Building closed exotic 4-manifolds by hand

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All 4mflds SM

X, Z 4-mflds

Y 3-mfld

Σ, Γ 2-mflds

X is <u>exotic</u> if $\exists\ X' \underset{\not\cong_{SM}}{\cong_{TOP}} X$ S4PC: S^4 is not exotic

(Freedman, Donaldson) $\exists$ exotic closed X $\pi_1(X)=1,\ b_2=23$

$\vdots$

(Akhmedov-Park '08) " $\pi_1=1\ \ b_2=3$

Sketch (old school): Need to build candidates X, X' and show
- $X \cong_{TOP} X'$
- $gauge(X) \neq gauge(X')$

Fact: can only compute gauge invts eg for
a) $\equiv 0$
b) geom info eg symplectic
c) result of certain cut+paste operations

Step 1: Build X w/ gauge $\equiv 0$ (eg $\#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$)
Step 2: Start w/ Z symplectic, gauge $\not\equiv 0$. Probably $b_2(Z)$ big. Try to make Z simpler using c) until get $X \simeq_{h.e.} X'$, maintaining gauge $\not\equiv 0$.
gnarly
Step 3: Show (using Freedman) $X \cong_{TOP} X'$. //

Sketch (L.L.P): Build X, X' explicitly out of 2-handles
Compute $HF(X) \not\equiv HF(X')$ explicitly from 2-handles
Show (using Freedman) $X \cong_{TOP} X'$ //

X is <u>exotic</u> if $\exists\ X' \underset{\not\cong_{SM}}{\cong_{TOP}} X$ S4PC: S^4 is not exotic

Thm: $\exists$ exotic closed X w/
- $\pi_1 = \mathbb{Z}$ $b_2 = 11$ In detail
- $\pi_1 = 1$ $b_2 = 10$
- (L.L.P) $\pi_1 = \mathbb{Z}/2$ $b_2 = 4$, <u>definite</u>

$\left.\right\}$ in less detail

first

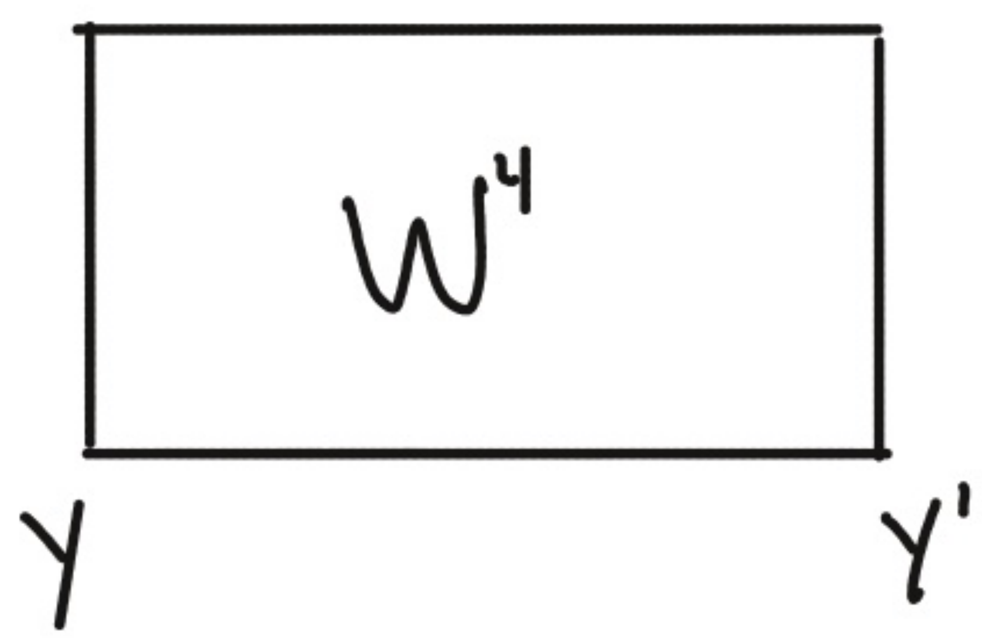
Sketch (L.L.P): Build X, X' explicitly out of 2-handles
Compute $HF(X) \not\cong HF(X')$ explicitly from 2-handles
Show $X \cong_{TOP} X'$ //

Heegaard Floer homology : $Y \rightsquigarrow HF^-(Y)$, a $\mathbb{Z}/2[U]$ module ($\oplus$ spin^c strs)

Oszvath - Szabo

$HF_{red}(Y) :=$ U-torsion submodule.

Fact: a cobordism induces



- $F_W^-: HF^-(Y) \to HF^-(Y')$ ($\oplus$ spin^c strs)
- $F_W^{red}: HF_{red}(Y) \longrightarrow HF_{red}(Y')$

Q: Why not your other favorite TQFT?

A: - explicitly computable

- surgery exact Δ

Old 3-mfld invt: $\alpha(Y) = \mathrm{rank}\big(HF_{red}(Y)\big)$

primitive H_3 class

New 4-mfld invt (L.L.P) For X w/ $b_1(X) > 0$, $\alpha(X, \eta) = \min\{\alpha(Y) : Y \hookrightarrow X \text{ reps } \eta\}$

suspicious...

eg: $\alpha((S^1 \times S^3) \# Z)) = 0$

$\pi_1 = 1$

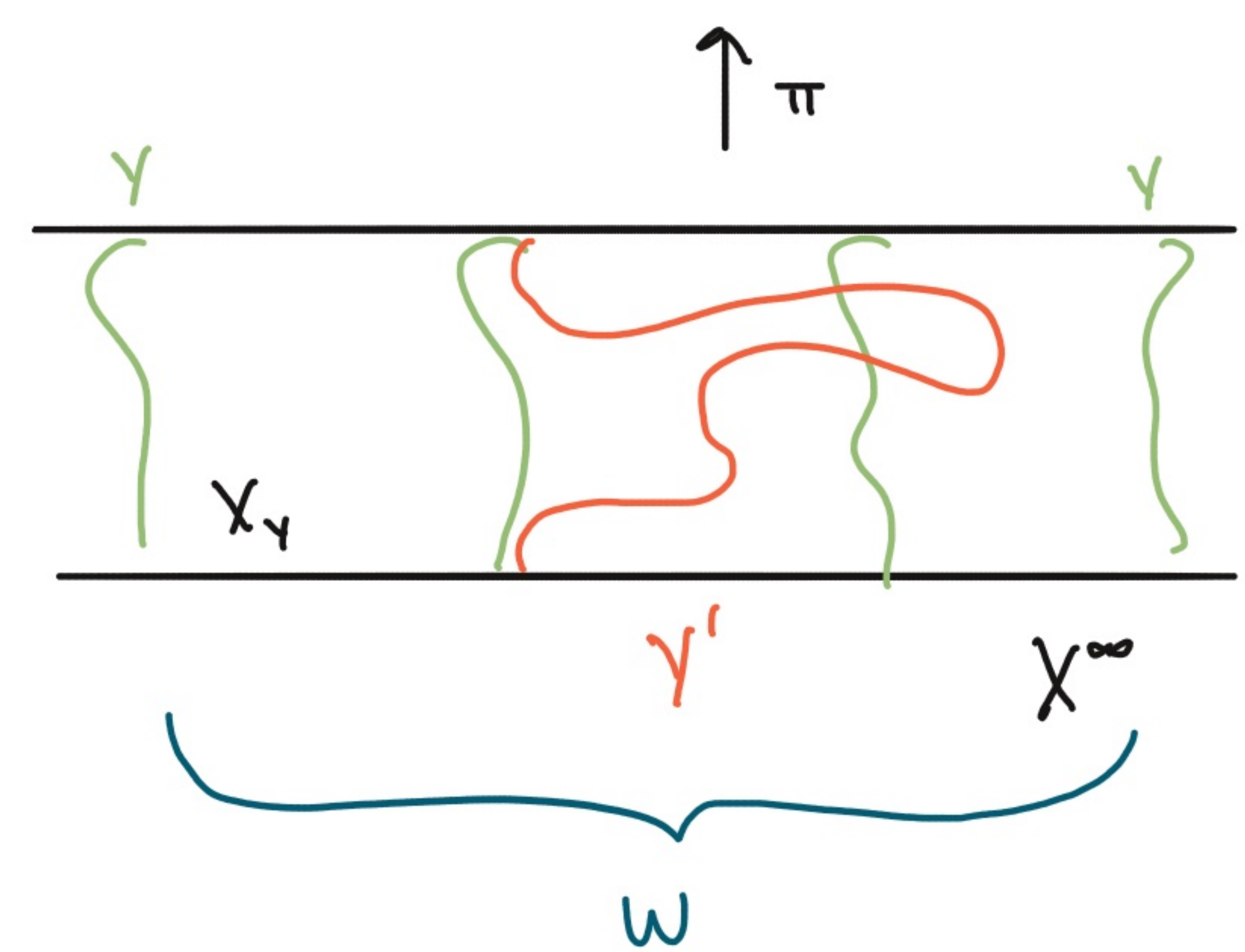
lemma: If $Y \hookrightarrow X$ reps η s.t $F_{X_Y} : HF_{red}(Y) \to HF_{red}(Y)$ is an isomorphism then $\alpha(X, \eta) = \alpha(Y)$

pf: Consider ∞ cyclic cover of X (corr. ker $PD[\eta]$)

Suppose $Y' \hookrightarrow X$ reps η, Y' lifts to X^∞.

$F_W : HF_{red}(Y) \to HF_{red}(Y)$ an isomorphism which factors through $HF_{red}(Y')$

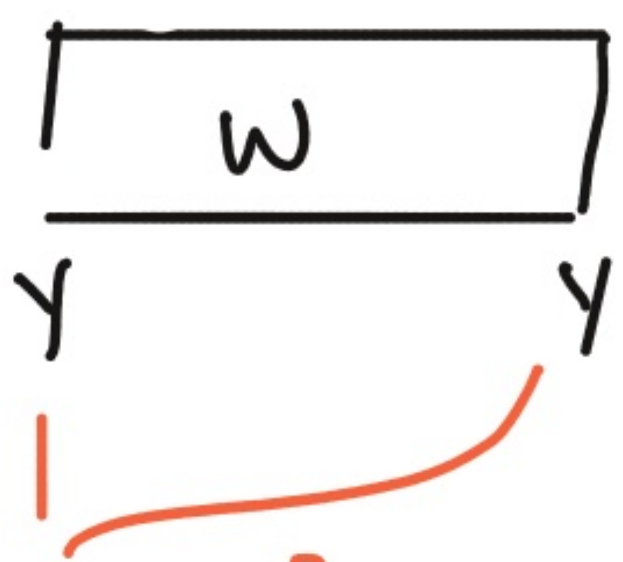
$\implies \alpha(Y') \geq \alpha(Y)$ //



To prove $S^1\times S^3 \# Z$ is exotic

some convenient $\pi_1 = 1$ mfld

Want: a cobordism



a $\mathbb{Z}HS$ S^3

- $\pi_1(W) = 1$ — Define $X := W/y^+\sim y^-$, $\pi_1(X) = \mathbb{Z}$
- $Q_W = Q_Z$ for some Z w/ $\pi_1 = 1$
- with ✱ built from -1 framed 2-handles

  $\xRightarrow{a}$ $X \cong_{TOP} (S^1\times S^3)\# Z$
- ✱ each 2-handle has 0-surgery v. simple

  $\xRightarrow{b}$ F_W an iso $\Longrightarrow$ $\alpha(X) = \alpha(Y)$ (lemma)
- $\alpha(Y) > 0 \Longrightarrow \alpha(X) \neq 0$. Since $\alpha(S^1\times S^3 \# Z) = 0$, $X \not\cong_{sm} S^1\times S^3 \# Z$

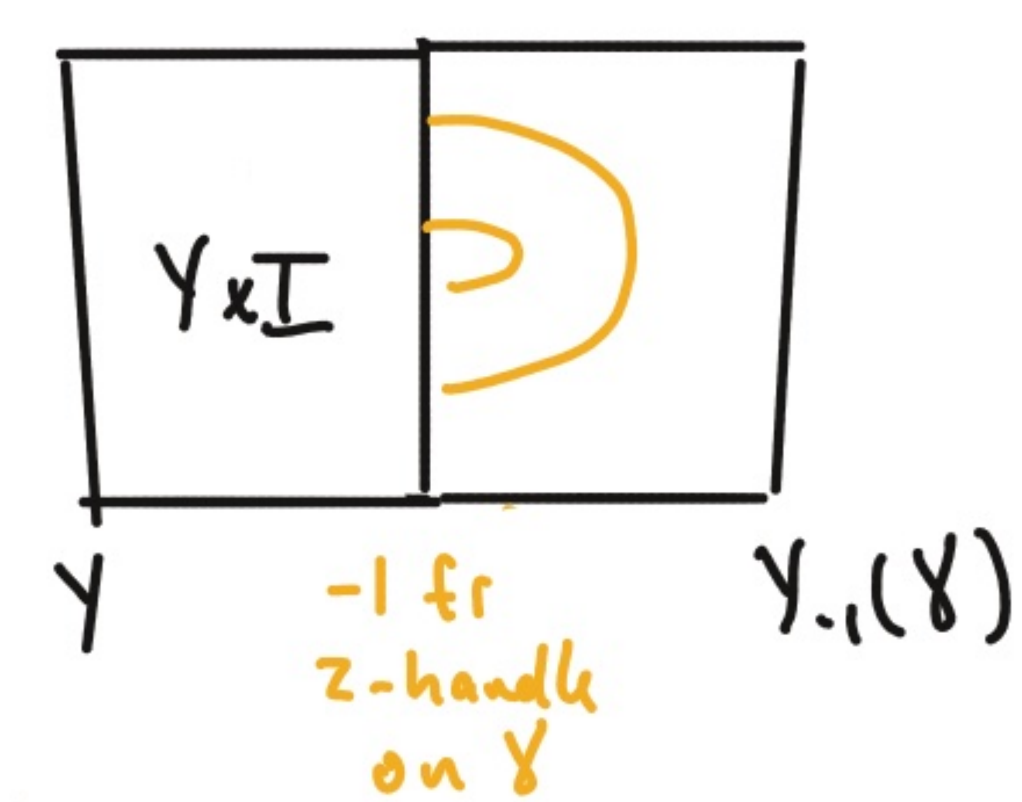
How to build such W? Dictionary of simple cobs w/ ✱, stack.

(a) (Freedman-Quinn '90) For X, X' closed w/ $\pi_1 = \mathbb{Z}$, $X \cong_{TOP} X' \Longleftrightarrow$ same equivariant intersection form

(b) (Oszvath-Szabo '03) Moral: for 2-handle cob W

If $HF_{red}(Y_0(\gamma))$ very small

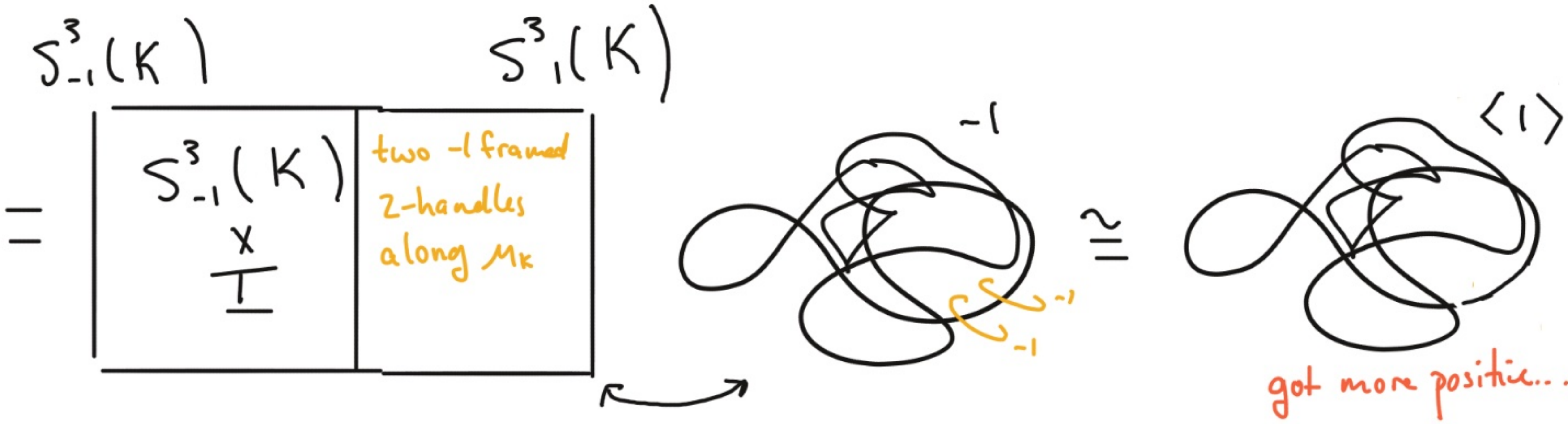
then $F_W^{red}: HF_{red}(Y) \to HF^-(Y_{-1}(\gamma))$ an isomorphism



$$\left(\begin{tikzcd} HF_{red}(Y) \arrow[rr, "F_W"] & \bullet & HF_{red}(Y_{-1}(\gamma)) \arrow[dl] \\ & HF_{red}(Y_0(\gamma)) \arrow[ul] & \end{tikzcd}\right)$$

Building Block #1

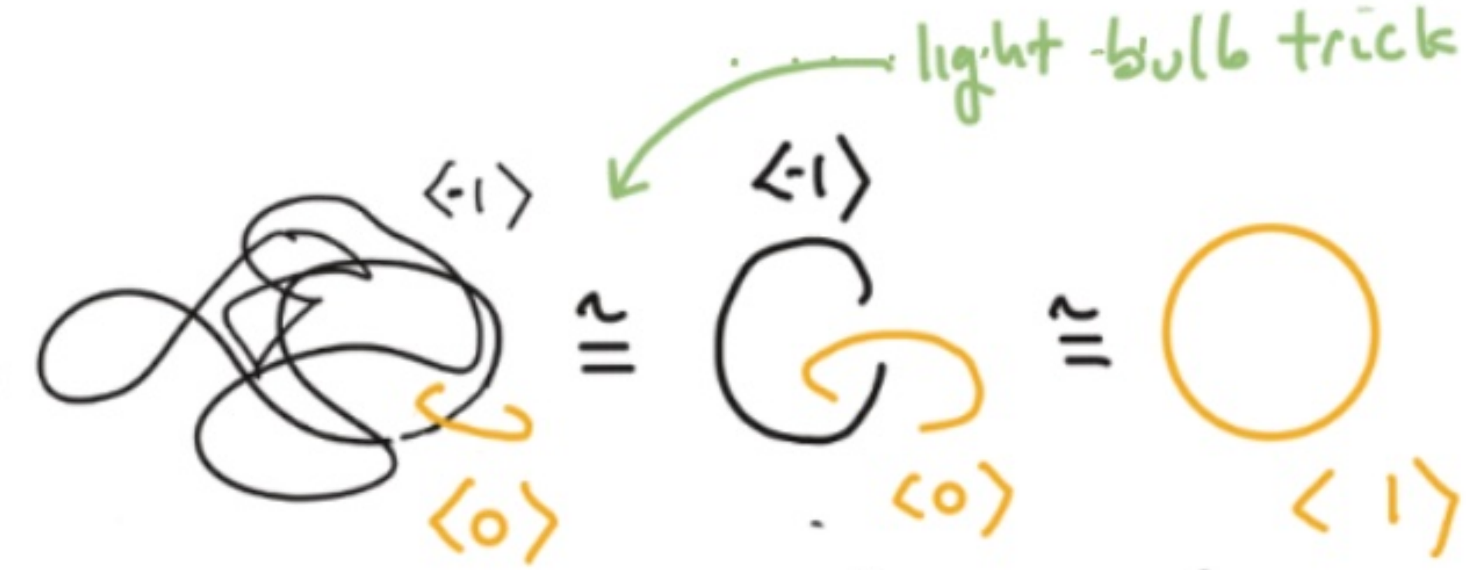
$T_K :=$



Facts

- $\pi_1(T_K) = 1$ (also $\cdot\ H_2(T_K) = \mathbb{Z} \oplus \mathbb{Z}$ $\cdot\ Q_{T_K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

- $\left(S^3_{-1}(K)\right)_0(\mu_K) \cong S^3$ (pf:



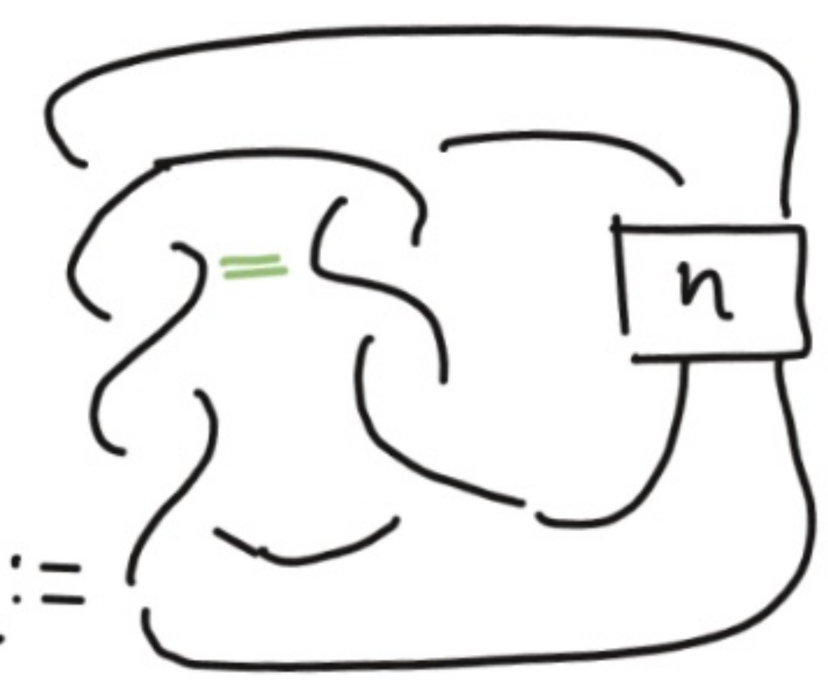
$\cong S^3$)

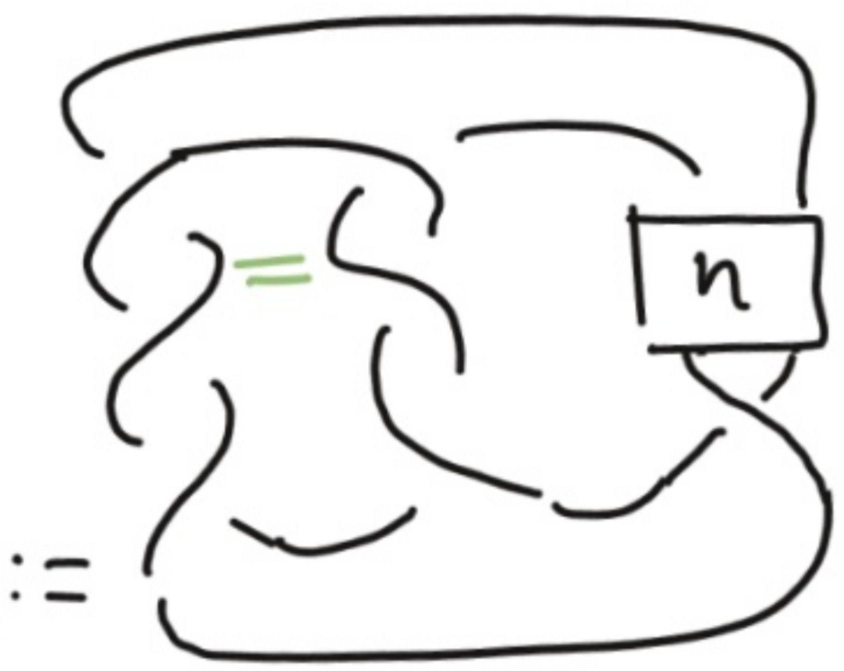
$\Longrightarrow F_{T_K} : HF_{red}(S^3_{-1}(K)) \to HF_{red}(S^3_1(K))$ an iso

Upshot: we like the π_1 and the HF of these T_K cobordisms

Concern: -1 framed 2-hs make ∂^+ "more positive"... not going to get back to Y.

Surgery homeomorphism workaround

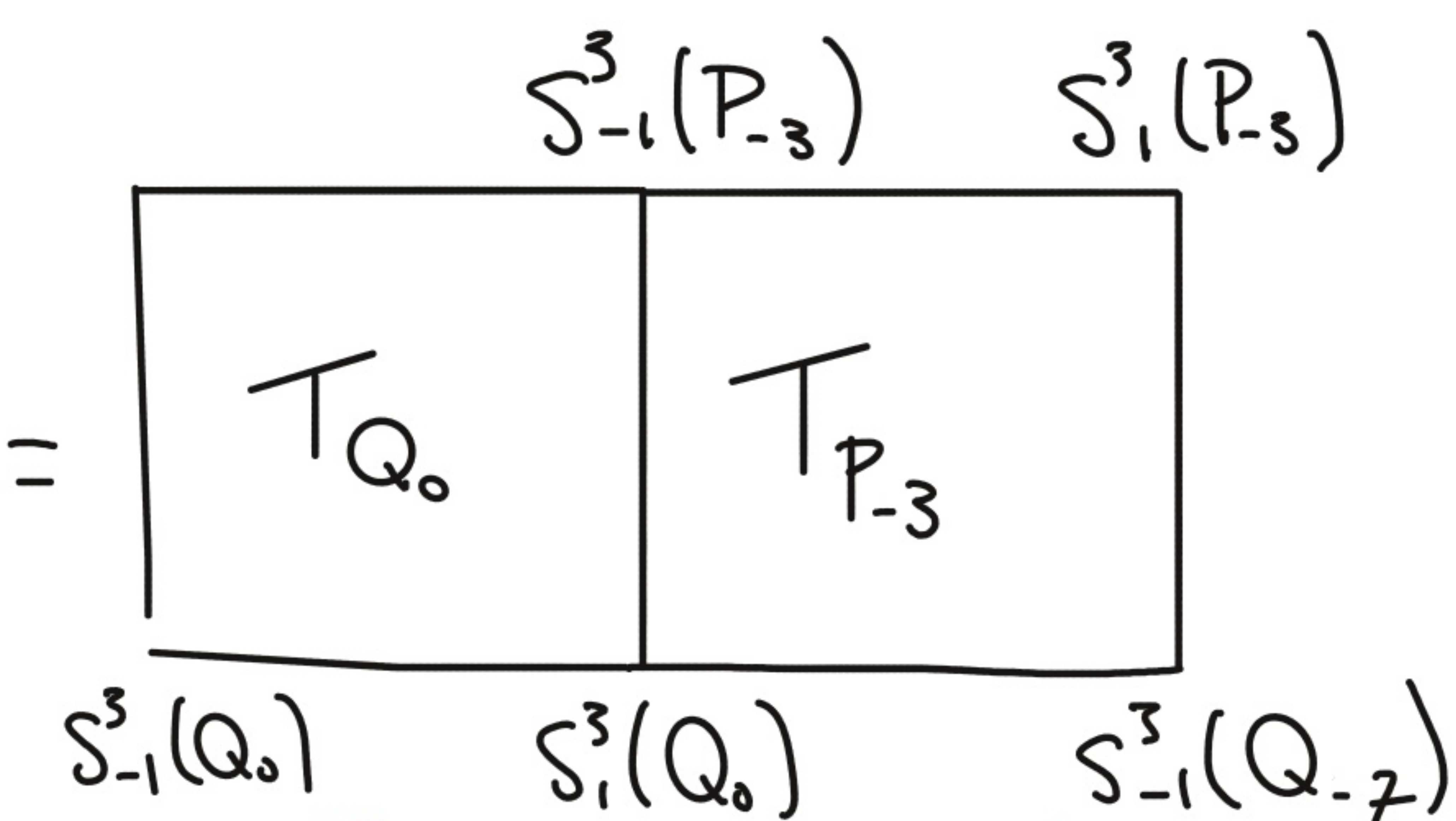
$Q_n :=$ 

$P_n :=$ 

<u>Thm</u> (L.L.P) $S^3_{-1}(Q_{n-4}) \cong S^3_1(P_n)$ and $S^3_{-1}(P_{n-3}) \cong S^3_1(Q_n)$

How does that help...?

- $F^{red}_{W'}$ an iso
- $\pi_1(W') = 1$
- $\alpha(S^3_{-1}(Q_0)) = 2$
- $Q_{W'} = \bigoplus_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

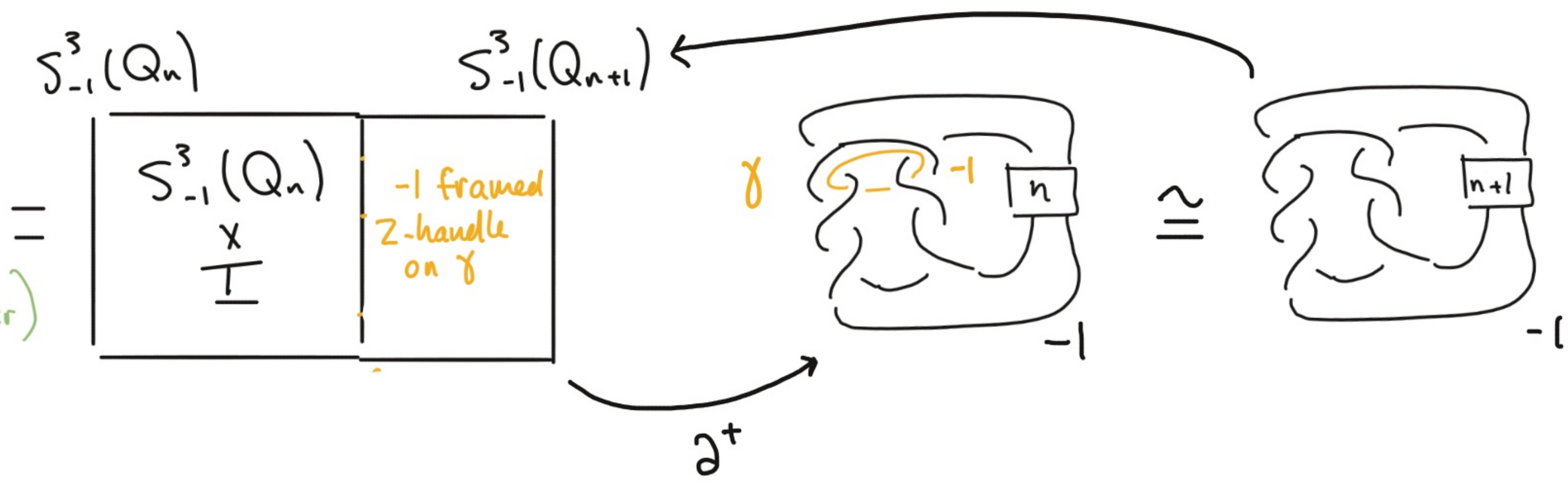
$W' :=$ 

not so different...

Building Block #2

$B_{Q_n} :=$

(Bp$_n$ similar)

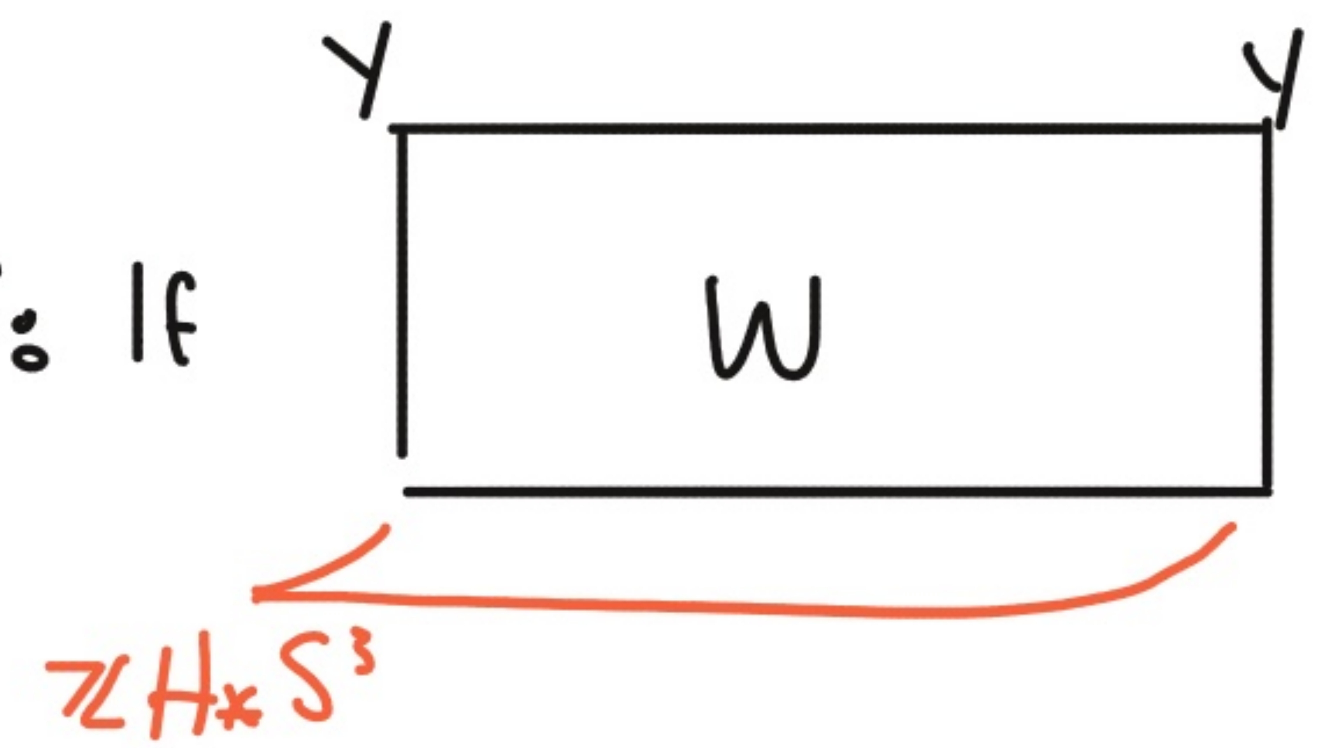


Facts

- $Q_{B_{Q_n}} = [-1]$
- $\left(S^3_{-1}(Q_n)\right)_0(\gamma) \cong S^1 \times S^2$

$\Rightarrow F_{B_{Q_n}} : HF_{red}(S^3_{-1}(Q_n)) \longrightarrow HF_{red}(S^3_{-1}(Q_{n+1}))$ an iso

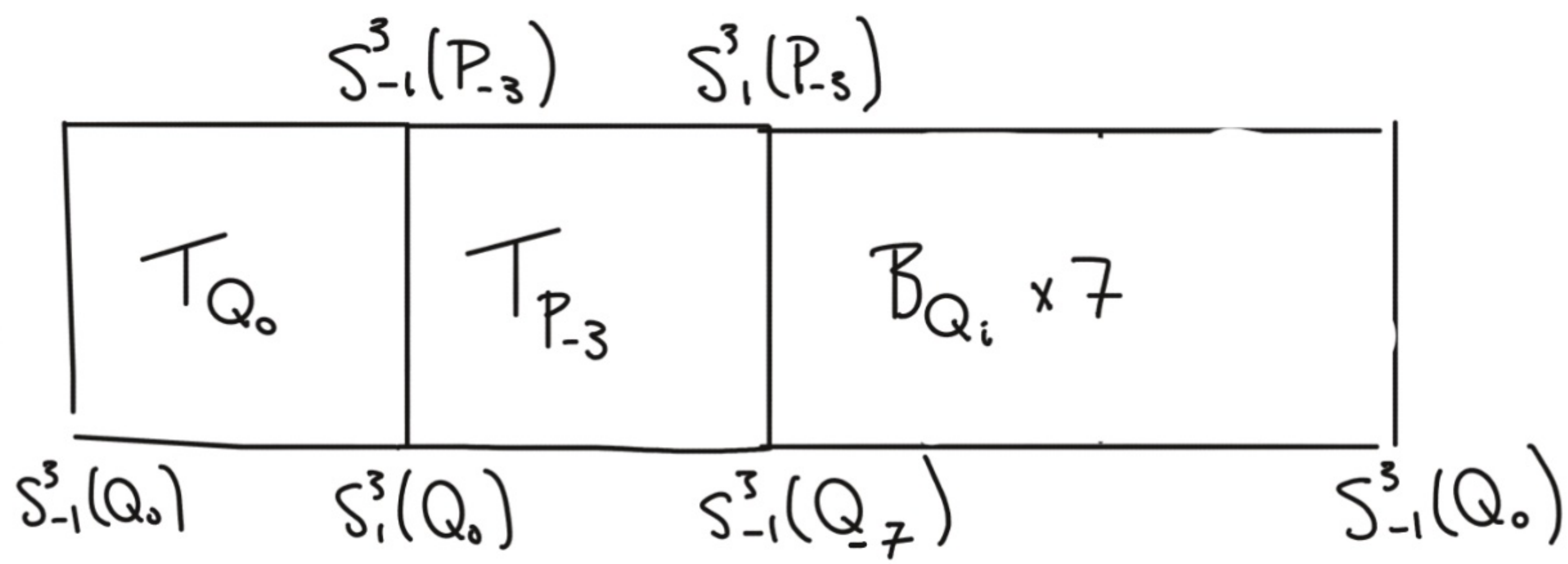
Thm: $\exists X^4 \; \underset{\not\cong_{SM}}{\cong_{TOP}} \; S^1 \times S^3 \#_2 \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$

pf: If 

satisfies

- $\pi_1(W) = 1$ ✓
- $Q_W = Q_{\#_2 \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2}$ ✓
- built from -1 framed 2-handles ✓
- each 2-handle has 0-surgery v. simple ✓
- $\alpha(Y) > 0$ ✓

then $X := W/\partial^+ \sim \partial^-$ has $X \; \underset{\not\cong_{SM}}{\cong_{TOP}} \; S^1 \times S^3 \#_2 \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$.

Define $W :=$ 

$$Q_W = \bigoplus_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_7 (-1) = Q_{\#_2 \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2}$$

//

Thm: $\exists X^4 \underset{\not\cong_{SM}}{\overset{\cong_{TOP}}{}} S^1 \times S^3 \#_2 \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P}^2$

Open: Build smaller ${}_{Y}\boxed{\;W\;}_{Y}$ satisfying wishlist.

Def^n: Knot surgery on $T^2 \hookrightarrow Z^4$ w/ knot K is a cut-paste operation

$$Z_{T,K} := (Z \setminus \nu(T)) \cup (S^3 \setminus \nu(K) \times S^1)$$

Thm (Fintushel–Stern '97): For any Z w/ $\pi_1(X)=1$, $b^+(Z)>0$

and any $T^2 \hookrightarrow$ fishtail nbhd $\overset{*}{\hookrightarrow} Z$, if K has $\Delta_K(t) \neq 1$

$$Z_{T,K} \underset{\not\cong_{SM}}{\overset{\cong_{TOP}}{}} Z$$

Open: what if $\Delta_K(t) = 1$? $(K \neq U)$

Thm (L.L.P): $\exists\ T^2 \hookrightarrow$ fishtail nbhd $\overset{*}{\hookrightarrow} X$ s.t. $\forall$ nontrivial K

$$X_{T,K} \underset{\not\cong_{SM}}{\overset{\cong_{TOP}}{}} X$$

$*$ $[T] \neq 0 \in H_2(X)$

Simply connected exotica

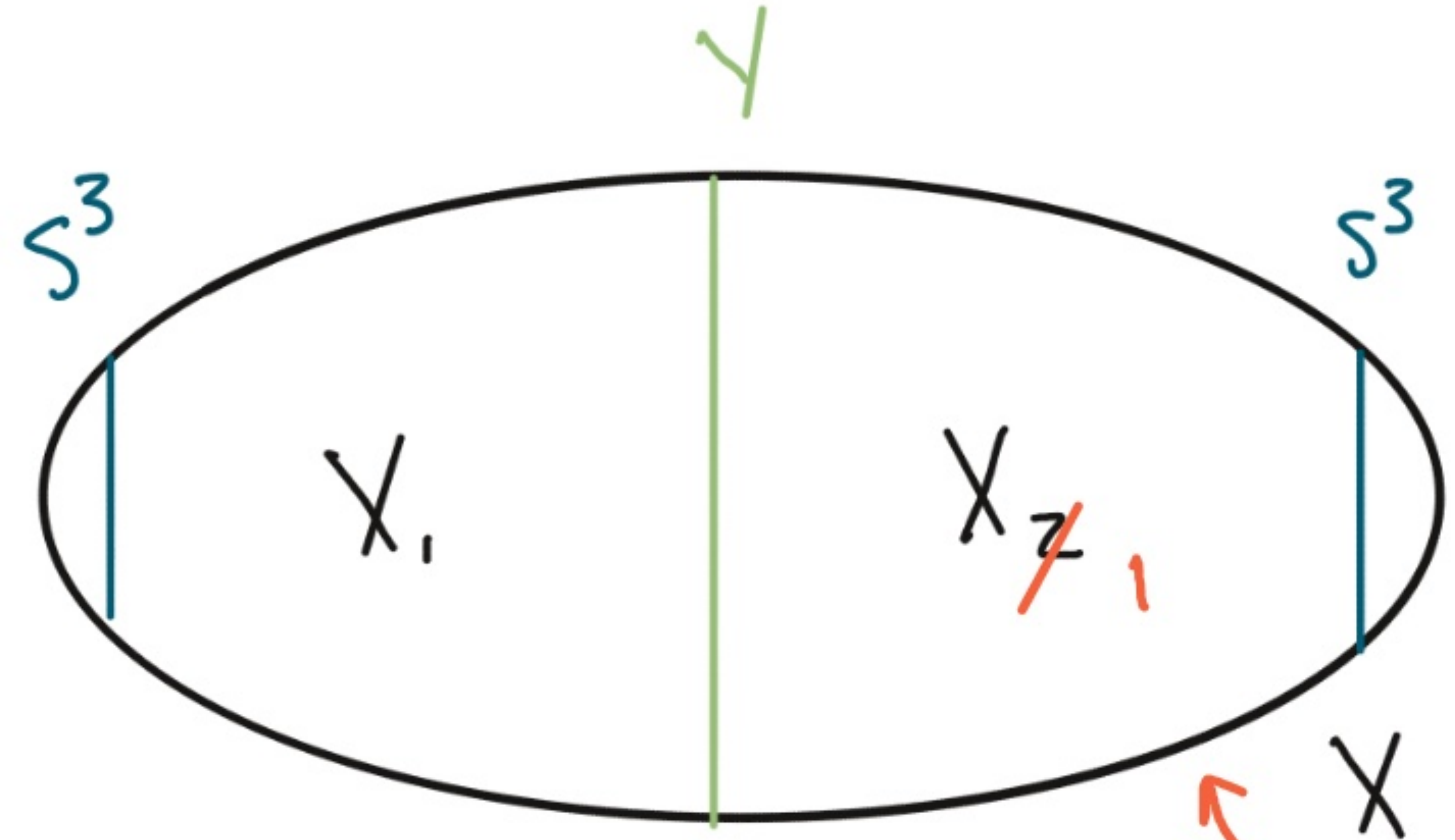
(Oszvath-Szabo)

For X^4 w/ $b_2^+ = 1$,

$\left(\begin{array}{l} \ell \in H_2(X) \text{ w/ } \ell\cdot\ell = 0 \\ + \ \text{spin}^c \text{ str } \mathfrak{s} \text{ on } X^{*} \end{array} \right)$

$\Theta_{OSz}(X, \ell, \mathfrak{s}) := b \circ a(1)$

is an invariant of $(X, \ell, \mathfrak{s})$



$\mathbb{Z}/2[U] \cong$

$F_{X_1} : HF^-(S^3) \xrightarrow{a} HF_{red}(Y, \mathfrak{s}|)$

$\cong$

$F_{X_2} : HF_{red}(Y, \mathfrak{s}|) \xrightarrow[a^*]{b} HF^+(S^3)$

$= \mathbb{Z}/2 \qquad \cong \mathbb{Z}/2[U]$

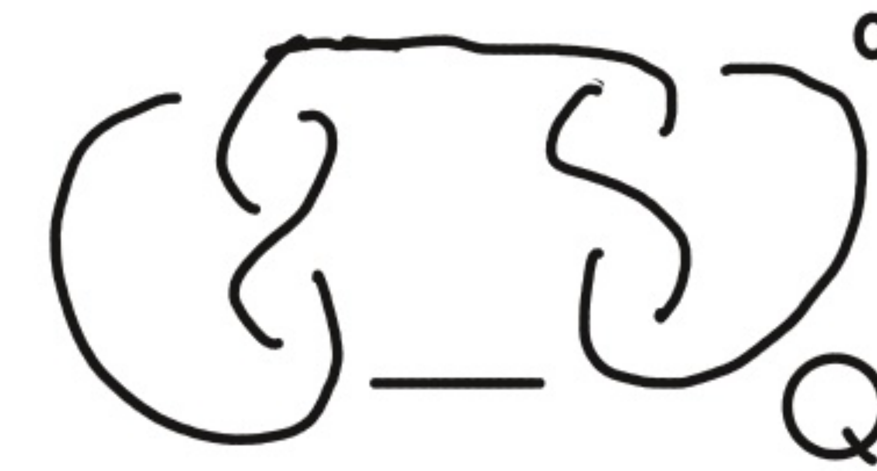
To make life easier

Goal: Build X, X' w/

- $\pi_1(X) = \pi_1(X') = 1$
- $X \overset{\sim_4}{\cong}_{TOP} X'$ $\left(\overset{\text{Freedman}}{\Longleftrightarrow} Q_X \cong Q_{X'}\right)$
- $\exists\ \ell, \mathfrak{s}$ w/ $\Theta(X, \ell, \mathfrak{s}) \neq 0$
  $\Theta(X', \ell', \mathfrak{s}') = 0 \ \forall\ \ell', \mathfrak{s}'$

$*$ $\mathfrak{s}|_Y$ nontorsion

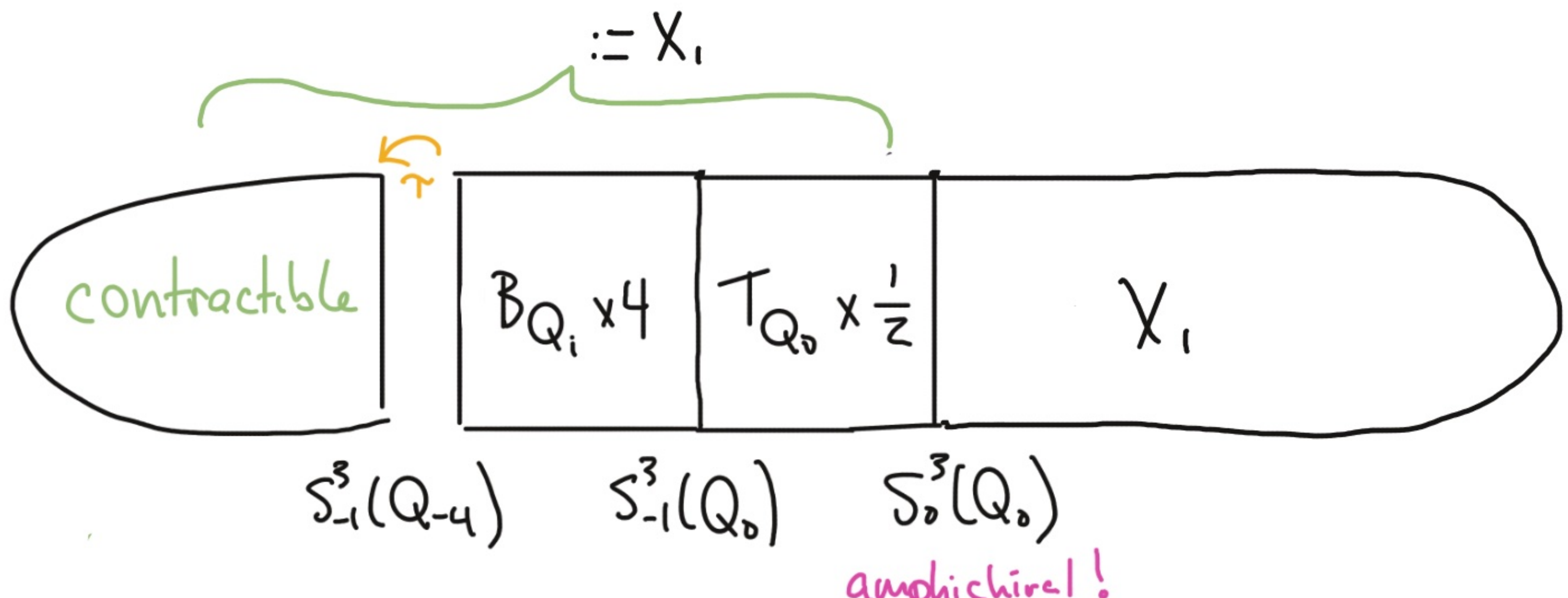
Thm $\exists\, X' \overset{\cong_{TOP}}{\not\cong_{SM}} \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}$



Q_0

sketch: • $\Theta(\mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}, \ell, \mathfrak{s}) \equiv 0 \quad \forall\, \ell, \mathfrak{s}$

• $X' :=$



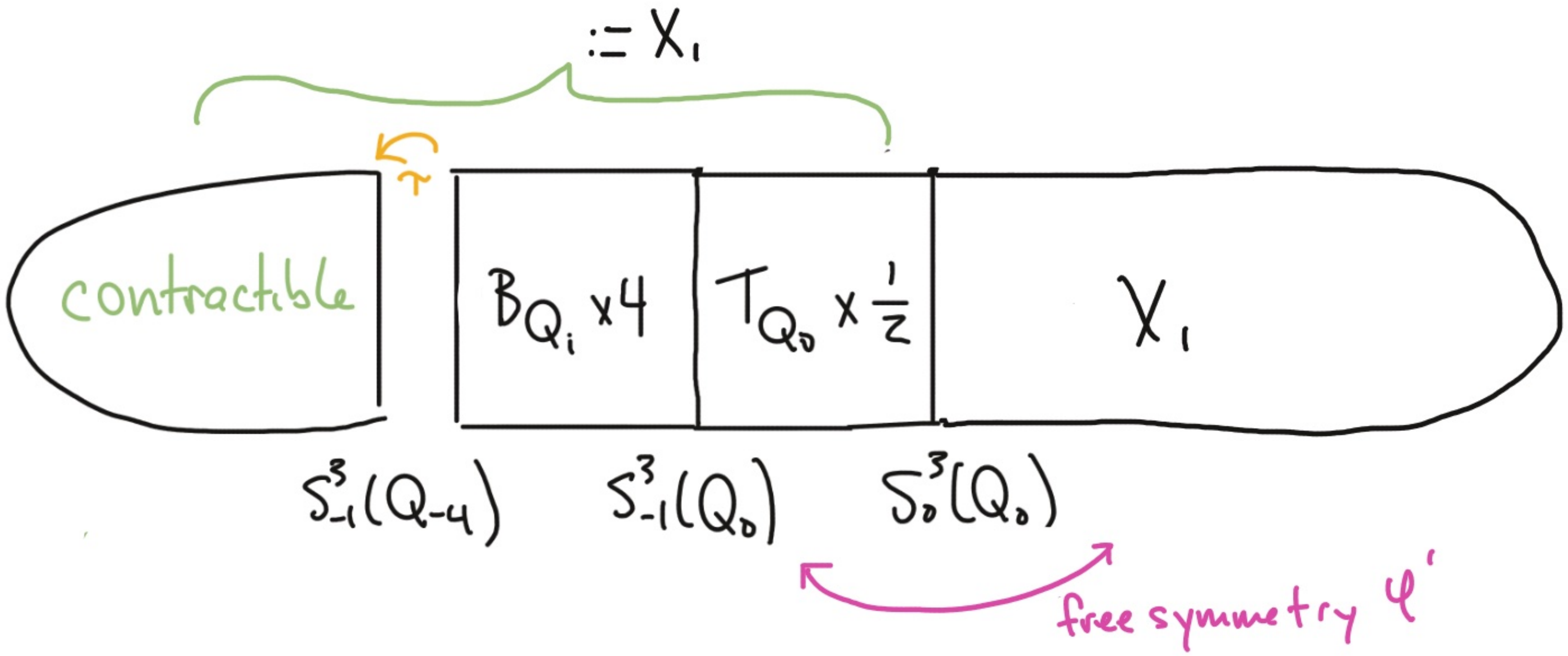
• compute $a(1) \neq 0 \implies \Theta_{OS_2}(X', \ell, \mathfrak{s}) \neq 0$

defined from setup.

• $\pi_1(X') = 0, \quad Q_{X'} = [1] \oplus_9 [-1] \overset{\text{Freedman}}{\implies} X' \cong_{TOP} \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}$. //

Thm $\exists\, X' \underset{\not\cong_{SM}}{\cong_{TOP}} \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}$

$X' :=$



Thm (L.L.P): $\exists$ closed definite exotic manifolds (w/ $\pi_1 = \mathbb{Z}/2$)

sketch: $Z' := X'/\varphi'$ $\quad Z := \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}/\varphi$ ← similar

check $\cdot\ Z' \cong_{TOP} Z$.

$\cdot\ Q_{Z'} = \bigoplus_4 [-1]$

$\widetilde{Z'} \not\cong_{SM} \widetilde{Z} \implies Z \not\cong_{SM} Z'$ //