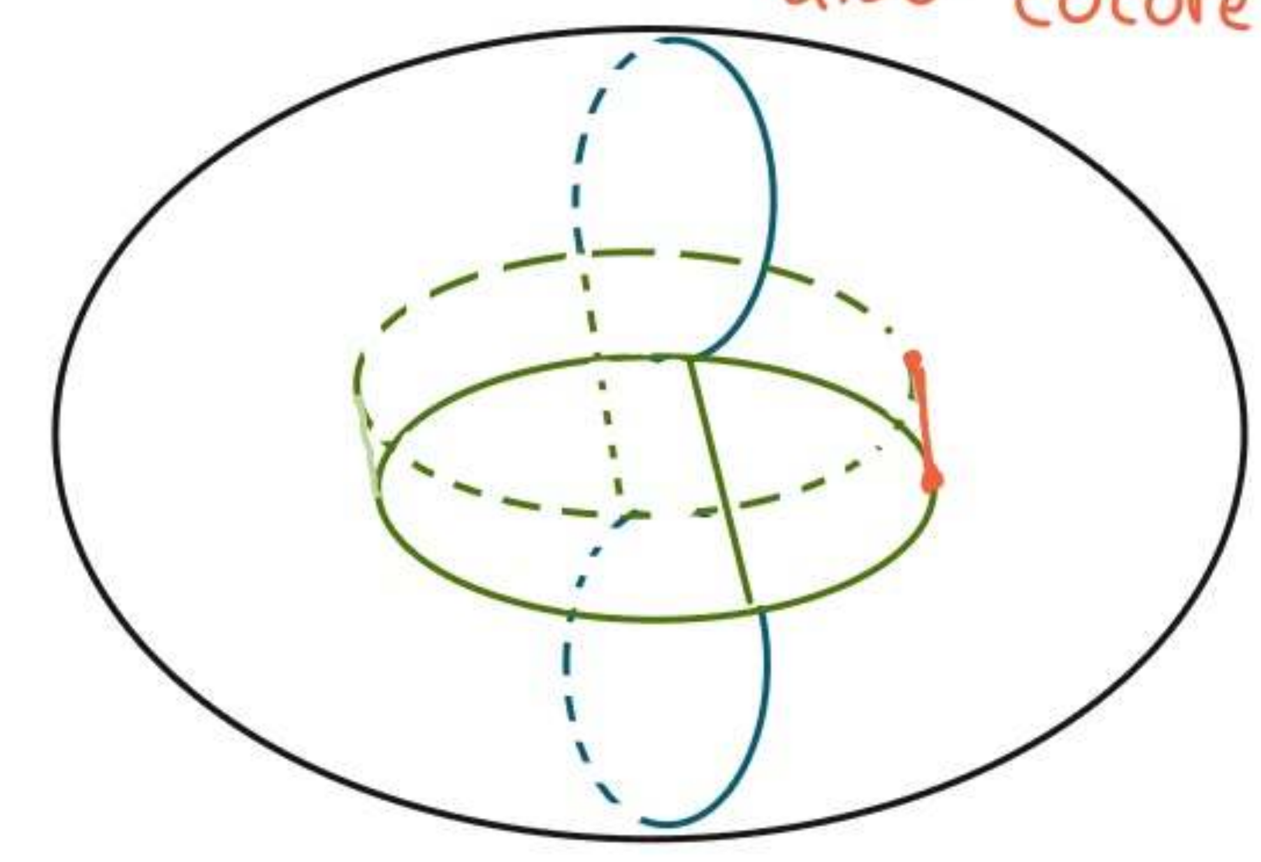
Cor: can think of adding a 1-handle as removing a 2-handle

How to add a 1-h by removing a 2-h in practice:

Removing 2-h $\rightsquigarrow$ cutting out $\nu(\underbrace{\text{cocore disk}}_{\uparrow \text{orthogonal to core}}) \cong D^{n-2}$



How to add a 1-h by removing a 2-h in practice:

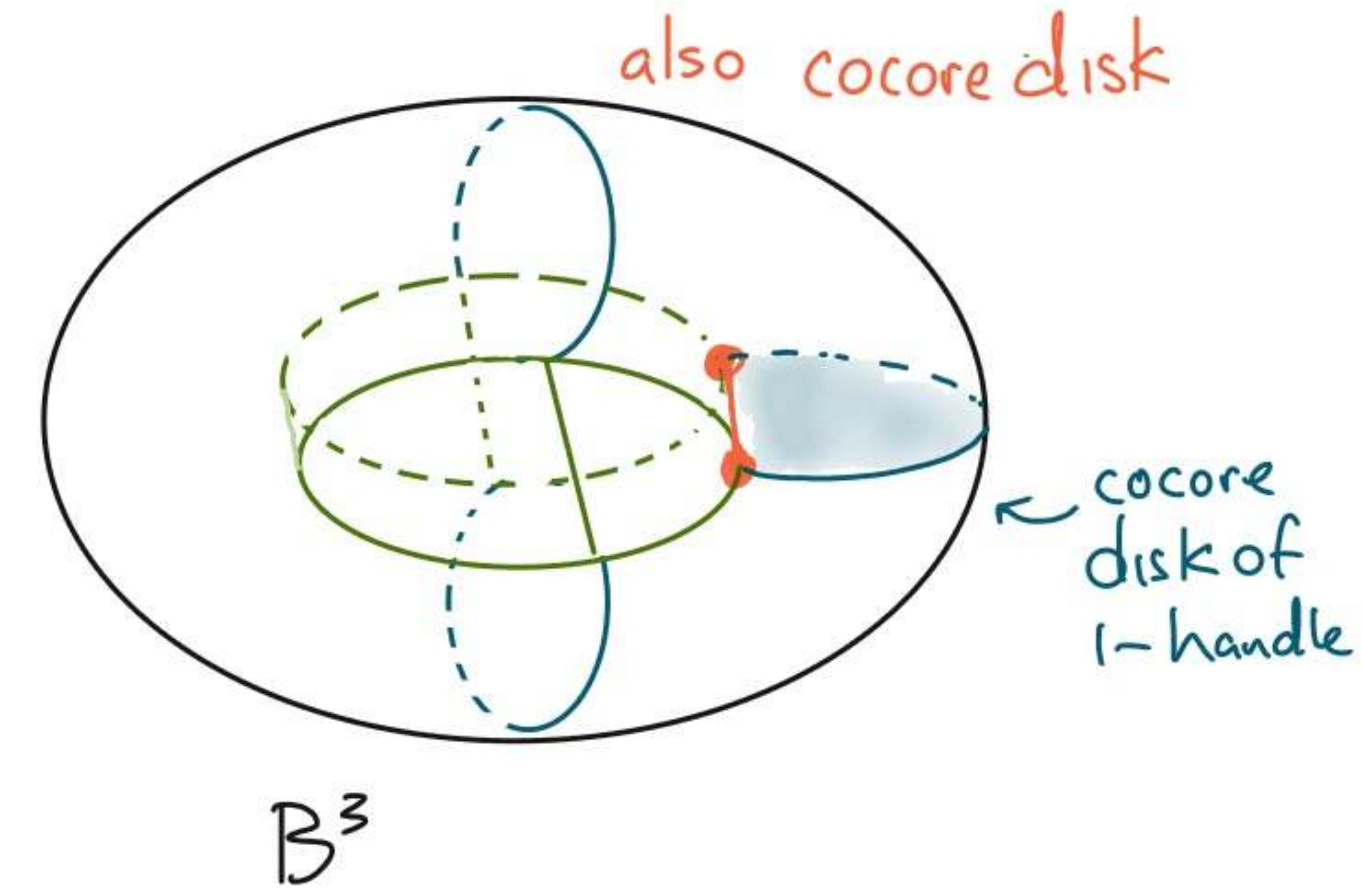


Removing 2-h $\rightsquigarrow$ cutting out ν(cocore disk) $\cong D^{n-2}$

orthogonal to core

cocore disk of 2-h (in cancelling pair) is boundary paralell.

How to add a 1-h by removing a 2-h in practice:



Removing 2-h $\leftrightsquigarrow$ cutting out ν(cocore disk) $\cong D^{n-2}$

orthogonal to core

cocore disk of ∂-h (in cancelling pair) is boundary paralell.

prop: adding a 1-h $\leftrightsquigarrow$ removing ∂ paralell D^{n-2} from B^n

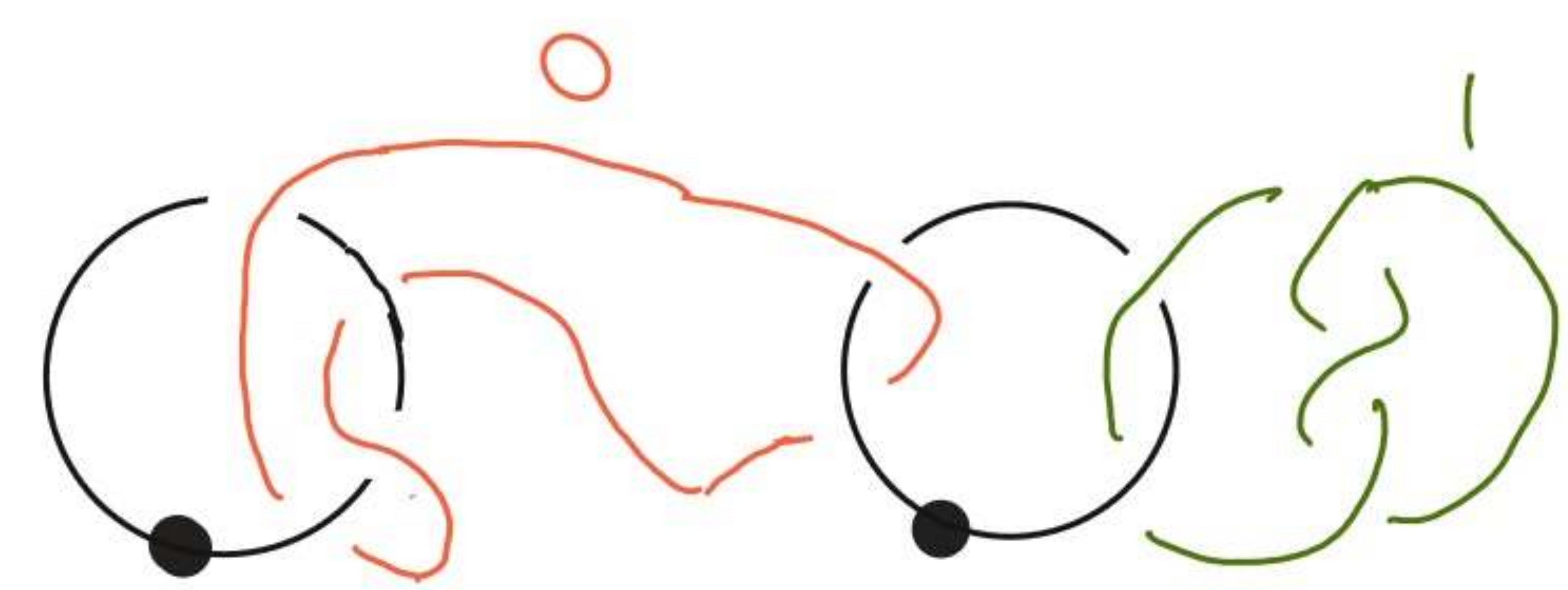
In dim 4, specify ∂ paralell $D^2 \hookrightarrow B^4$ via

this means take D^2 for this U, push slightly in to B^4

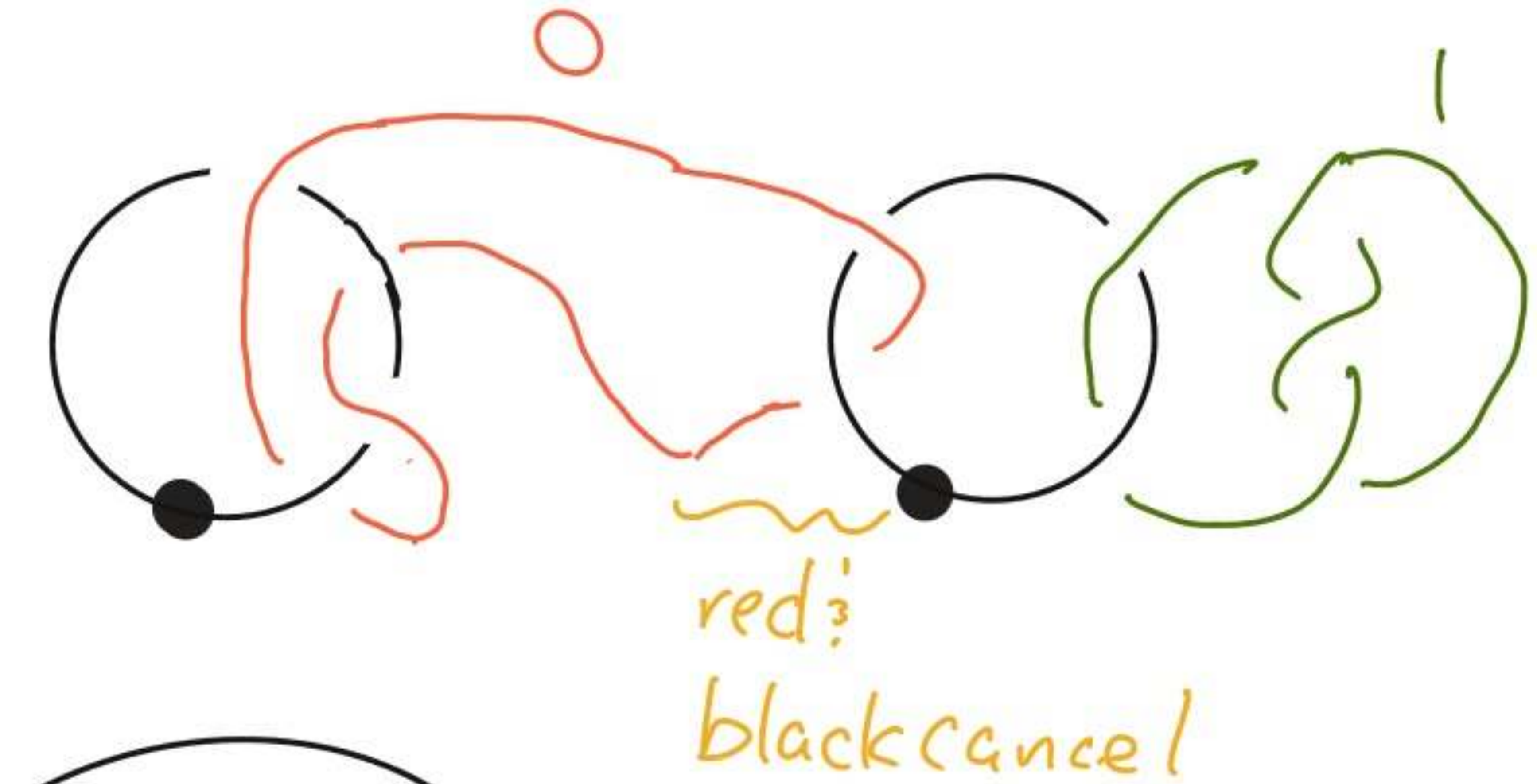
Handle diagram of 4-manifold (w/ 0, 1, 2-handles)

- Start w/ B^4, just look @ S^3 boundary.
- for 1-handles, specify dotted unknots.
- for 2-handles, specify a framed link

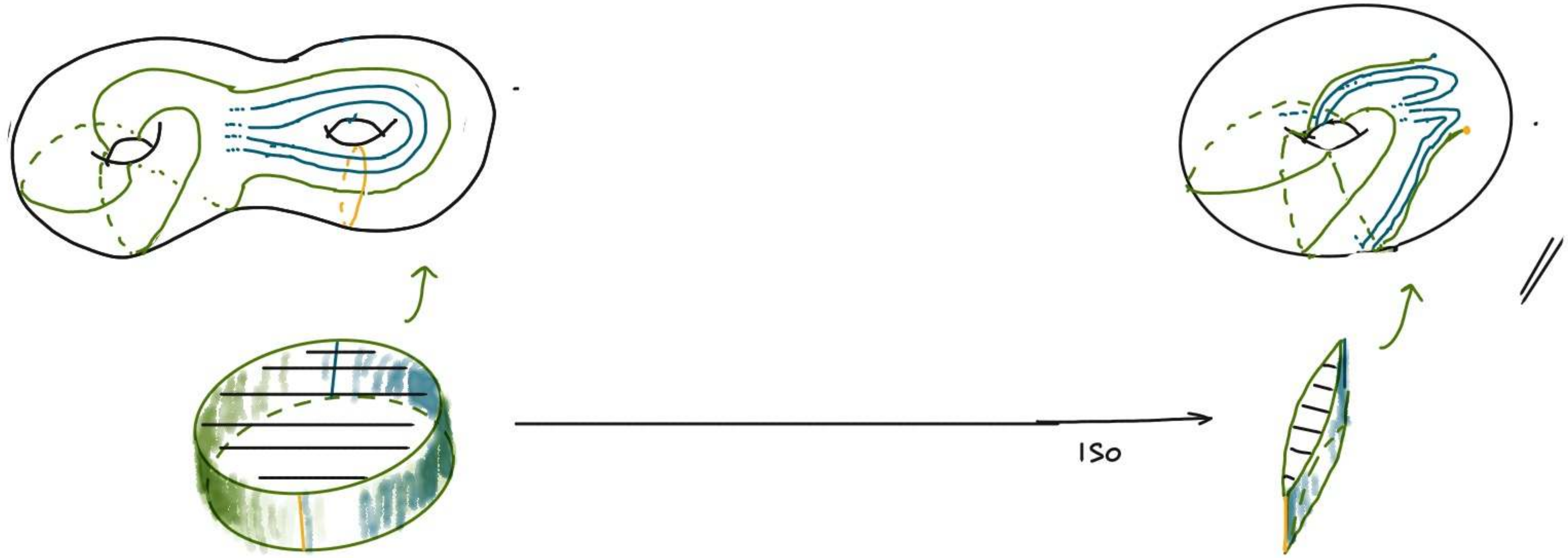
link may link unknots $\longleftarrow$ linking w/ unknots $\leftrightsquigarrow$ running over 1-h



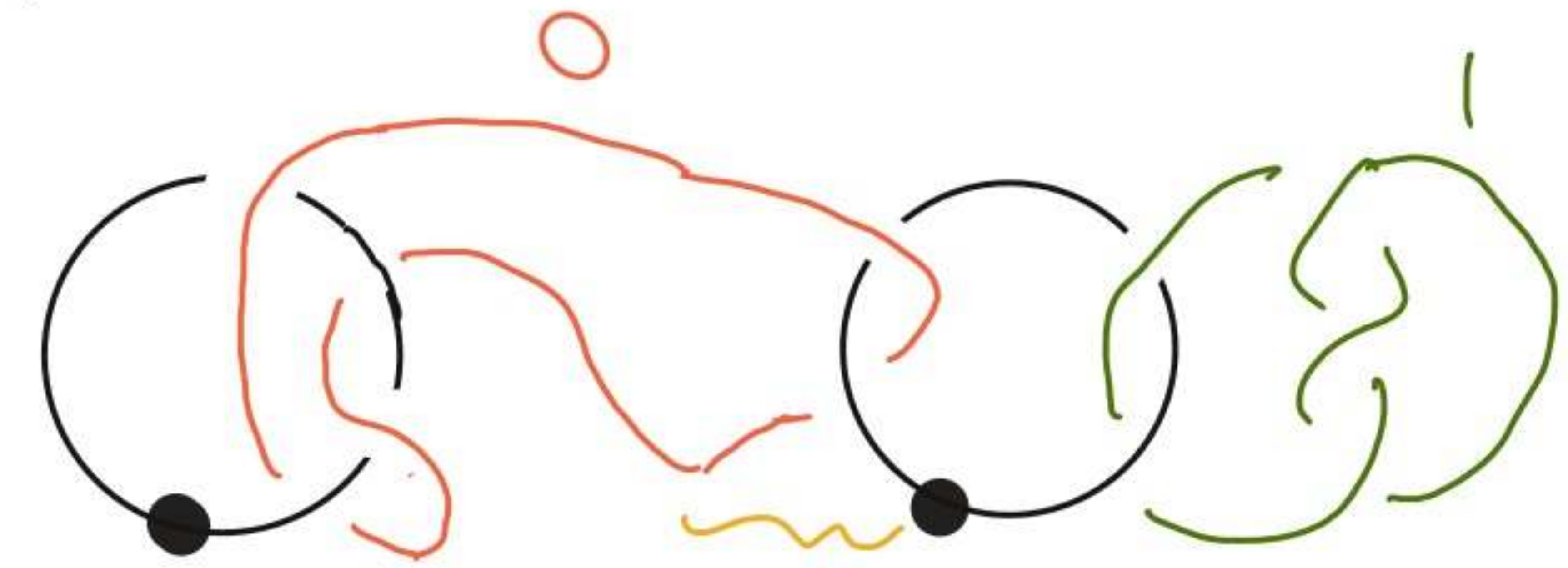
Still a lemma: If a 2-handle runs 1x geometrically over a 1-handle then the pair of handles is cancelling



How to actually cancel:

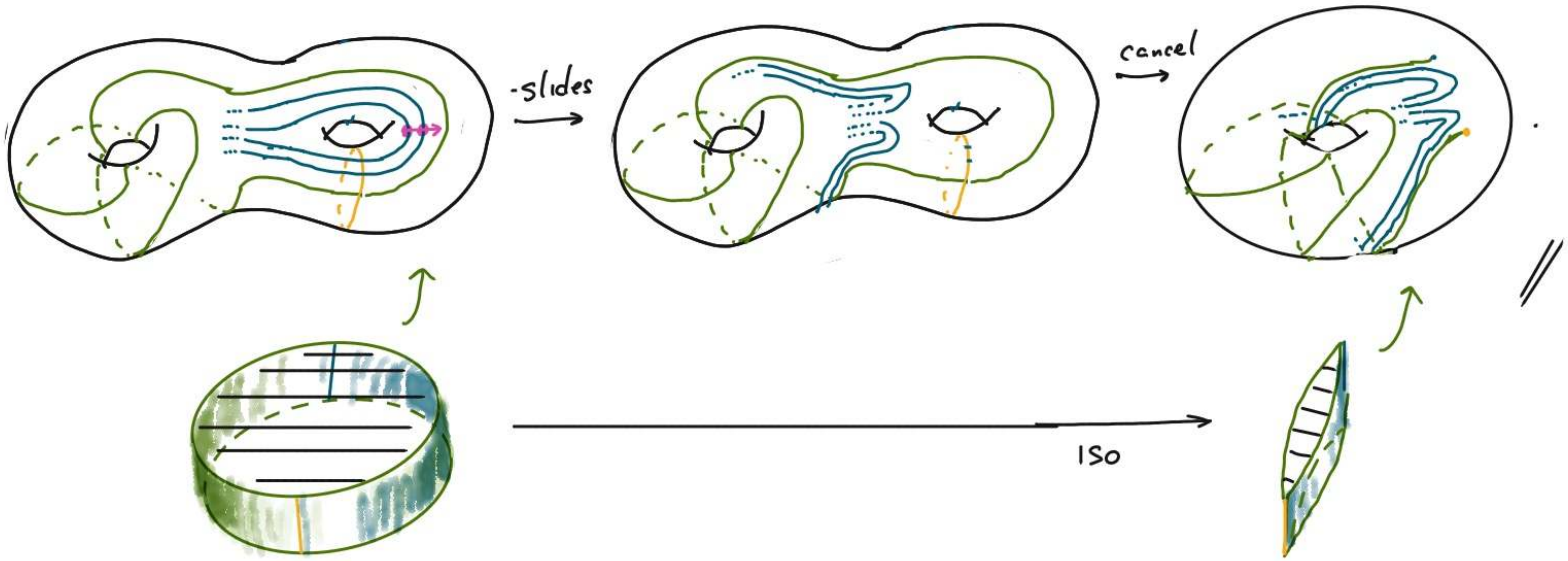


Still a lemma: If a 2-handle runs 1x geometrically over a 1-handle then the pair of handles is cancelling

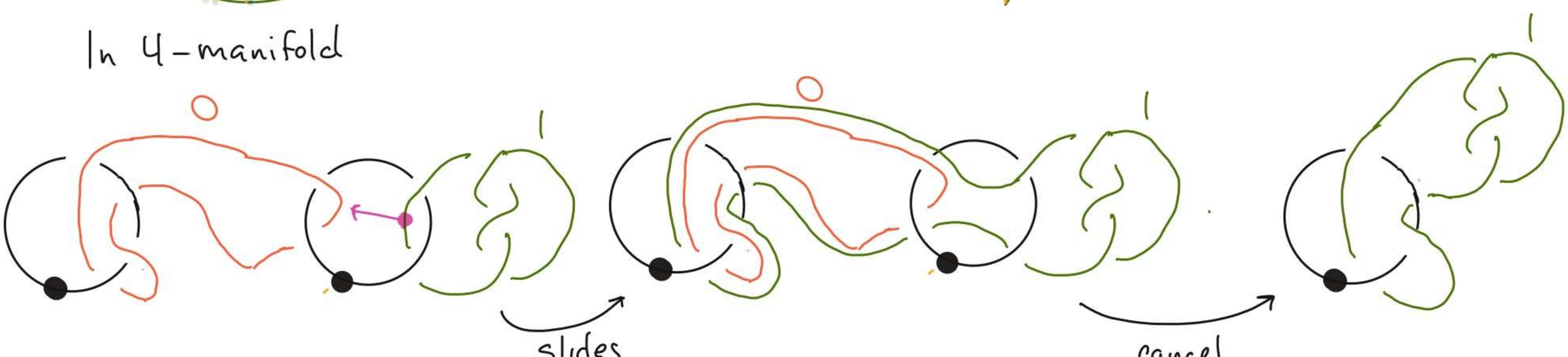


red ; black cancel

How to actually cancel:

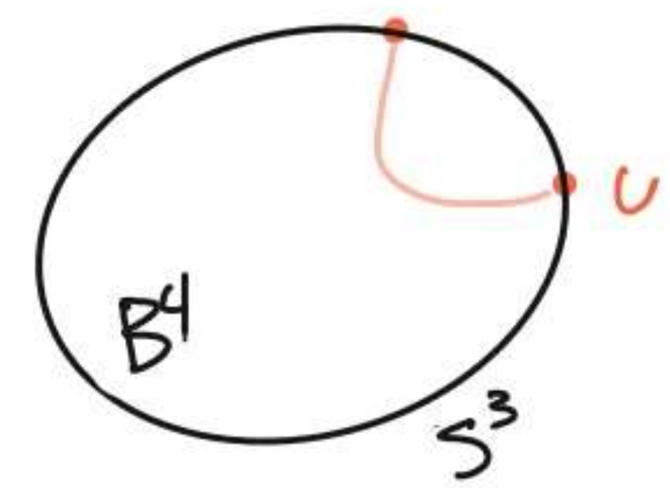


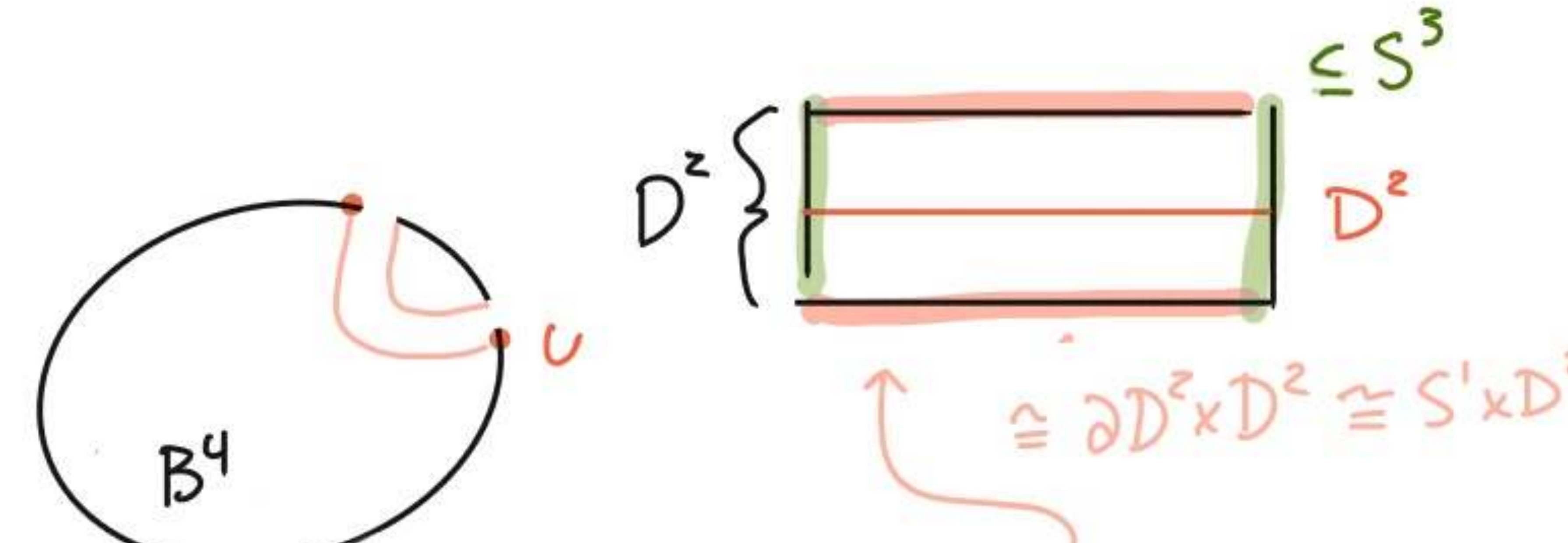
In 4-manifold



(once red is the only thing running over 1-h)

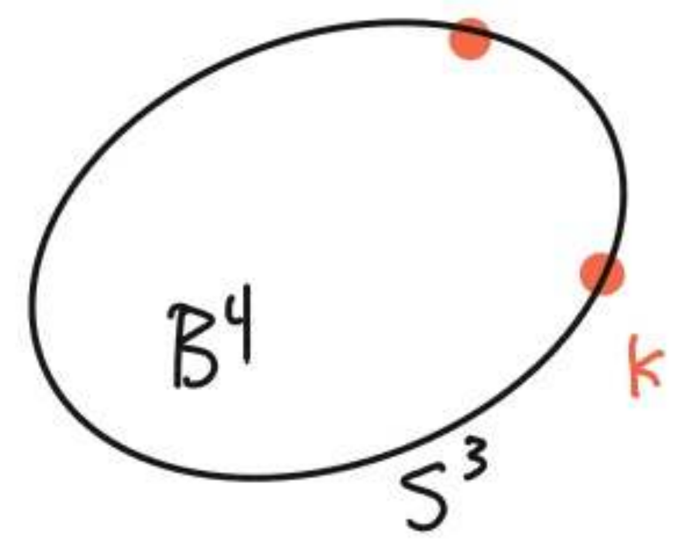
Effect of adding 1-h on ∂:

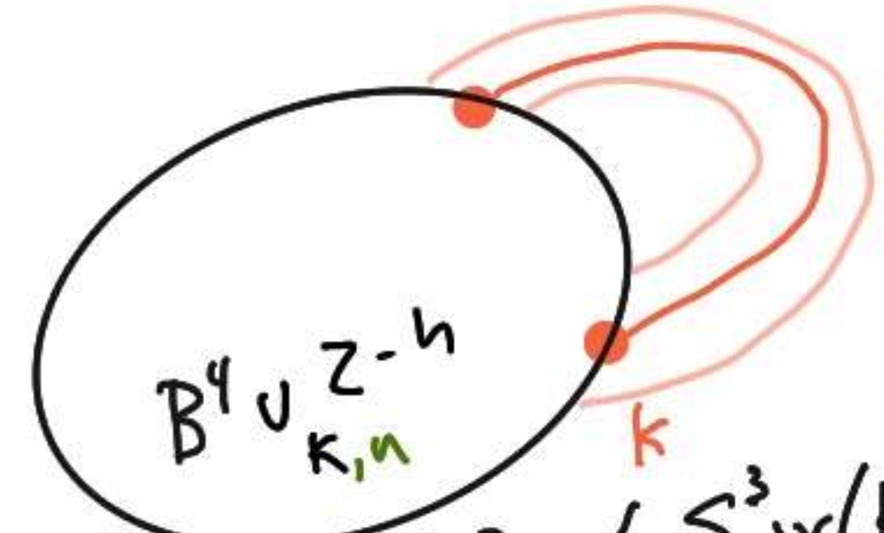




$\partial = (B^4 \setminus \nu(U)) \cup (\text{rest of } \partial \text{ of disk we removed}) \cong S^3_0(U)$

Effect of adding a 2-h to ∂:





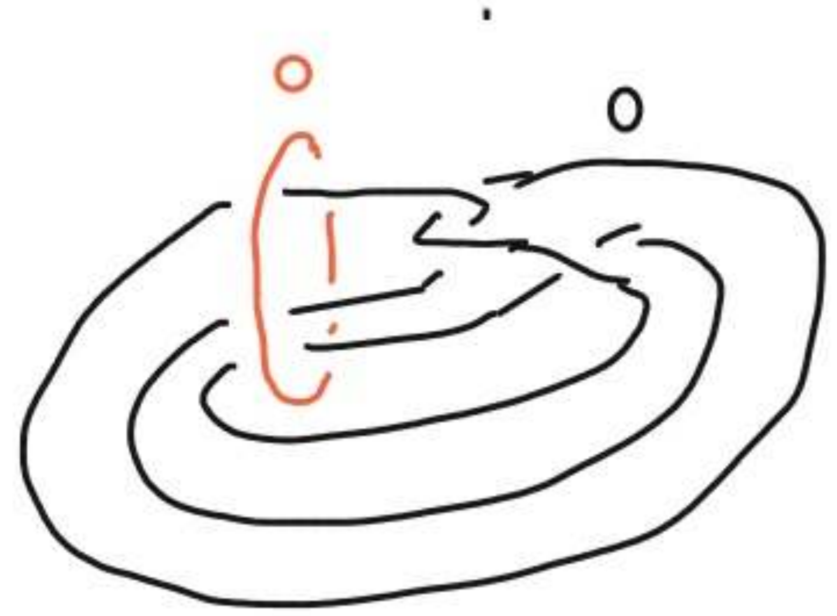
This 4-manifold is a <u>knot trace</u>, $X_n(K)$, or $X(K) = X_0(K)$

$\partial = (S^3 \setminus \nu(K)) \cup (\text{rest of } \partial \text{ of disk we added})$
$\cong S^3_n(K)$

Silly observation:
∂ can't tell difference between adding a 1-h & adding a 0-f 2-h along U.

Cor (Poenaru '60, Mazur '61): $\exists$ pairs of contractible 4-mflds W_1, W_2 w/ $\partial W_1 \cong \partial W_2$

pf: Take $\partial M_i =$



$M_1 :=$

$(\pi_1 = 1)$

$M_2 =$

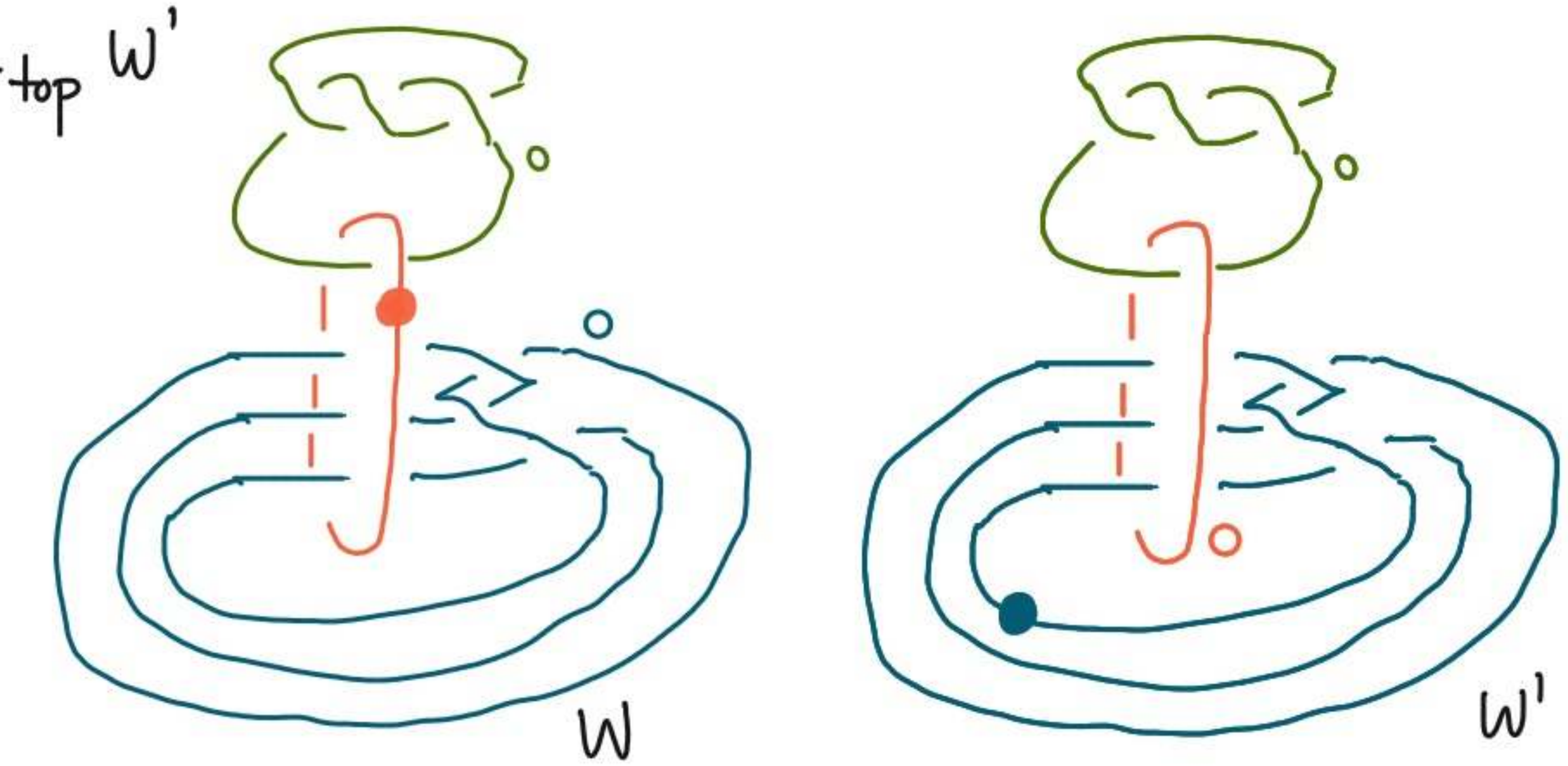
this is an unknot!

//

Thm: (Yasui '15) $\exists$ exotic W^4 homotopy equiv. to S^2, (w/ $Q_W = [0]$)

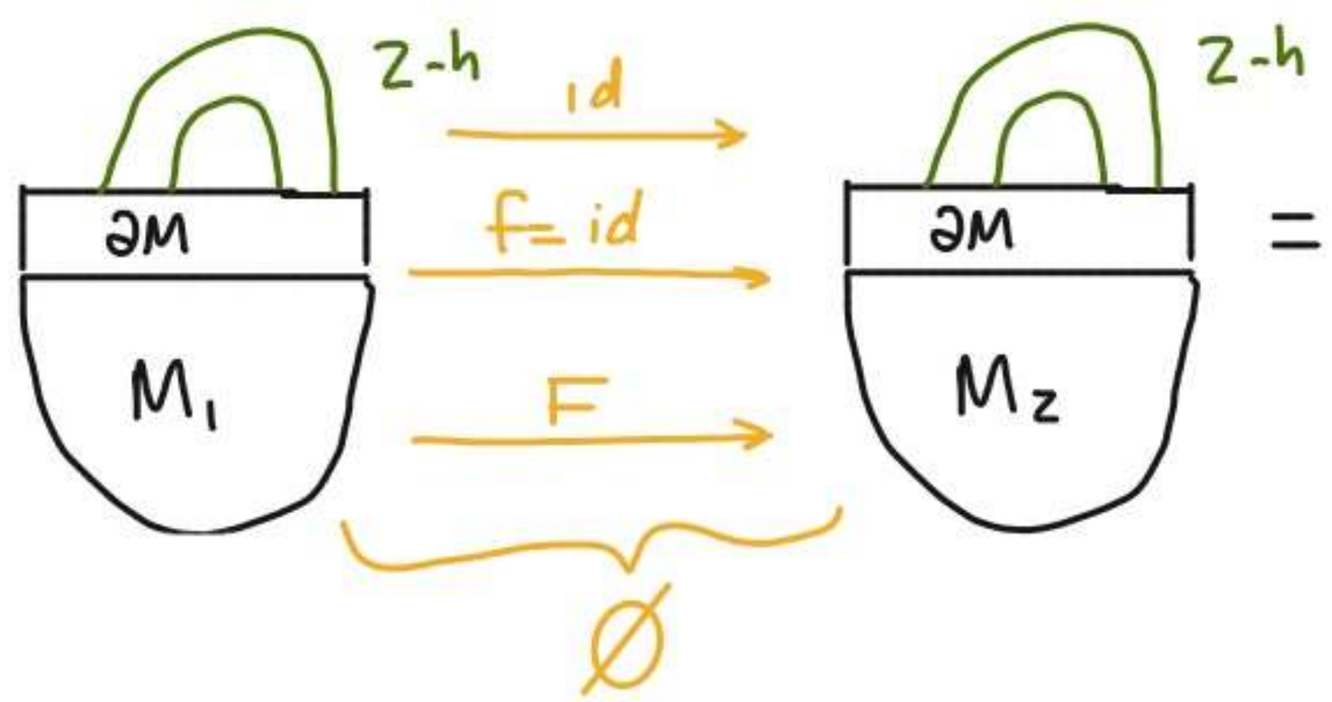
pf: a) build W, W' w/ correct alg top & $W \cong_{top} W'$

b) show $W \not\cong_{sm} W'$.



a) W, W' at right. certainly $\partial W \cong \partial W'$.

Both contractible $\cup$ 2-h $\Longrightarrow$ homotopy S^2s.

In fact $W_1 =$  $= W_2$

Theorem (Freedman, '82): For any W_1, W_2 contractible & $f: \partial W_1 \to \partial W_2$, $\exists$ homeo $F: W_1 \to W_2$, $F|_\partial = f$.

$F \rightsquigarrow \emptyset_{TOP} : W \to W'$ as above.

Historical note: this method of constructing homeomorphic pairs orig. in Akbulut '92.

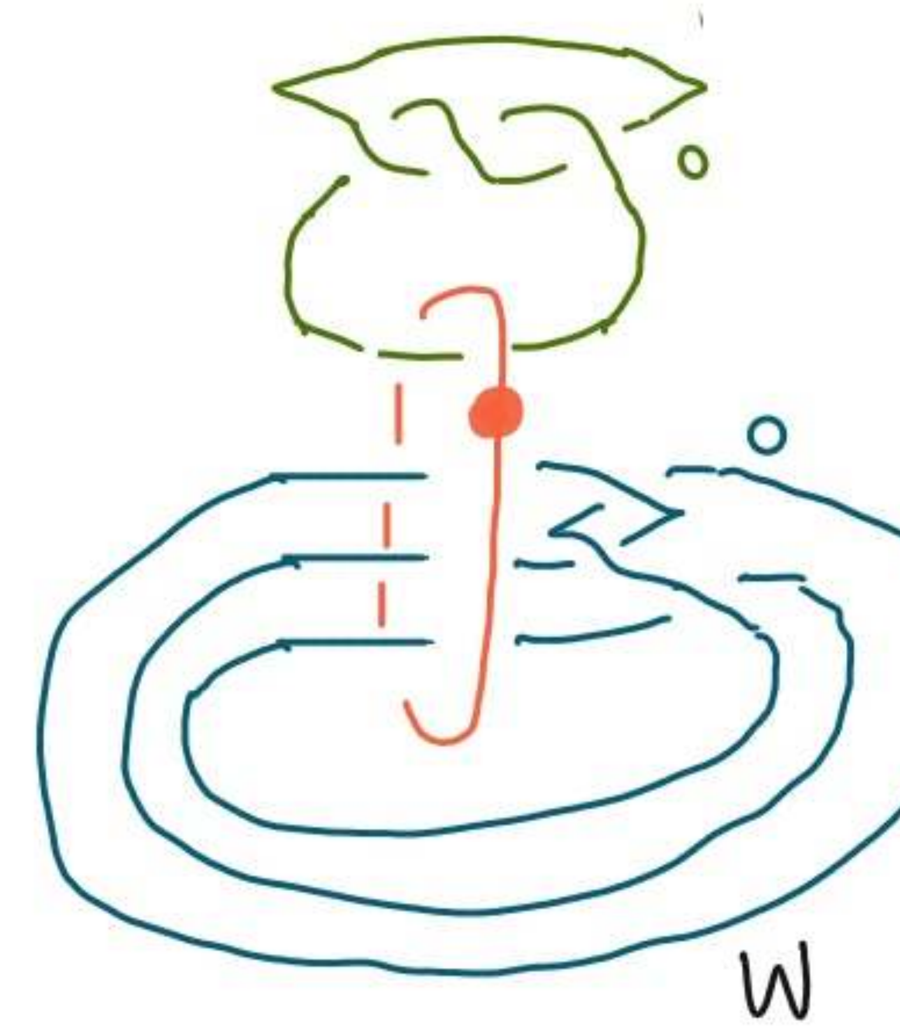
↑ dot-zero surgery, cork twisting

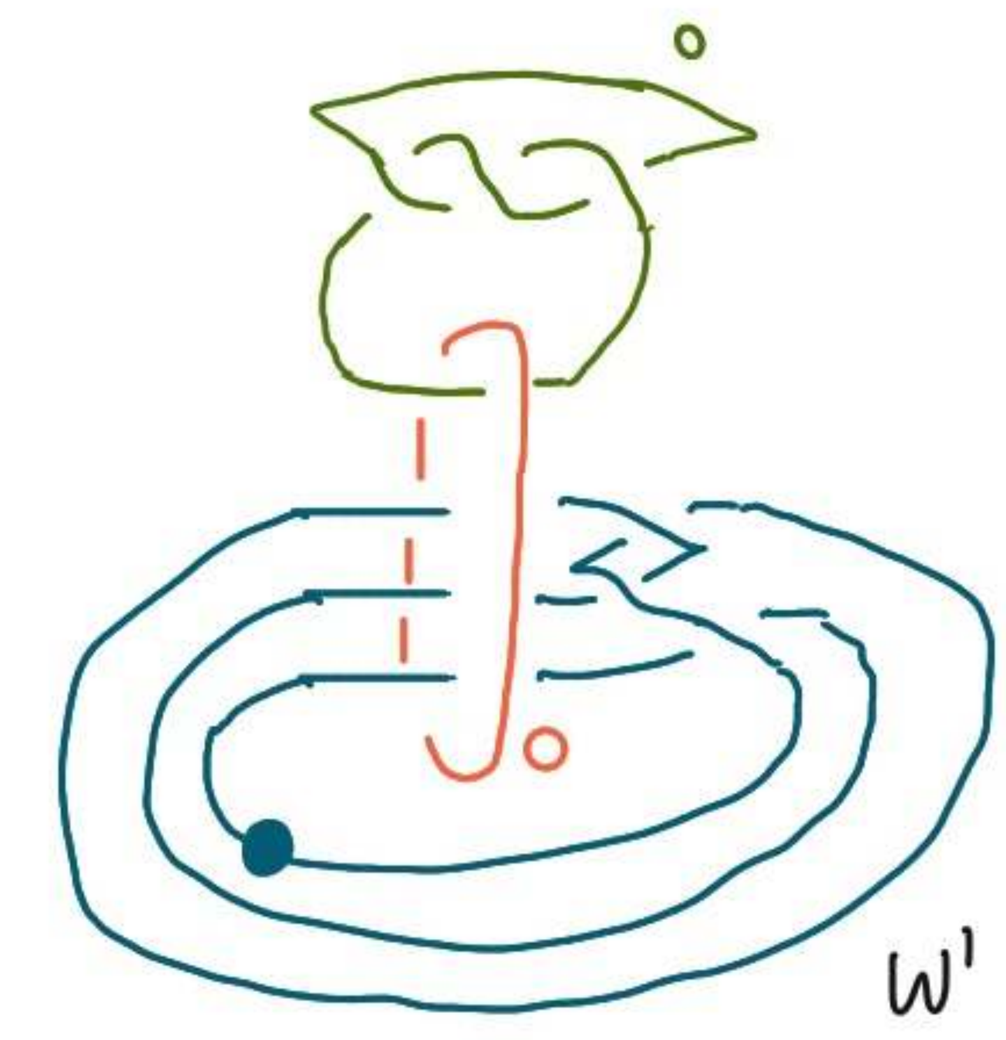
Thm: (Yasui '15) $\exists$ exotic W^4 homotopy equiv. to S^2, (w/ $Q_W = [0]$)

pf: b) show $W \not\cong_{sm} W'$.

We'll show 1) $\exists\ T^2 \xhookrightarrow[sm]{} W'$ gen H_2 $(=2)$

2) $\min\{g(\Sigma) : \Sigma \hookrightarrow W, [\Sigma] \text{ gen } H_2\} \geq 2$



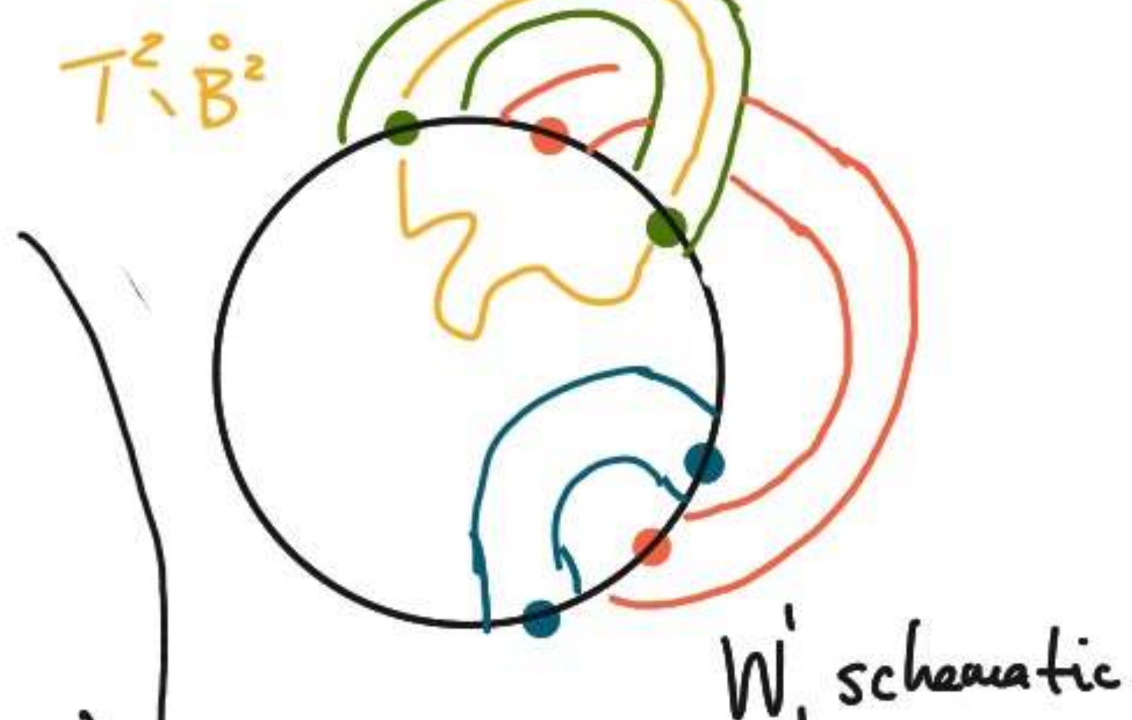


1) RHT bounds genus 1 Seifert surface in S^3

hence bounds $T^2 \setminus \mathring{B}^2 \xhookrightarrow[sm]{} B^4$.

In W', cap off to T^2 gen H_2

2) Need some inherently sm obstruction $\left(F^{-1}(T^2) \xhookrightarrow[TOP]{} W \text{ gen } H_2\right)$



Thm: (Eliashberg '90, Lisca-Matić '98,

either
- K-M '94, M-S-T '98, O-Sz '00 (Gauge/SW)
- Lambert-Cole '20 (symplectic top & Khovanov/Lee/Rasmussen))

"Stein adjunction inequality"

I'll tell you later

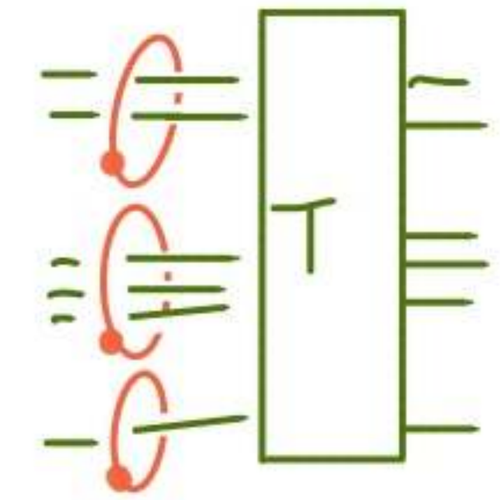
If some handle diagram of W satisfies some conditions and $\Sigma^2 \xhookrightarrow[sm]{} W$

w/ $[\Sigma]$ non-torsion then $\chi(\Sigma) \leq \langle c_1(W), \Sigma \rangle - [\Sigma][\Sigma]$

can be read off from nice handle diagram

self intersection #

<u>Thm</u>: If h.d. of W satisfies some conditions & $[\Sigma]$ nontorsion then $\chi(\Sigma) \leq \langle c_1(\omega), \Sigma \rangle - [\Sigma]\cdot[\Sigma]$

- all 1-hs isotoped to  & digram isotoped to closure of 

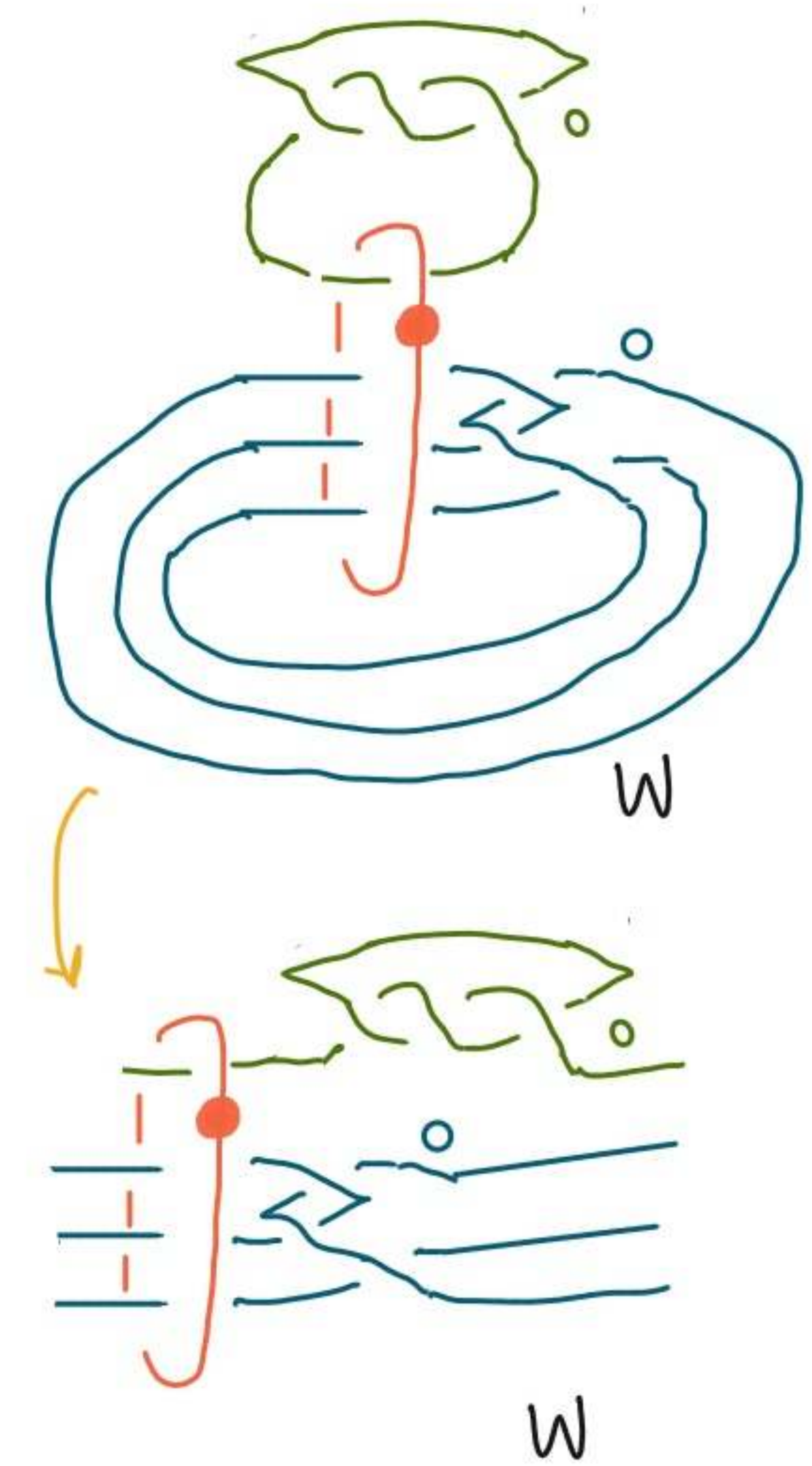
- all 2-h crossings look like  & all vertical tangencies are  or 

(both achievable for every W)

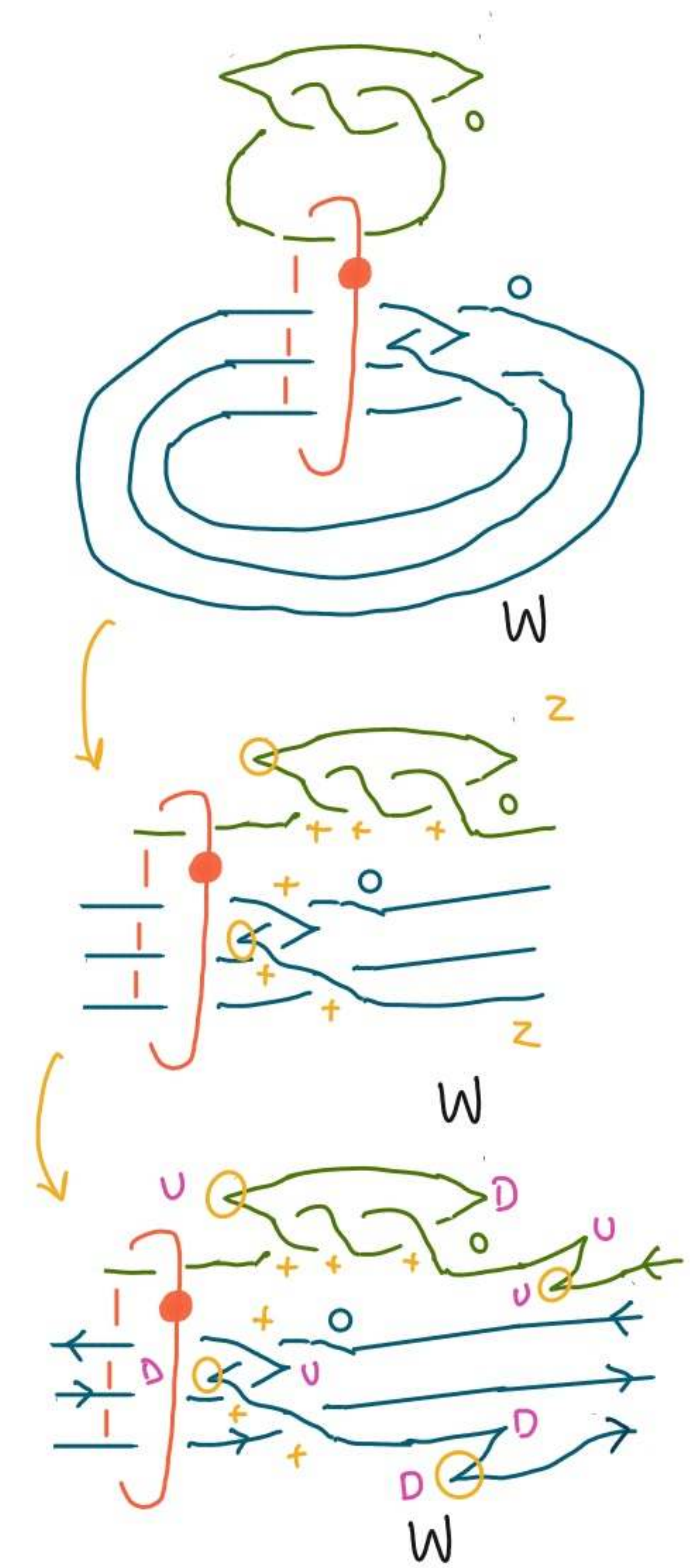
your diagram of h

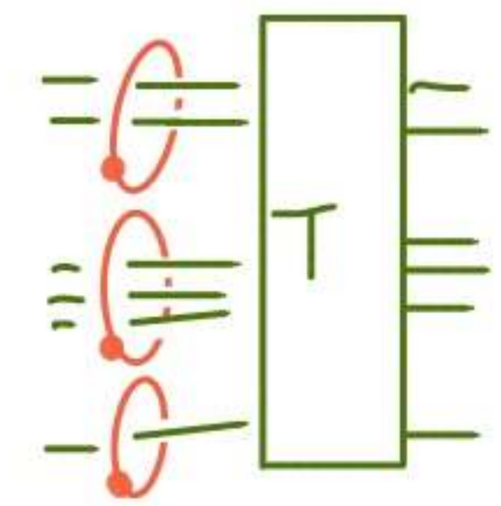
- each 2-h h has framing $= tb(h) - 1$

$\quad\quad\quad\quad$ $\parallel$ writhe(h) − # left cusps of h



Thm: If h.d. of W satisfies some conditions & $[\Sigma]$ nontorsion then $\chi(\Sigma) \leq \langle c_1(\omega), \Sigma \rangle - [\Sigma]\cdot[\Sigma]$



- all 1-hs isotoped to  & digram isotoped to closure of 

- all 2-h crossings look like  & all vertical tangencies are $<$ or $>$

(both achievable for every W)

your diagram of h

- each 2-h h has framing $= tb(h) - 1$

$= \text{writhe}(h) - \#\text{ left cusps of } h$

(this condition is generally hard to meet)

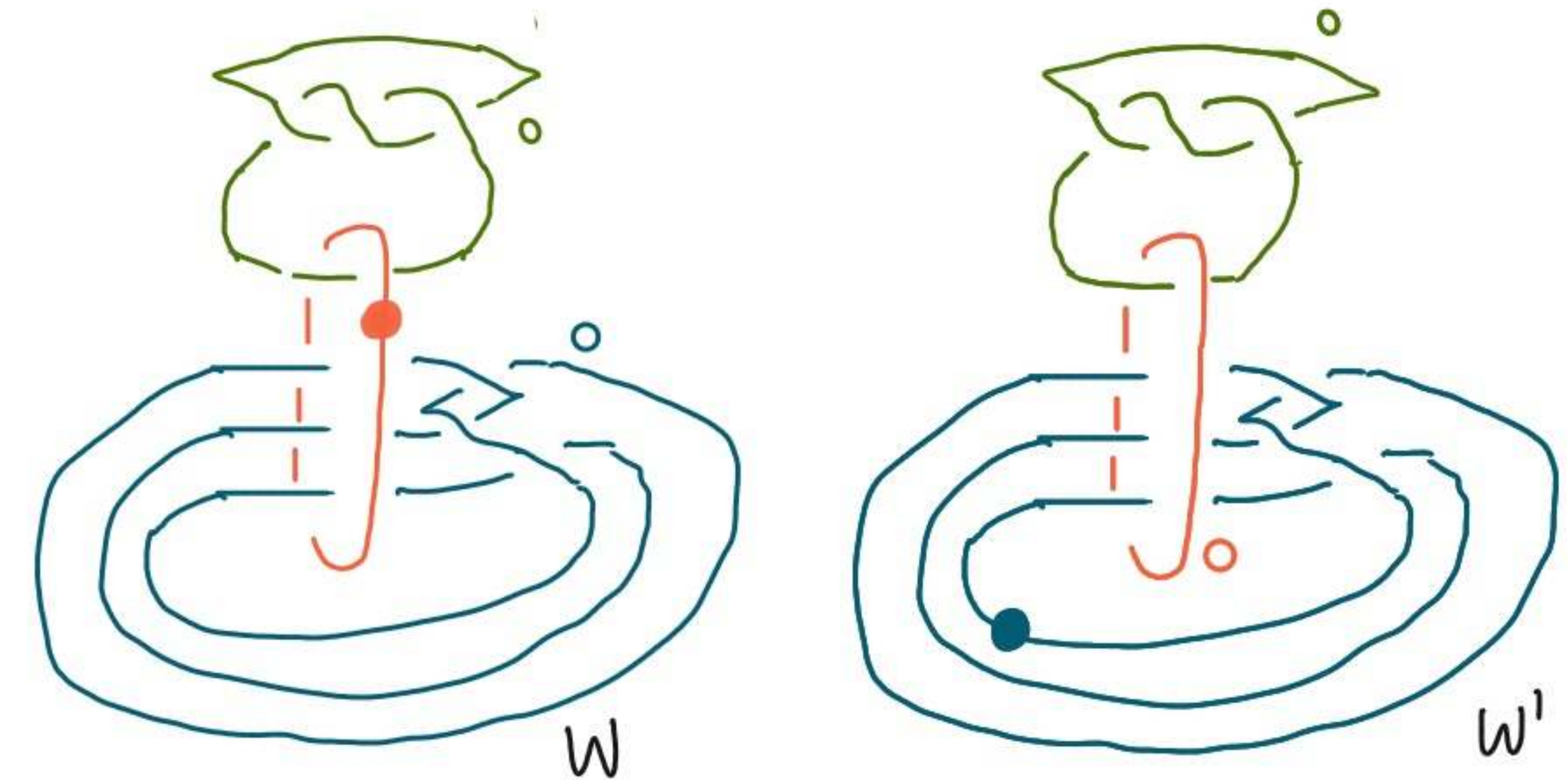
If $\Sigma^2 \xrightarrow[\text{sm}]{} W$ rep H_2 $\quad \chi(\Sigma) \leq \underbrace{\langle c_1(\omega), \Sigma \rangle}_{= r(B) - r(G) = -1-1 = -2} - \underbrace{[\Sigma]\cdot[\Sigma]}_{=0} \implies \chi(\Sigma) \leq -2$

$r(B) = \frac{1}{2}(\#\text{ up cusps} - \#\text{ down cusps})$

Historical remark: this method of distinguishing pairs due to Akbulut-Matveyev, '00

<u>Remarks</u>: • W has a cancelling 1-2 pair, hence

$$W \cong_{sm} B^4 \cup_{0,K} 2\text{-}h =: X(K)$$



(not obvious) • After a slide, W' has a cancelling 1-2 pair, hence $W' \cong_{sm} X(K')$

(historically, Akbulut)

Cor (Yasui): $\exists\, K, K'$ w/ $\partial X(K) \cong \partial X(K')$ s.t. $K \not\cong K'$ (Kirby 4.19)

$S^3_0(K) \cong S^3_0(K')$

Take aways: Dot-zero trick + Freedman to give homeomorphism

doesn't make sense w/out H_2

Distinguish genus functions to give non-diffeo

- Stein adjunction to get lower bds

easy to use when it can be used

only can be used in special settings

From a handle theoretic perspective, this is simplest exotica possible ($\simeq S^2$)

can we work in other htpy types? (pt, S^1)

Thm (Hayden–Mark–P.) $\exists$ $W \underset{TOP}{\cong} W' \simeq pt$ s.t. $W \not\cong_{sm} W'$ s.t.

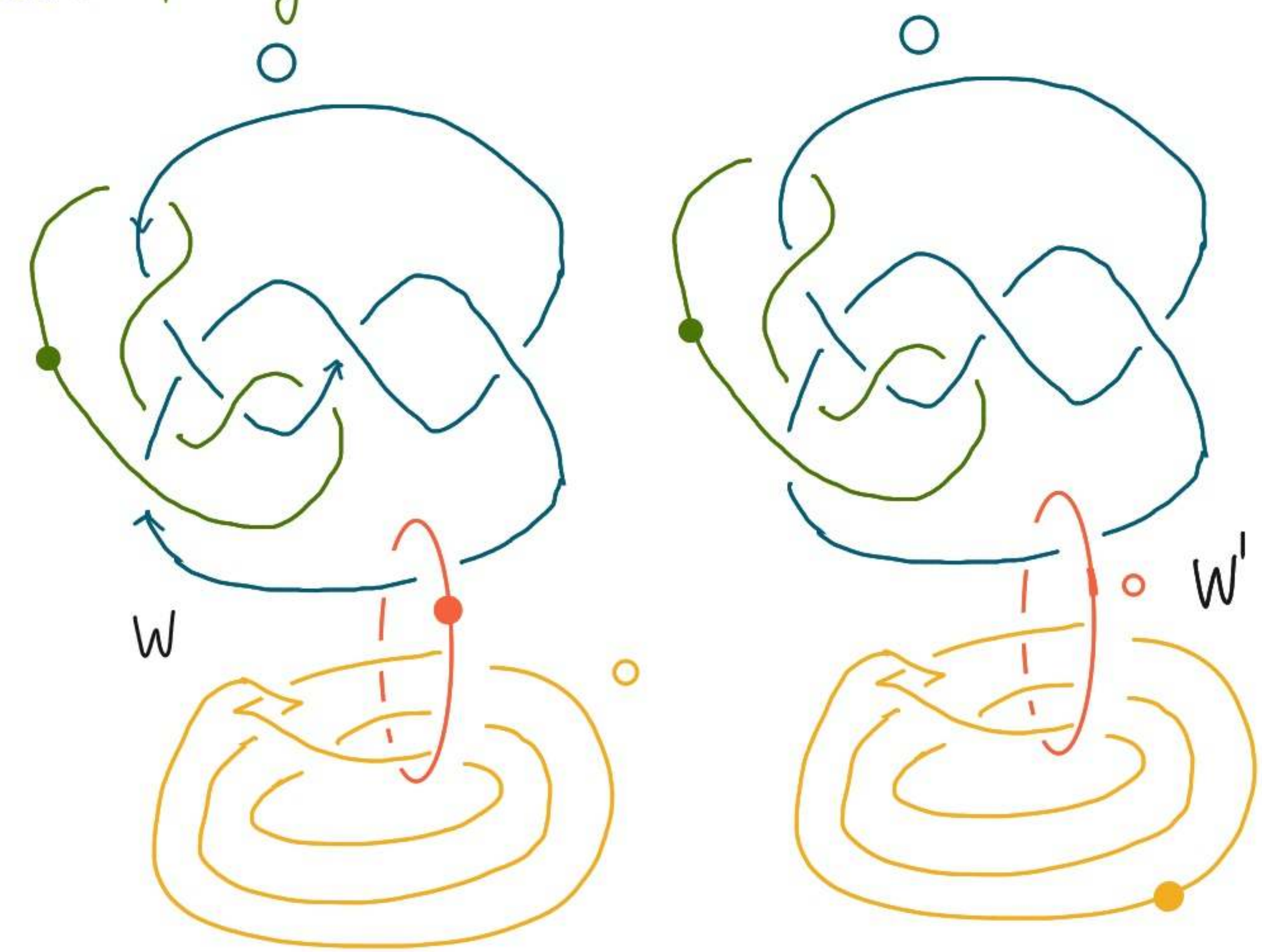
both W, W' as simple as possible from handle theoretic perspective.

(both W, W' can be built w/ single 1 & 2 handle) $\leftarrow$ Mazur-type

Pf: a) Build $W, W' \simeq pt$, show $W \cong_{TOP} W'$

b) Show W, W' both Mazur-type

c) Show $W \not\cong_{sm} W'$



a) W, W' as shown.

$\pi_1(W) = \langle r, g ; r_b, r_y \rangle = 1$

$\pi_1(W') = \langle y, g : r_b, r_r \rangle = 1$

$id : \partial W \longrightarrow \partial W'$

Theorem (Freedman, '82): For any W_1, W_2 contractible & $f : \partial W_1 \to \partial W_2$, $\exists$ homeo $F : W_1 \to W_2$, $F|_\partial = f$.

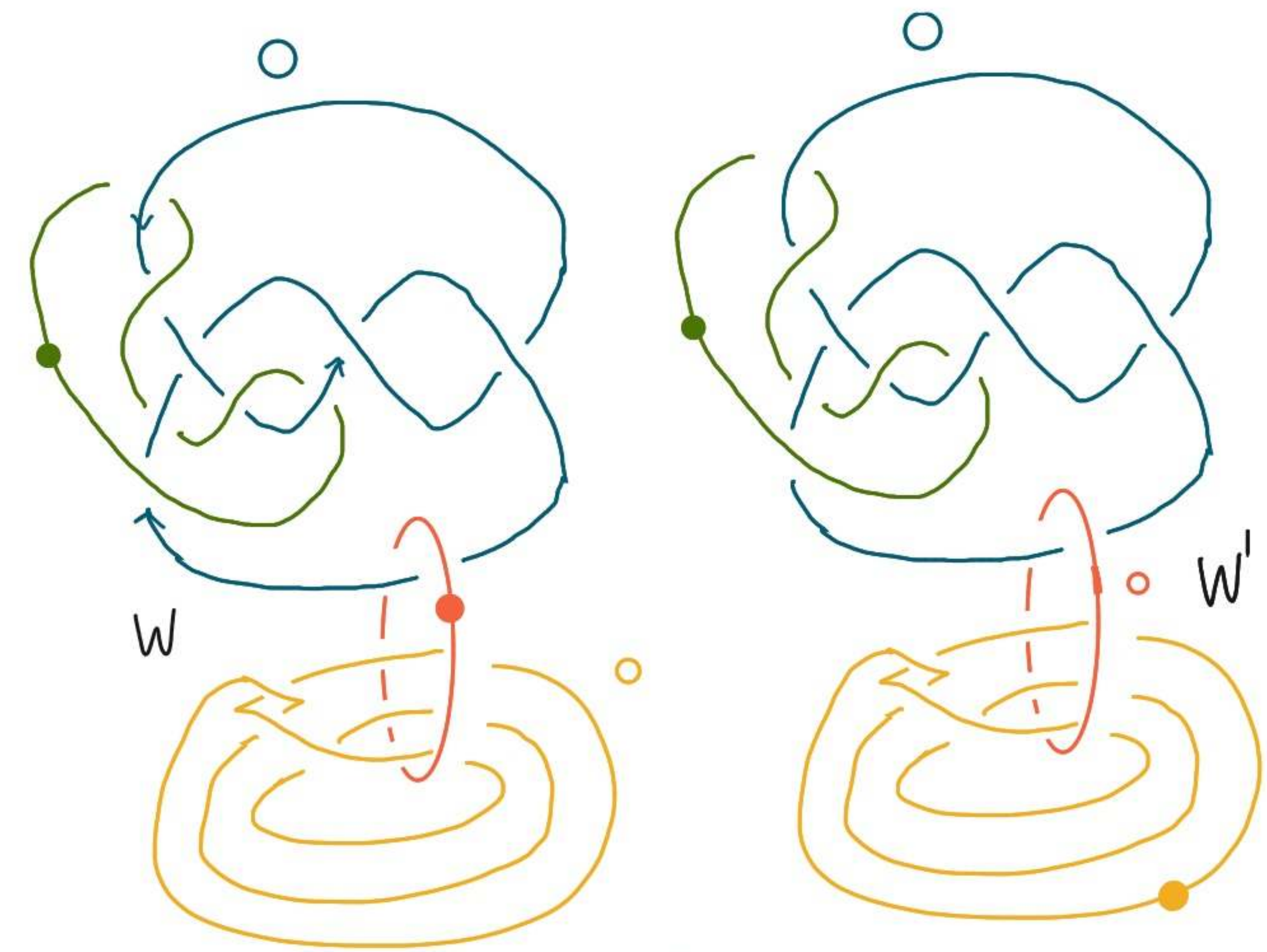
Hence $\exists$ homeo $F : W \longrightarrow W'$.

b) Show both W, W' are Mazur-type

- W has a cancelling 1-2 pair (red, blue), hence

  $W \cong_{sm} B^4 \cup 1\text{-h} \cup 2\text{-h}$ (green, yellow)

- After a slide, W' has a cancelling 1-2 pair (yellow, red), hence $W' \cong_{sm} B^4 \cup 1\text{-h} \cup 2\text{-h}$ (green, blue)

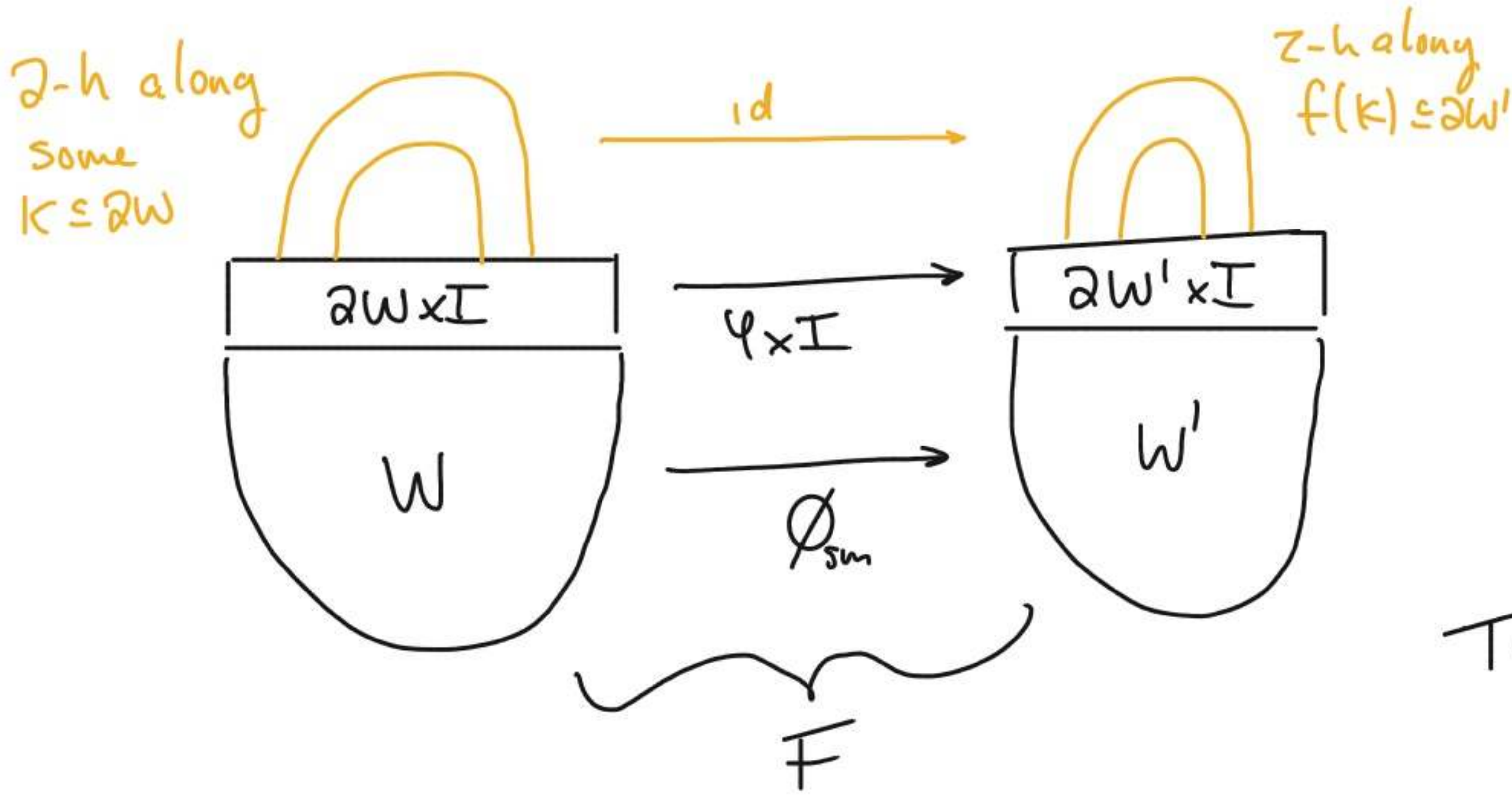


c) Show $W \not\cong_{sm} W'$

If $\phi_{sm}: W \to W'$, induce $\varphi: \partial W \to \partial W'$.

Given a fixed $\varphi: \partial W \to \partial W'$, sometimes can show there is no ϕ_{sm} w/ $\phi|_\partial = \varphi$

φ does not extend smoothly



If $\phi_{sm}: W \to W'$ then get

$F_{sm}: W \cup 2\text{-h} \longrightarrow W' \cup 2\text{-h}$.

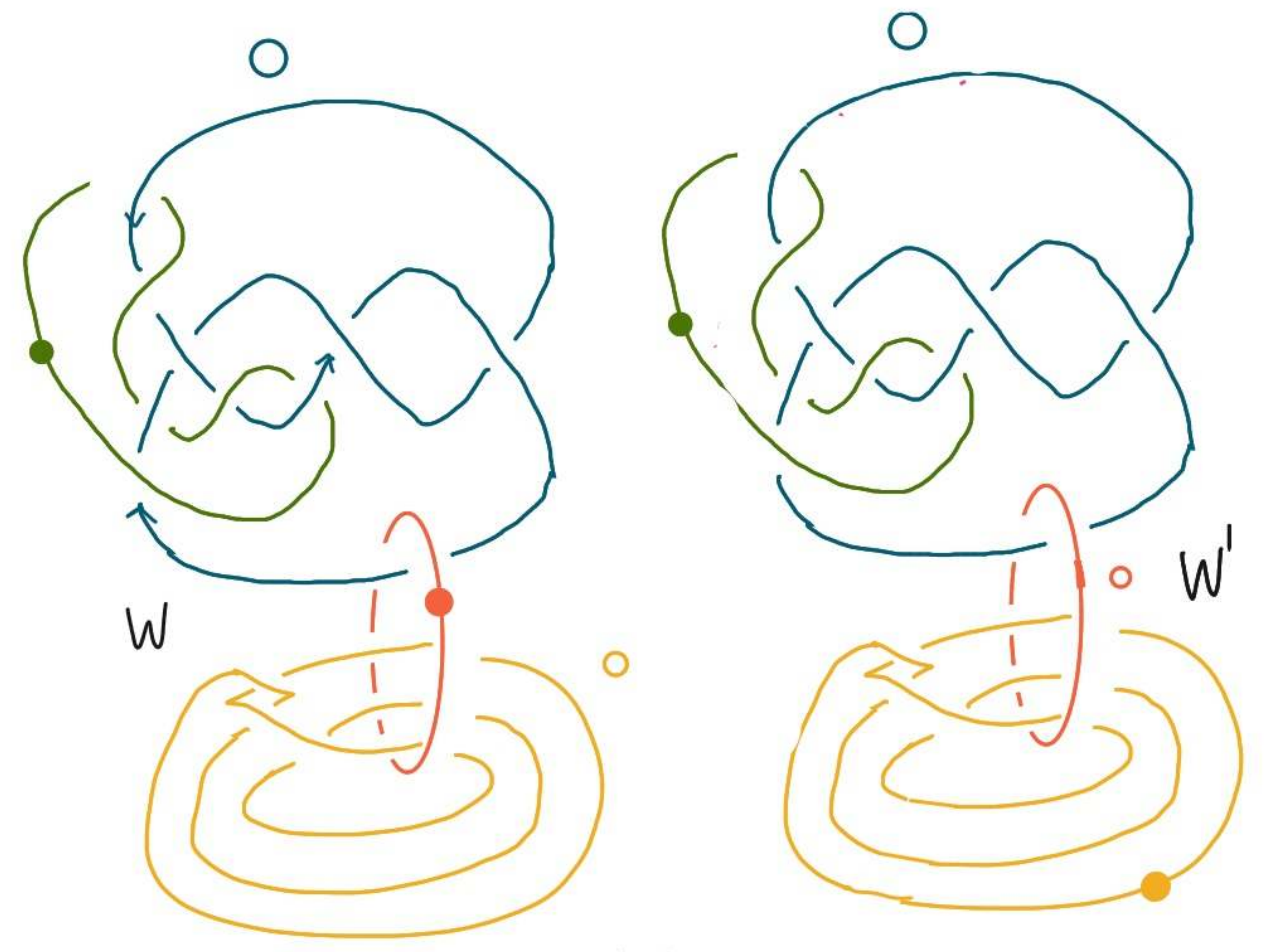
Maybe you can show are not diff?

These are $\simeq S^2$, good news.

To show $W \not\cong_{sm} W'$, show <u>every</u> $f: \partial W \to \partial W'$ does not extend

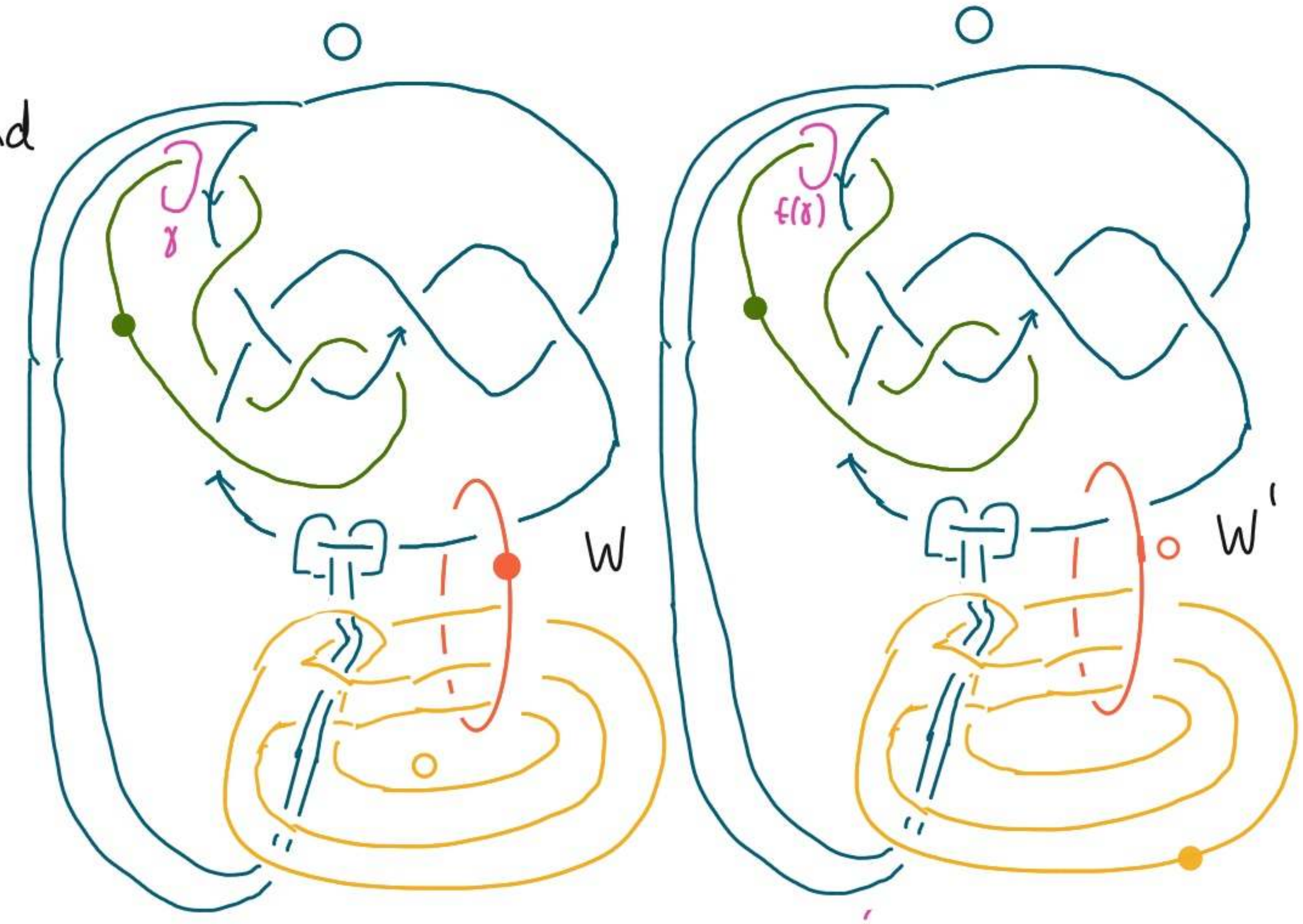
c) Show every $\varphi : \partial W \to \partial W'$ does not extend

wouldn't it be great if $MCG(\partial M) = 1$? it's not.



c) Show every $\varphi : \partial W \to \partial W'$ does not extend

wouldn't it be great if $MCG(\partial M) = 1$? it is!



Consider $W \cup_\gamma 2\text{-h}$ & $W' \cup_{\gamma'} 2\text{-h}$,
in both cases pink & green are cancelling 1-2 pair.

Recall: b) Show both W, W' are Mazur-type

- W has a cancelling $\overset{\text{red}}{1}\text{-}\overset{\text{blue}}{2}$ pair, hence $W \cong_{sm} B^4 \cup \underset{\text{green}}{1\text{-h}} \cup \underset{\text{yellow}}{2\text{-h}}$ $\Big\}$ $W \cup_\gamma 2\text{-h} \cong B^4 \cup \underbrace{2\text{-h}}_{\text{yellow}} = X(J)$
- After a slide, W' has a cancelling $\overset{\text{yellow}}{1}\text{-}\overset{\text{red}}{2}$ pair. hence $W' \cup B^4 \cup 1\text{-h} \cup 2\text{-h}$ $\Big\}$ $W' \cup_{\varphi(\gamma)} 2\text{-h} \cong B^4 \cup \underbrace{2\text{-h}}_{\text{blue}} = X(J')$

I'm not gonna draw them, but these are explicit

Then suffices to show $X(J) \not\cong_{sm} X(J')$

a concordance invariant from Heegaard Floer homology (+)
(knot)

c) Show $X(J) \not\cong_{sm} X(J')$

<u>Theorem</u> (Hayden-Mark-P.) $\nu(K)$ is an invariant of $X(K)$

Comments
- $\exists$ computer program for computing $\nu(K)$ for $c(K) < 100$ (Szabo)
- Compare $s \in Kh(K)$ – Theorem is false (P)
  – computer program for $c(K) < 40$ (Morrison)
- Concordance invts which are not knot trace invariants are useful for stuff

//

pf of c) show $\nu(J) = 2$, $\nu(J') = 1$ (we actually dont use computer)

Take aways: Dot-zero trick + Freedman to give homeomorphism

Cancelation tricks to get Mazur-type

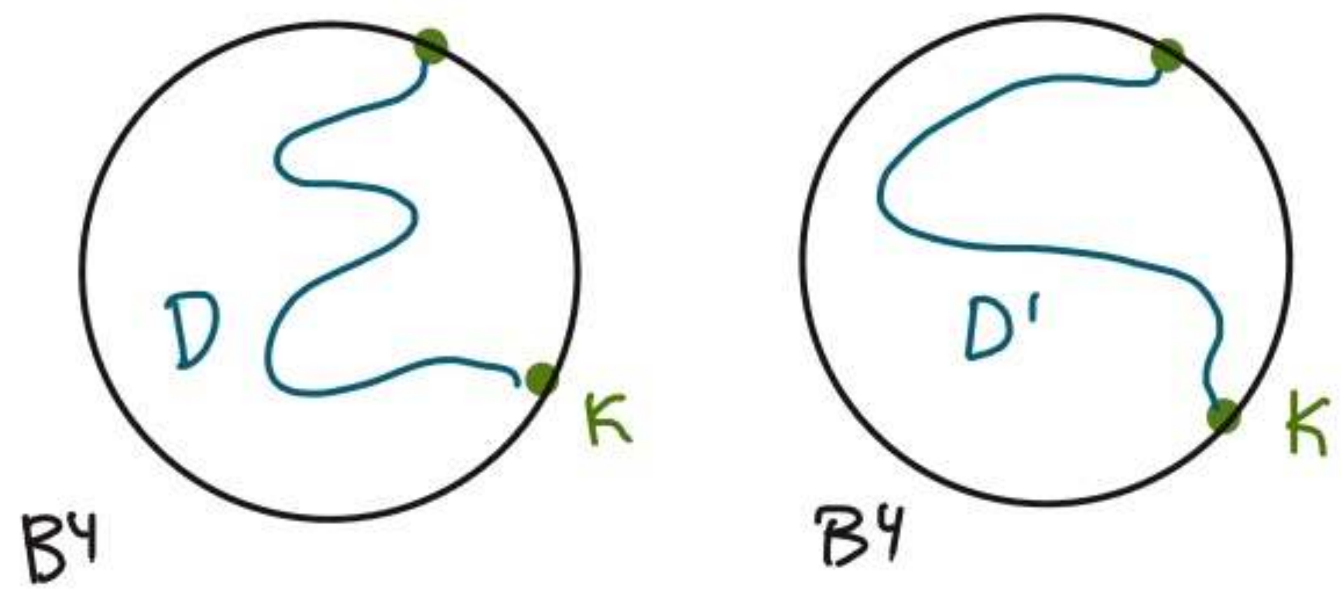
dont require, work harder

Show non diffeo by showing ! ∂ homeo doesn't extend.
- show that ∂ homeo extends $\Longrightarrow X \cong_{sm} X' \simeq S^2$
- have tools for obstructing diffeo of htpy S^2s

Stein adjunction, ν

Theorem: (Hayden, '20): $\exists\ W \underset{TOP}{\cong} W' \simeq S^1,\ W \not\cong_{sm} W'$ and

$\exists$ disks $D, D' \hookrightarrow B^4$ s.t. $\partial D = \partial D' = K$ & $W = B^4 \setminus \nu(D),\ W' = B^4 \setminus \nu(D')$



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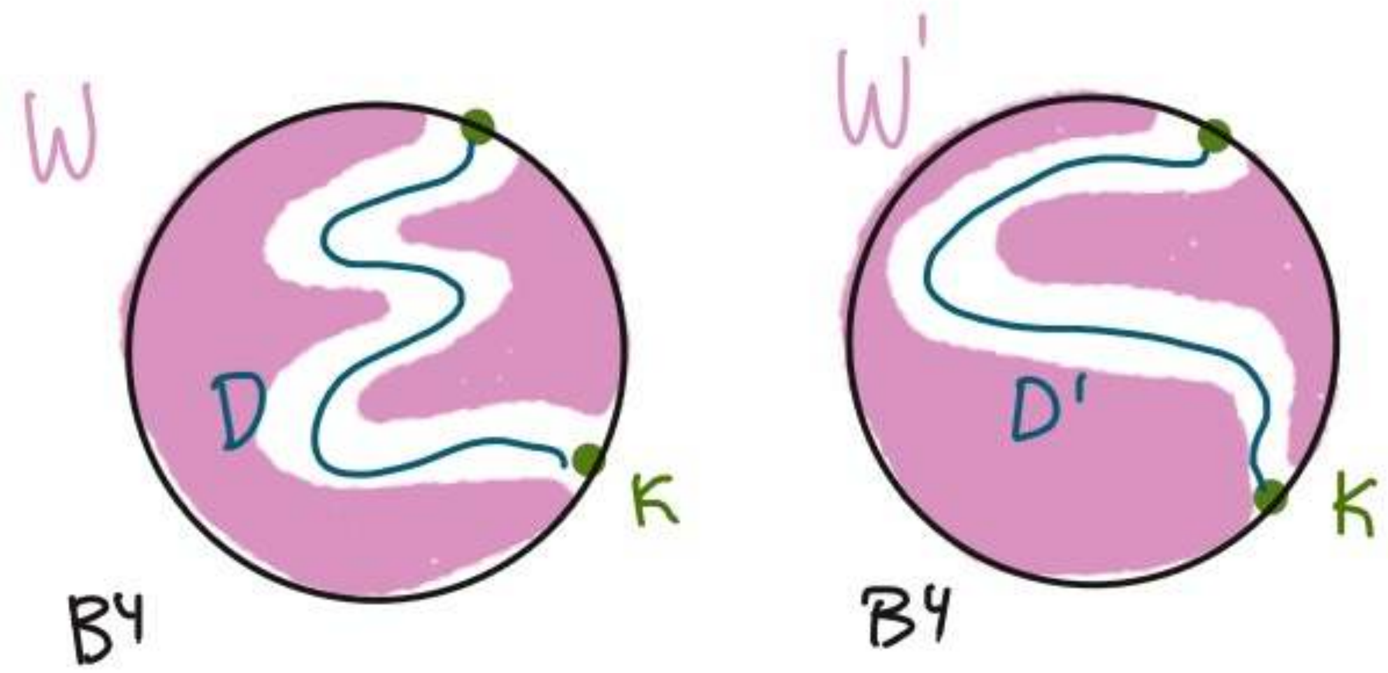


Theorem (Conway-Powell '19): Any $D, D' \xhookrightarrow[top]{} B^4$ w/ $\pi_1(B^4 \setminus \nu(D)) = \pi_1(B^4 \setminus \nu(D')) = \mathbb{Z}$ are top isotopic (rel ∂)

Cor: $\exists$ pairs of disks in B^4 which are top isotopic (rel ∂) but not smoothly isotopic (rel ∂)

dot-zero surgery

pf: a) Build D, D' w/ same ∂, $\pi_1(B^4 \setminus \nu(D^{(\prime)})) = \mathbb{Z}$. C-P $\Longrightarrow$ $W := B^4 \setminus \nu(D) \cong_{TOP} W' := B^4 \setminus \nu(D')$

b) Show W, W' not diffeomorphic

some knot

i) Check $MCG(\partial W) = 1$

ii) Show $W \cup_L$ two 2-hs $\cong_{sm} X(J)$ (for good choice of L)

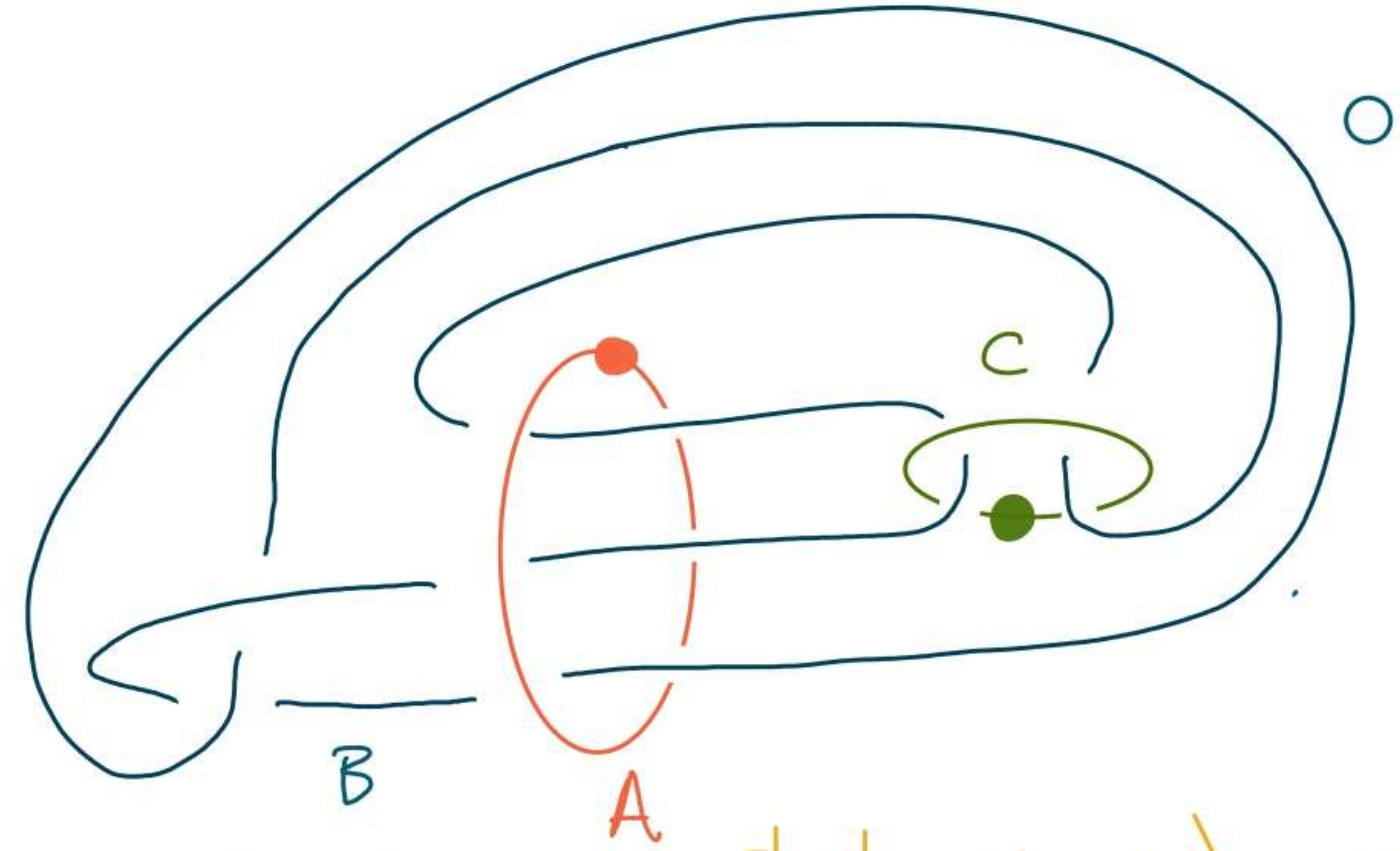
& $W \cup_{f(L)}$ two 2-hs $\cong_{sm} X(J')$ some knot

iii) Show $X(J) \not\cong_{sm} X(J')$

- Many examples where v works, Stein adjunction doesn't apply
- are exs w/ Stein adjunction

a) Let $A \cup B \cup C$ a 3-comp link in S^3 s.t.

- $A \cup B \cong$ hopf
- $B \cup C \cong U^2$
- $A \cup C \cong U^2$



$\pi_1(N_{D'}) = \langle a, c : c a^{-1} c^{-1} a . a^{-1} = 1 \rangle = \mathbb{Z}$

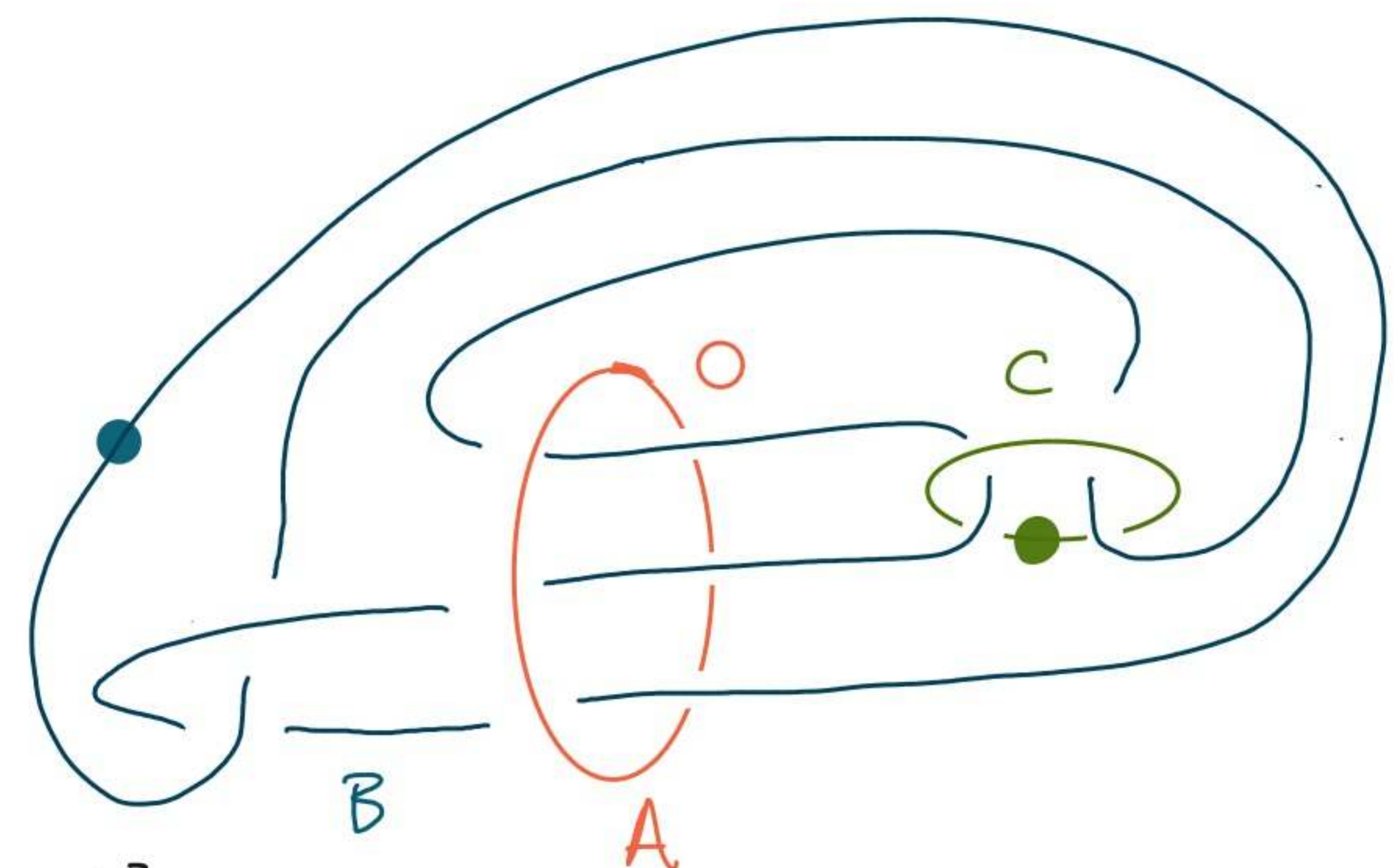
Consider ∂ manifolds det. by A:

Observe that $\partial N_{D'} \cong \partial N_D$.

Looking at $N_{D'}$, obs $A \cup B \cong B^4$.

So $N_{D'}$ is B^4 w/ a disk (D') removed.

Sim: N_D is B^4 is B^4 w/a disk (D) removed.

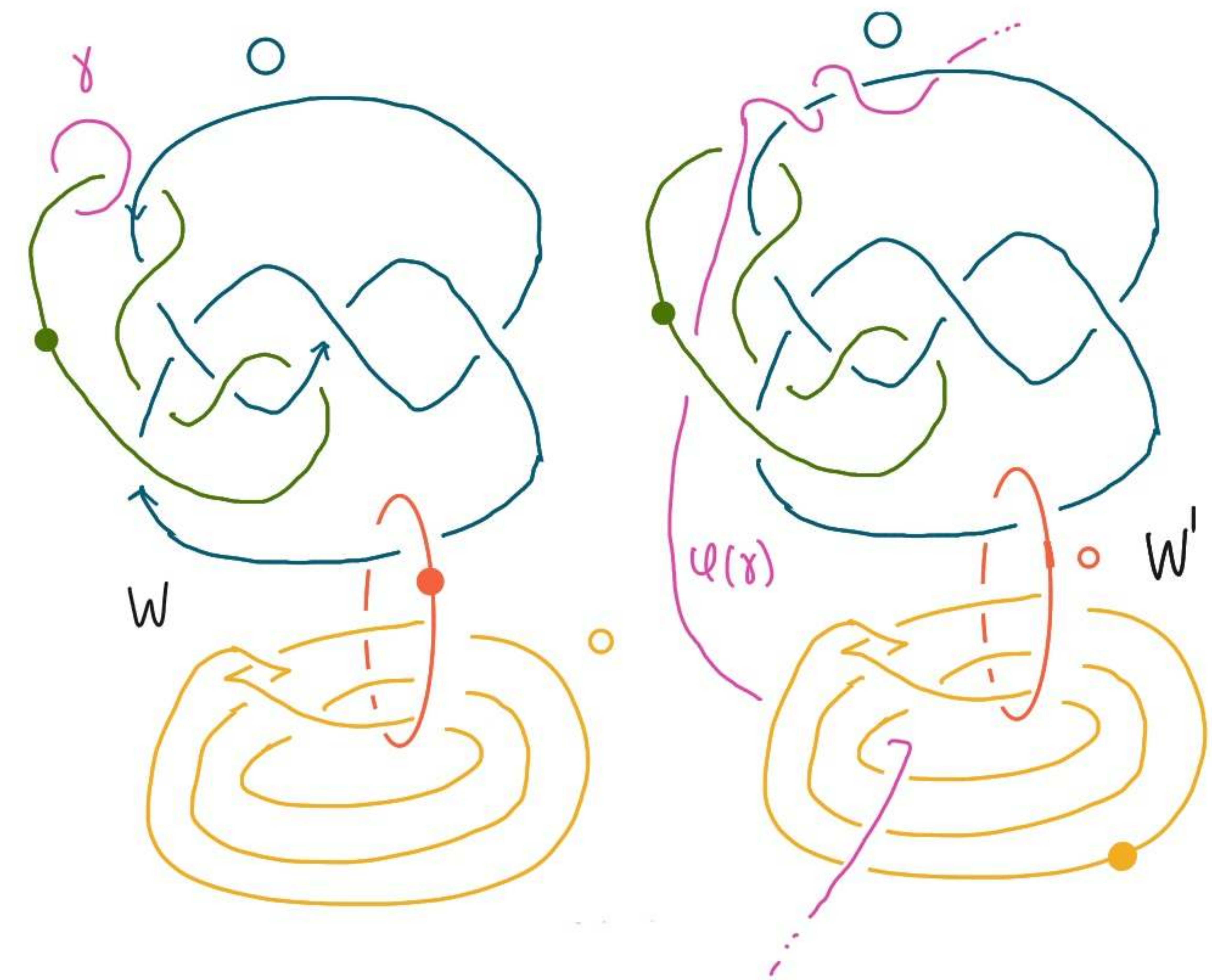


Why is $\partial D \cong \partial D'$?: $\partial D = C \subseteq S^3_{0,0}(A,B) \cong S^3$

$\partial D' = C \subseteq S^3_{0,0}(A,B) \cong S^3$

c) Show every $\varphi : \partial W \to \partial W'$ does not extend

(wouldn't it be great if $MCG(\partial M) = 1$?)



- Show every $\varphi : \partial W \to \partial W'$ has $\varphi(\gamma) \cap D_g = \{pt\}$

  3mfld topology, JSJ decomp

- Consider $W \cup_\gamma$ 2-h, $W' \cup_{\varphi(\gamma)}$ 2-h

  in both, pink & green are cancelling 1-2 pair.

Recall: b) Show both W, W' are Mazur-type

- W has a cancelling $\underset{\text{red}}{1}$-$\underset{\text{blue}}{2}$ pair, hence $W \cong_{sm} B^4 \cup \underset{\text{green}}{1\text{-h}} \cup \underset{\text{yellow}}{2\text{-h}}$ $\Big\}$ $W \cup_\gamma 2\text{-h} \cong B^4 \cup \underset{\text{yellow}}{2\text{-h}} = X(J)$

- After a slide, W' has a cancelling $\underset{\text{yellow}}{1}$-$\underset{\text{red}}{2}$ pair, hence $W' \cong_{sm} B^4 \cup \underset{\text{green}}{1\text{-h}} \cup \underset{\text{blue}}{2\text{-h}}$ $\Big\}$ $W' \cup_{\varphi(\gamma)} 2\text{-h} \cong B^4 \cup \underset{\text{blue}}{2\text{-h}} = X(J')$

I'm not gonna draw them, but these are explicit

- Show that J, J' indep of φ (3mfld topology, JSJ decomp)

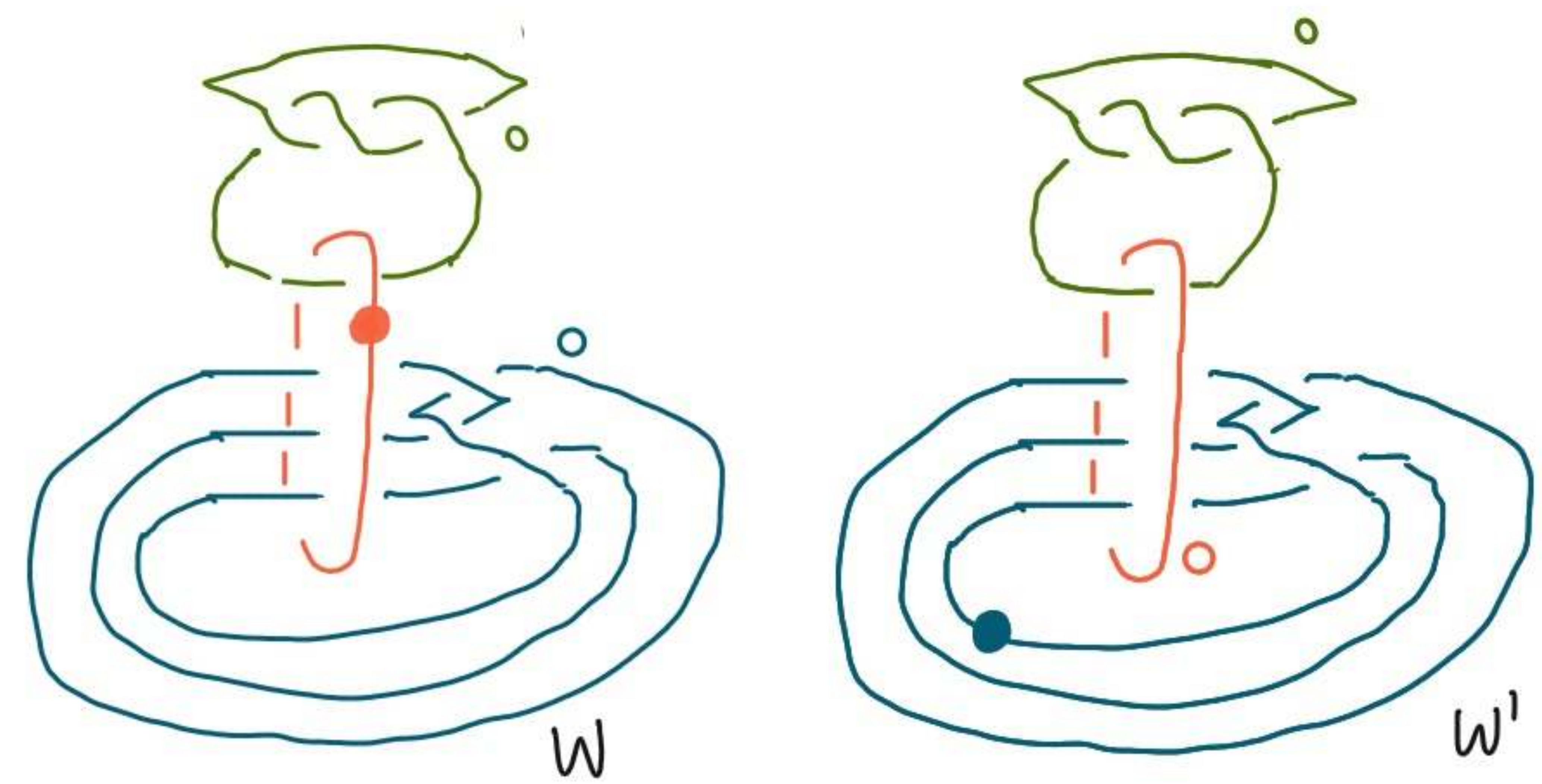
Then suffices to show $X(J) \not\cong_{sm} X(J')$

Thm: (Yasui '15) $\exists$ exotic W^4 homotopy equiv. to S^2, (w/ $Q_W = [0]$)

pf: b) show $W \not\cong_{sm} W'$.

We'll show 1) $\exists\ T^2 \xhookrightarrow[sm]{} W'$ gen H_2 $\quad$ $(=2)$

2) $\min\{g(\Sigma) : \Sigma \hookrightarrow W, [\Sigma] \text{ gen } H_2\} \geq 2$

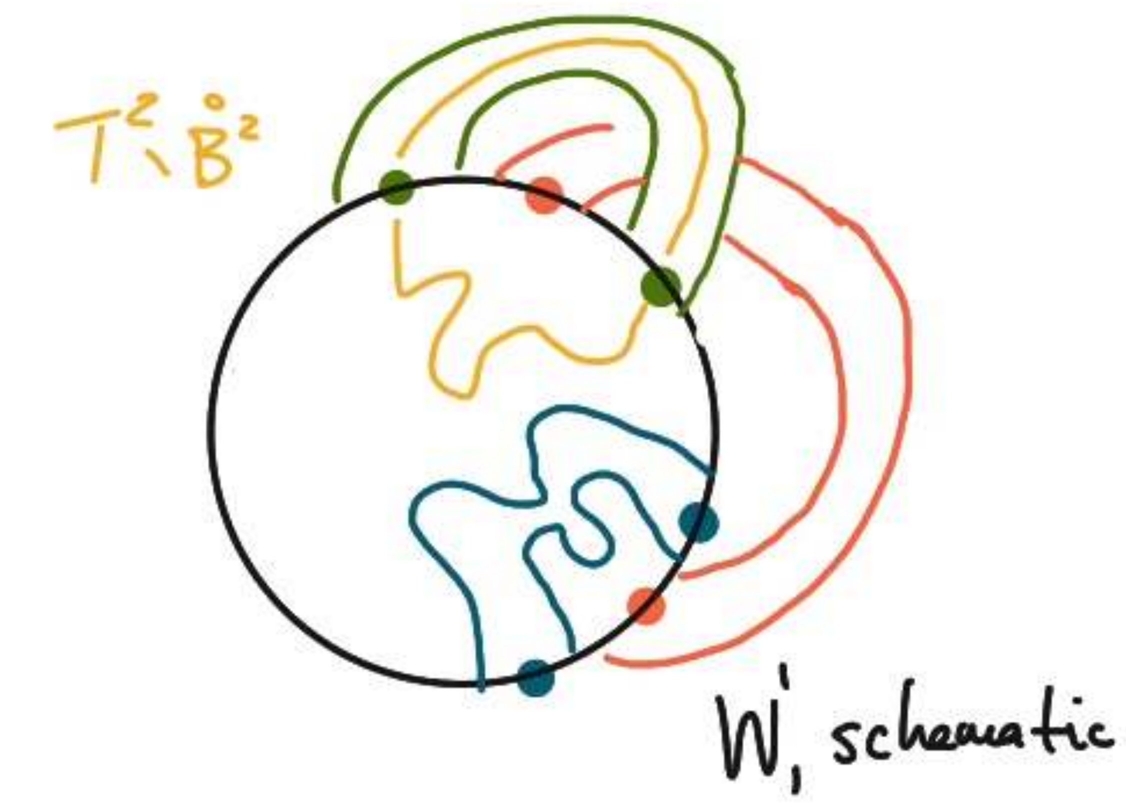


1) RHT bounds genus 1 Seifert surface in S^3

hence bounds $T^2 \setminus \mathring{B}^2 \xhookrightarrow[sm]{} B^4$.

In W', cap off to T^2 gen H_2

2) Need some inherently sm obstruction $\left(F^{-1}(T^2) \xhookrightarrow[TOP]{} W \text{ gen } H_2\right)$



- Eliashberg, '90: If handle diagram of W satisfies some conditions then W admits a Stein structure

has a nice symplectic str

- Lisca-Matić '98 + either
  - Kronheimer-Mrowka '94, Morgan-Szabó-Taubes '96, (Gauge via SW)
  - Lambert-Cole '20 (combinatorial via Khovanov/Lee/Rasmussen)

If W Stein & $\Sigma^2 \xhookrightarrow[sm]{} W$ w/ $[\Sigma]$ non-torsion then $\chi(\Sigma) \leq \langle c_1(\omega), \Sigma \rangle - [\Sigma]\cdot[\Sigma]$

can be read off of Eliashberg diagram

"Stein adjunction inequality"

Thm: (Yasui '15) $\exists$ exotic W^4 homotopy equiv. to S^2, (w/ $Q_W = [0]$)

pf: b) show $W \not\cong_{sm} W'$.

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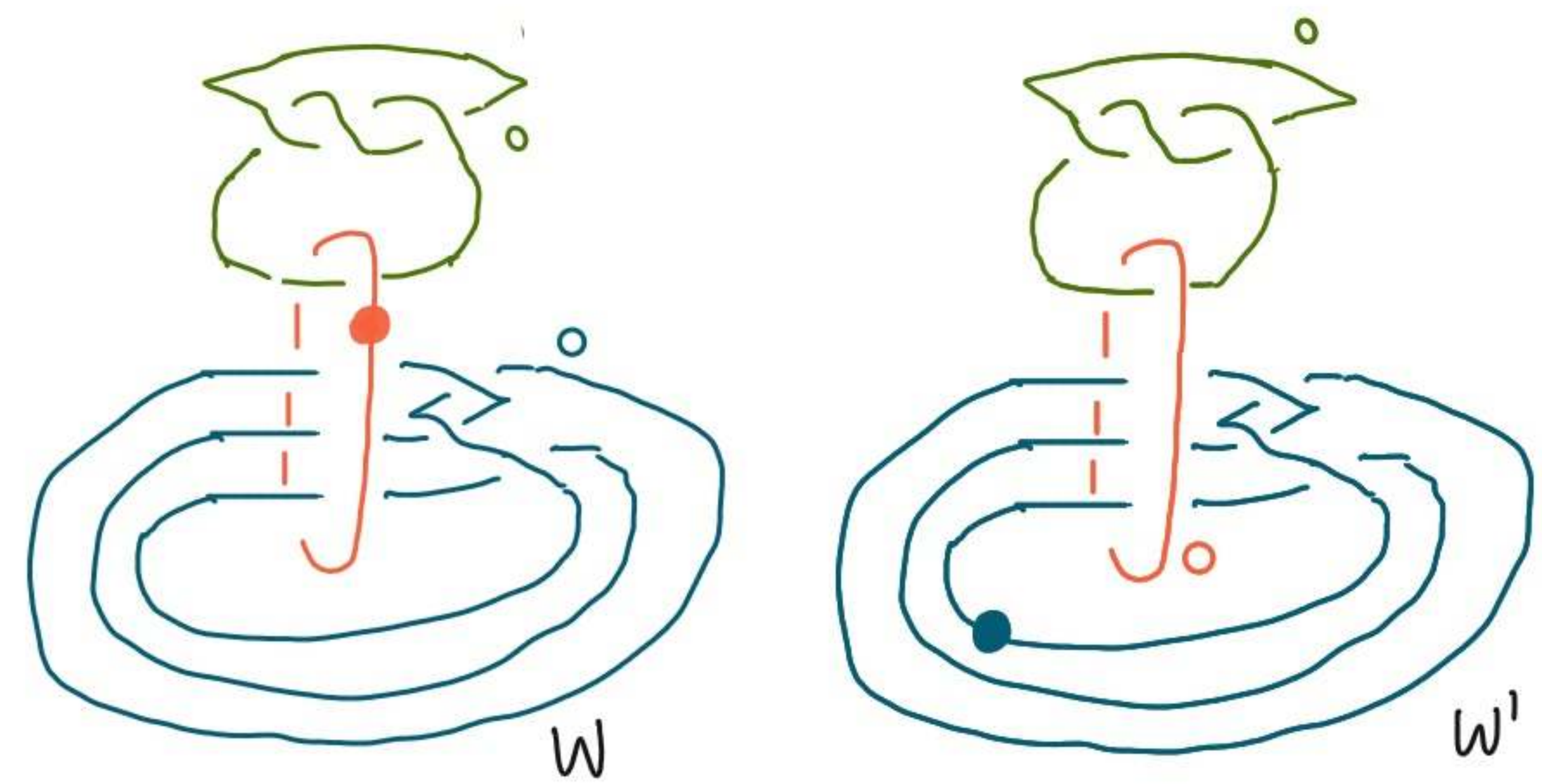

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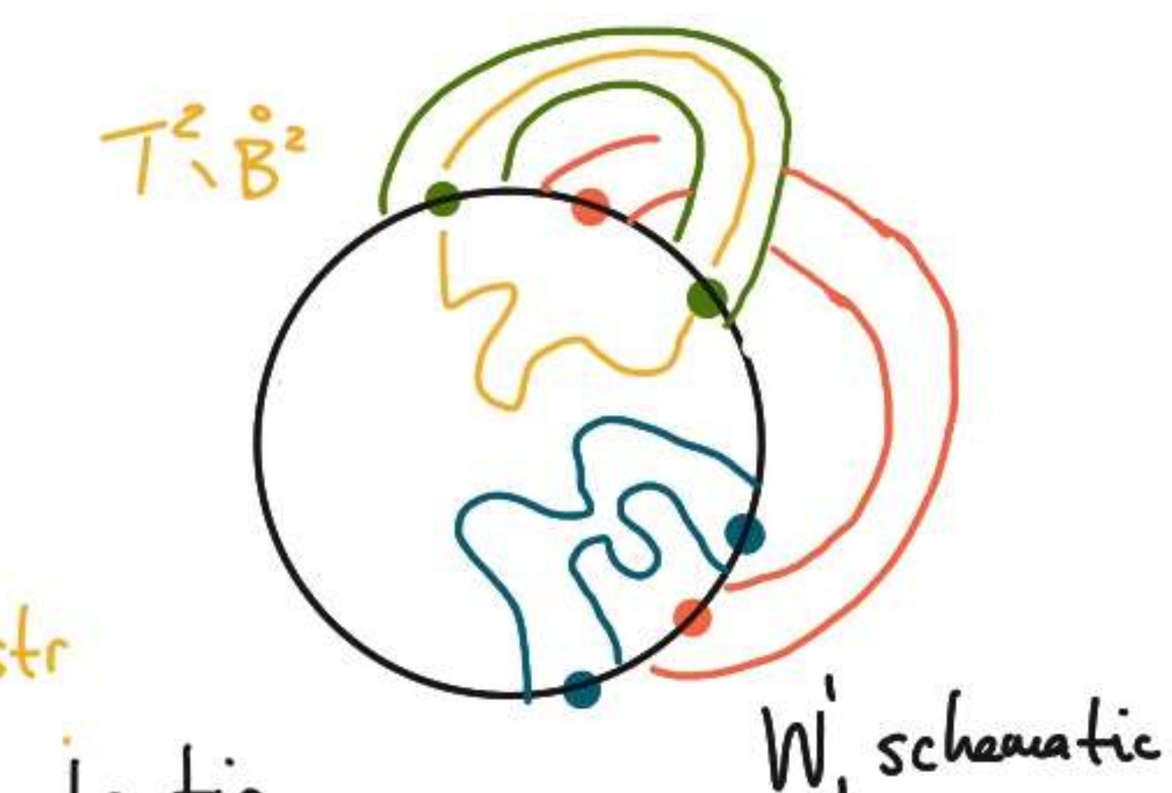

- Eliashberg, '90: If handle diagram of W satisfies some conditions then W admits a Stein structure

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- Lisca-Matić '98: Stein W admits $W \xhookrightarrow[sm]{i} X$ for X v. nice closed symplectic, and for $W \simeq S^2$, $i_* : H_2(W) \to H_2(X)$ is inj.

- Kronheimer-Mrowka '94, Morgan-Szabó-Taubes '96, Ozsváth-Szabó '00 (Gauge via SW)

Lambert-Cole '20 (combinatorial via Khovanov/Lee/Rasmussen)

If $\Sigma^2 \xhookrightarrow[sm]{} X$ nice closed symp. & $[\Sigma]$ non-torsion then $\chi(\Sigma) \leq \langle c_1(\omega), \Sigma\rangle - [\Sigma]\cdot[\Sigma]$

- Crash course in handle things, in partic cancelling 1-2 pairs & dots & ∂
- cork/dot-zero, absolute vs/rel,
- $X_n(K)$, adjunction w/ & w/out gauge. $\vee$.
- Mazur
- disks
- strong corks?

Theorem (Lisca–Matić '98): If $n < \overline{tb}(K)$ then $n + r(\mathcal{K}) \leq 2g_n^{sh}(K) - 2$ for $\mathcal{K}$ any Legendrian rep of K with $n < tb(\mathcal{K})$

- Lisca–Matić '98 ∃ either
  - Kronheimer–Mrowka '94, Morgan–Szabó–Taubes '96, (Gauge via SW)
  - Lambert-Cole '20 (combinatorial via Khovanov/Lee/Rasmussen)

If W Stein ∃ $\Sigma^2 \underset{sm}{\hookrightarrow} W$ w/ $[\Sigma]$ non-torsion then $\chi(\Sigma) \leq \langle c_1(\omega), \Sigma \rangle - [\Sigma]\cdot[\Sigma]$

can be read off of handle diagram