Links of complex surface singularities: symplectic vs. algebraic fillings.

w/ Laura Starkston

singularities, deformations (alg. geom) $\overset{??}{\rightsquigarrow}$ topology & sympl. geometry of related manifolds

Setting: $(X,0) \subset \mathbb{C}^N$ $\quad \dim_{\mathbb{C}} X = 2$

complex surface w/ isolated singularity at 0 (normal sing.)

example: hypersurface sing. $F(x,y,z)=0$

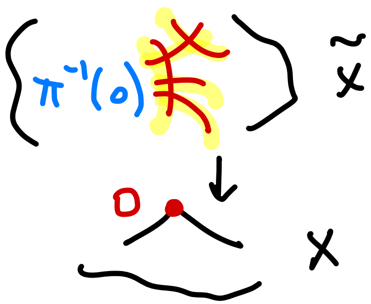
eg $x^2+y^3=z^5=0$ $\quad \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$

$Y = X \cap S^{2N-1}_{\varepsilon}$. link of singularity (indep of choices)

← smooth 3-mfld!

Fact The link determines "topology" of the singularity

Resolution



$\tilde{X}$ smooth

$\pi \downarrow$ 1:1 away from 0

$(X, 0)$

unique up to blow-ups.

exceptional divisor

$\pi^{-1}(0) = \bigcup_i C_i$

$\boxed{\text{normal crossings}}$

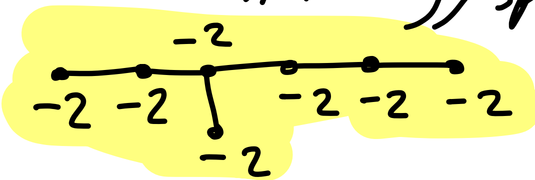
C_i : smooth complex curves,

$C_i \pitchfork C_j$, meeting at double points only

Example : $x^2 + y^3 + z^5 = 0$

link $\Sigma(2,3,5)$ = Poincaré homology sphere

spheres $C_i \cdot C_i = -2$



link = ∂(plumbing of disk bundles)

vs analytic structure

resolution graph (intersection data)
+ self-intersect. $C_i \cdot C_i$
+ genera $g(C_i)$

$\longrightarrow$ topology of the link.

Symplectic/contact structures!

Y = link of $(X, 0)$

Y is a contact manifold!

contact structure given by complex tangencies

$\underline{TY \cap J(TY)} = \xi \leftarrow$ non-integrable 2-plane field.

canonical cont. str. on link of singularity

Resolution of $(X, 0)$ gives a symplectic filling

(Y, ξ) = convex bdry of the standard symp. nbhd of the plumbing

C_i - complex curves

If a minimal resolution, can get a Stein structure.

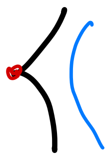
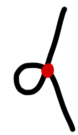
So ξ is always fillable

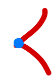
some fillings come from $(X, 0)$

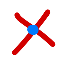
topology, cont/symp. geom. $\overset{???}{\longleftrightarrow}$ singularities & their deformations

Deformations, smoothings, adjacencies:

deformation: family $\{F_t\}$

$\underline{y^2 = x^3}$  $\longrightarrow$ $\underline{y^2 = x^3 + tx^2}$ 

At time $\underline{t=0}$: cusp 

for <u>all</u> small $t \neq 0$: node 

[More generally: 1-param. deformation is a "flat" family X_t of singularities, s.t. $t=0$: $(X, 0)$

all $t \neq 0$ $(X', 0)$

we say $(X, 0)$ deforms into $(X', 0)$]

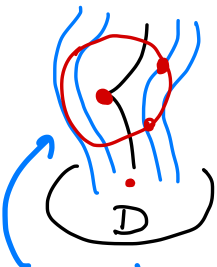
Smoothing: get a <u>smooth</u> point for $t \neq 0$

= "deform away" the singular point

DEFORM. THEORY QUESTIONS: ? understand smoothings/def.

Example

$$x^2 + y^3 + z^5 = 0$$



in $\mathbb{C}^3$

$x^2 + y^3 + z^5 = t$, small $t > 0$

smooth surfaces $X^t \neq 0$

1-param. Deformation over $D \subset \mathbb{C}$

smoothing

X^t = Milnor fiber

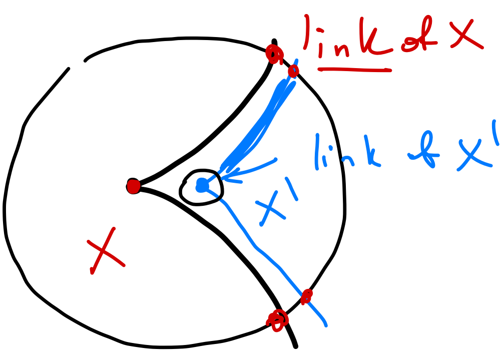
$\boxed{Y = \partial X^t}$

Note: • smoothings do not always exist!

• When they do, there can be different smoothing components (different topology)

• Each Milnor fiber gives a symplectic (Stein) filling for (Y, ξ)

• Deformation → Stein cobordism



cobordism from link of X' to link of X

$(X, 0)$ - sing. $\qquad (Y, \xi)$ - link

smoothings $\longrightarrow$ Stein fillings of (Y, ξ)

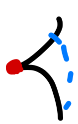
deformations $\longrightarrow$ Stein cobordisms between links

Can we use symp. topology to get info about deformations/smoothings?

Compare : similar questions about plane curve singularities & knots, slice genus, cobordisms

algebraic plane curve $C \subset \mathbb{C}^2$,

eg $\quad x^p + y^q = 0 \quad$ isolated sing. at 0

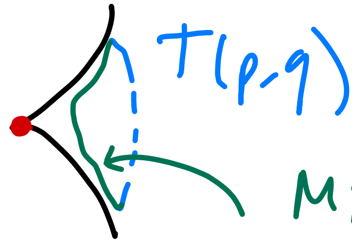


link of the sing. point

gcd $(p,q)=1$ $\qquad C \cap S^3 = T(p,q)$ torus knot!

(embedded in S^3)

Smoothings:



Milnor fiber minimizes genus among all such orientable smooth surfaces!

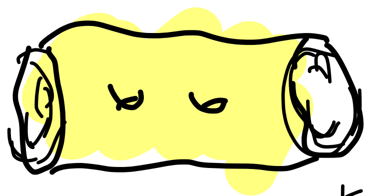
S embedded in D^4

$\partial S = T(p,q)$

$$g = \frac{(p-1)(q-1)}{2}$$

Deformations

adjacency $\rightsquigarrow$ cobordism of optimal genus



knots = links of plane curve sing

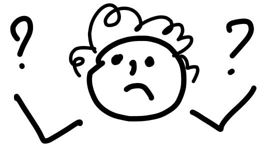
A number of results Feller, Borodzik-Livingston, Bodnár-Celoria-Golla ....

topological obstructions / existence of certain smooth cobordisms

$\Updownarrow$

info on algebraic invariants, adjacency problem

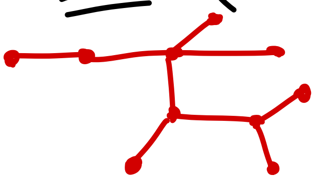
Can we get new results on deformations/smoothings of surface singularities from symplectic topology?



Can reprove some known results using knowledge of symplectic/Stein fillings.

Examples

① Prop (P. Starkston, P) $(X, 0)$ surface singularity s.t. for each exceptional curve, $g(C) = 0$, $C \cdot C \leq -5$



no "bad" vertices

Then link (Y, ξ) has a unique minimal symp./Stein filling

$\Downarrow$

$\exists!$ Milnor fiber, up to diffeo/Stein homotopy

[known: T. de Jong, D. van Straten ~ 1995]

$\exists!$ smoothing

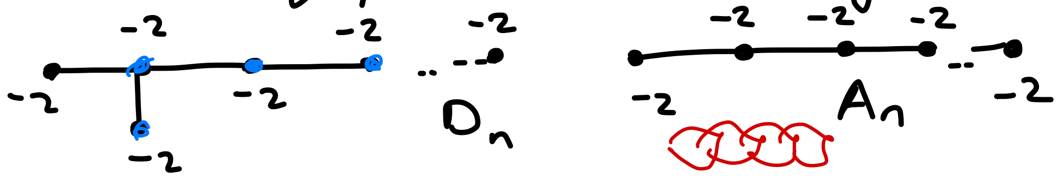
② Bodnár-P: certain properties of the contact invariant of (Y,ξ) in Heegaard Floer hom.
$\Downarrow$
rational singularity cannot be deformed into non-rational

Elkik classical ~1975

* * *

③ Simple singularities: A_k, D_k, E_6, E_7, E_8

resolution graph = Dynkin diagram



known : adjacency exists $\iff$ one diagram embeds into the other
Arnold ~1973

(Ohta-Ono)
known: the link of a simple sing. has a unique min symplectic filling

can use this to show: Stein cobordism between links exists $\iff$ Diagram embeds
$\iff$ adjacency

But in a sense, symplectic questions are harder!

A <u>very</u> simple example:

A_2 : $x^2+y^2+z^2=0$ link $=\mathbb{R}P^3 = L(2,1)$

ALGEBRAIC

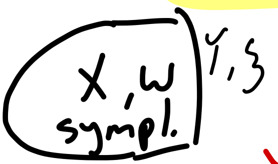
- deformations of this polynomial - easy

SYMPLECTIC

- unique filling, but proof requires holomorphic curves!

<u>QUESTION</u>:

DO ALL STEIN / min symp fillings arise from smoothings (Milnor fibers) and the resolution?



$$(Y,\xi) = \partial(X,\omega)$$

YES in a number of simple cases!

- S^3 (no singularity) Eliashberg McDuff
- lens spaces (cyclic quotient sing) Lisca
- links of simple singularities $(A_n, D_n, E_6, E_7, E_8)$ Ohta-Ono

+ some more ...

But in general:

Stein fillings $\supsetneq$* Milnor fibers (for all compatible analytic structures)

$\exists$ exotic fillings, $b_1 \neq 0$ [Akhmedov-Ozbagci]

Question: when does $\boxed{\text{Milnor} = \text{Stein}}$ break down?

"unexpected" fillings = Stein but not Milnor

P. Starkston: compare Milnor vs Stein

for RATIONAL SINGULARITIES w/ REDUCED FUNDAMENTAL CYCLE

- simple algebraic geometry
- simple contact/symp geometry

planar open books

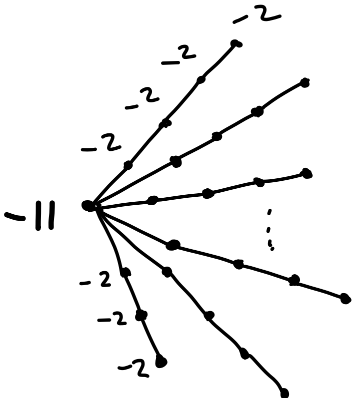
bounded topology, conjectural finiteness of fillings

can do detailed comparison

still find a lot of unexpected fillings!

Unexpected fillings exist even for "reasonably small" Seifert fibered spaces!

$b_1 = 0$



+ MANY MORE!

star-shaped resolution graph w/ 10 legs

8 legs: 5 vertices each

2 legs: 4 vertices

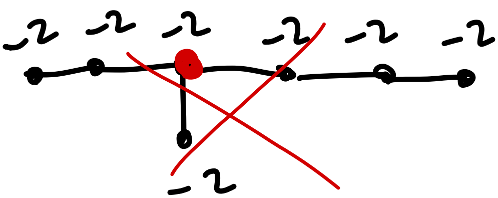
all self-intersections $= -2$ except central vertex

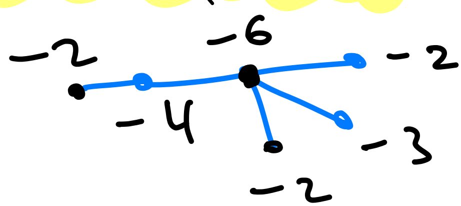
WHAT'S THIS CLASS of SINGULARITIES?

RATIONAL SINGULARITIES w/ REDUCED FUNDAMENTAL CYCLE

Definition via the resolution graph

- res. graph is a tree
- each except. curve has genus 0
- $-v \cdot v \geq$ valency of v (for each vertex)





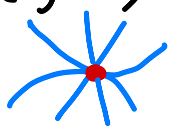
Why this class of singularities?

ALG. GEOM: deformation theory known

T. de Jong - D. van Straten ~1998

"sandwiched singularities"

deformations of surface singularity $(X,0)$ $\longleftrightarrow$ deformations of an associated singular plane curve (w/ weights) $(C,0)$ germ



topology of Milnor fibers! $\leftarrow$

We described symplectic fillings in a similar way, via curve arrangements & their smooth homotopies

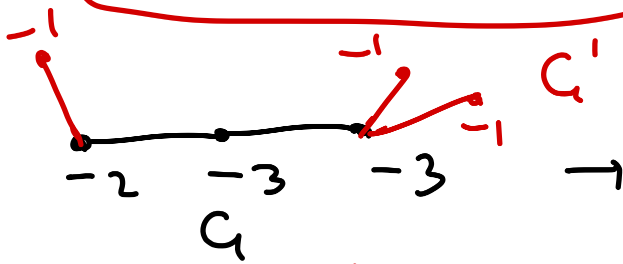
$\uparrow$

our setting allows for more general arrangements than dJvS $\rightarrow$ get "unexpected" fillings!

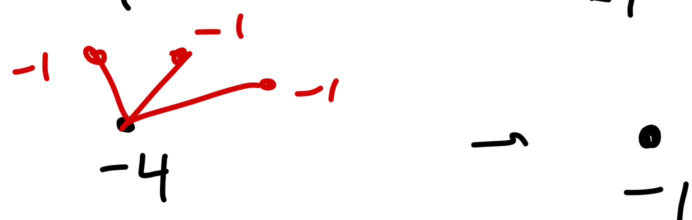
Brief survey of dJvS:

- associate a decorated curve germ to the singularity

$\boxed{\begin{array}{c} G\text{ - resol. graph} \\ -v\cdot v \geq \text{valency}(v) \\ \text{for each vertex} \end{array}}$ $\Rightarrow$ G can be "augmented" to G' that gives a blow-up of $\mathbb{C}^2$!



$\longrightarrow$

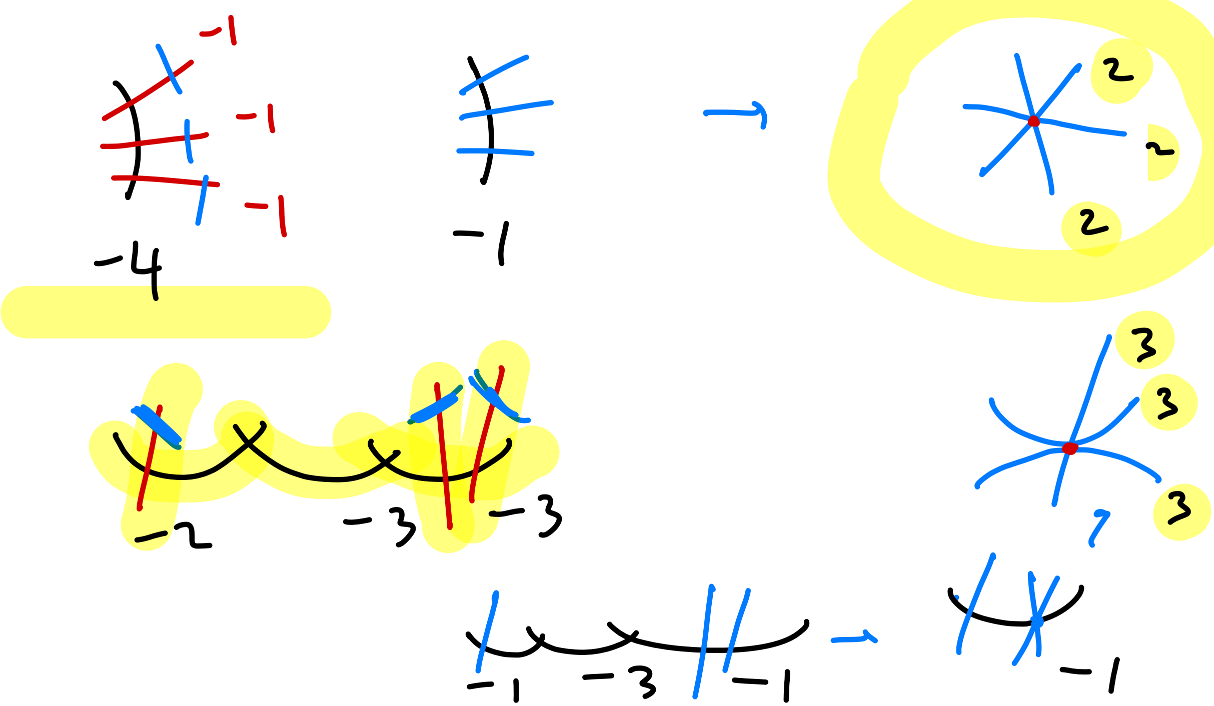


<u>link</u> $L(4,1)$

Mark the <u>additional (-1) curves</u> of $g=0$ by placing a transverse cpx disk on each

Blow down G' $\rightsquigarrow$ <u>configuration of the transverse</u> disks projects to a reducible cpx curve in $\mathbb{C}^2$

record "weights" = # blow-downs



for our class of singularities –

can <u>always</u> get

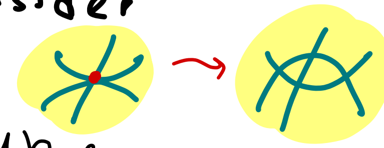
a reducible curve w/ <u>SMOOTH</u> components

- can reconstruct the singularity from this associated germ

- can read off all smoothings & topology of Milnor fibers!

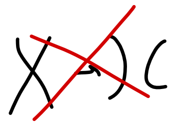
[dJvS: more general setting, deformations]

To get smoothings: consider germ deformations



subject to weight restrictions such that:

- no merging of components

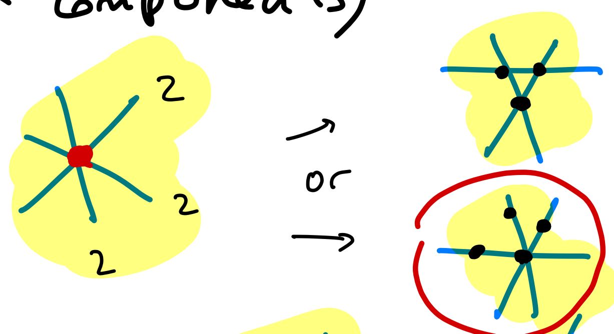


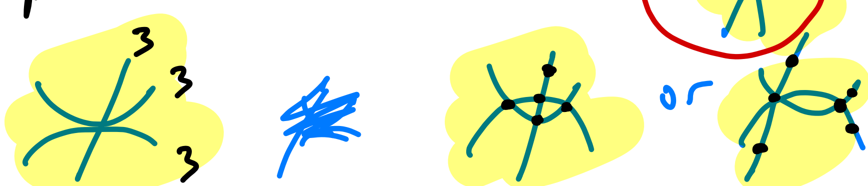
- only transverse multiple points (resolve all tangencies!)
- for each curve component:

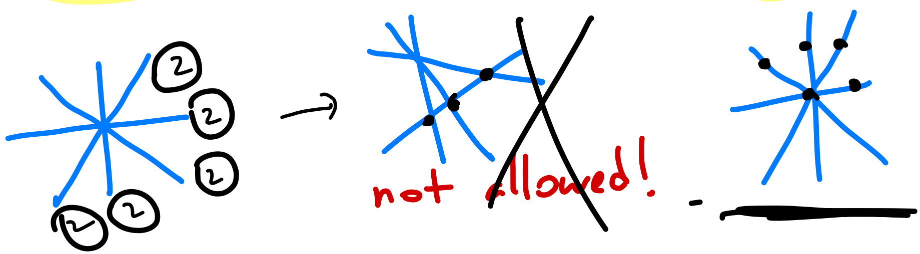
  # (intersections w/ other components) $\leq$ weight

Examples

weight ↔ marked pts

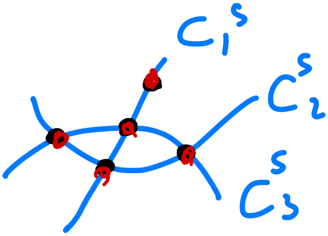
① 

② 

③ 

L(5,1)

$\mathbb{C}^2$



$\longrightarrow$ reconstruct Milnor fiber!

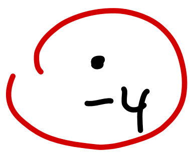
Blow up at all marked pts

take complement of proper transforms

dJvS (tubular nbhds) of C_i^s

$$\left(\underline{\underline{\mathbb{C}^2 \# n\,\overline{\mathbb{CP}}^2}}\right) \setminus \bigcup_i C_i^s$$

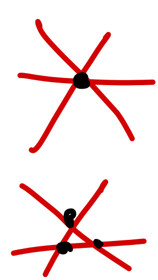
Milnor fiber of the corresp. smoothing of $(X,0)$

For  get two smoothings

one w/ Milnor fiber = disk bundle $e=-4$

the other Milnor fiber = rational homology ball.

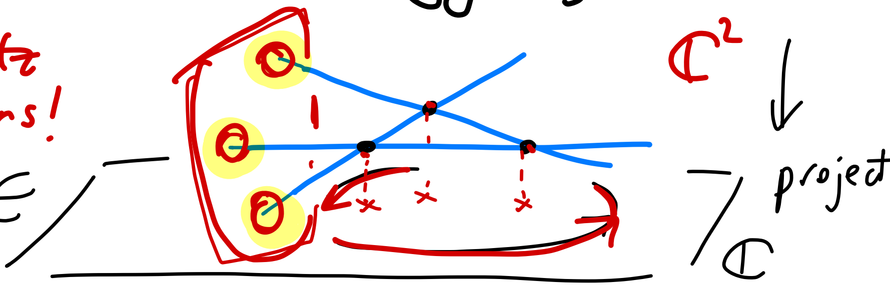
these are two Stein fillings of $L(4,1)$!



How to see the topology + Stein str.

Lefschetz fibrations!

graphical arrangement $\not\subset$

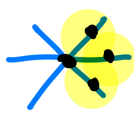


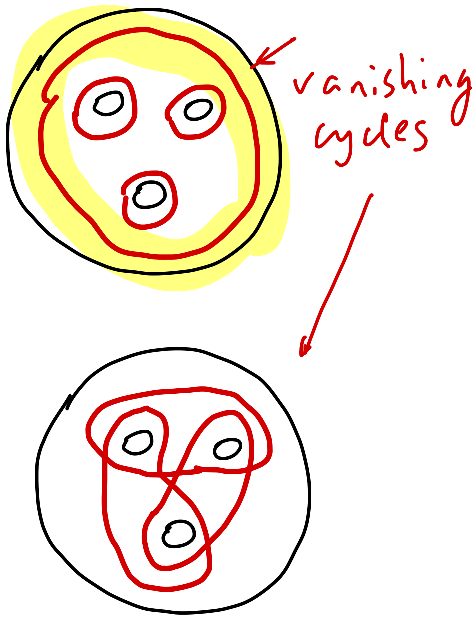
get compatible Lefschetz fibration.

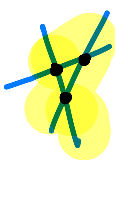
- planar fiber (disk w/ holes)
- holes $\longleftrightarrow$ germ branches
- monodromy $\longleftrightarrow$ intersections/marked pts

For $L(4,1)$:

$\underset{-4}{\bullet}$



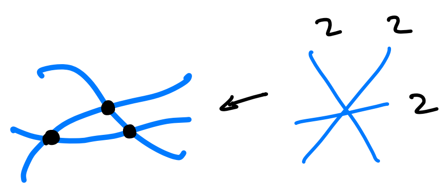




This was de Jong-van Straten story
for deformations/smoothings/
Milnor fibers

what we did for Stein fillings
& min symplectic fillings:

input from symp. topology: (Wendl)
every filling is given
by some Lefschetz fibration same planar fiber
corresp. to positive monodromy
factorization!



construct
smooth graphical disk arrangement
(related to germ by
smooth graphical homotopy)
THINK BRAID MONODROMY!

WHAT'S THE DIFFERENCE?

Milnor fibers
algebraic germ deformations

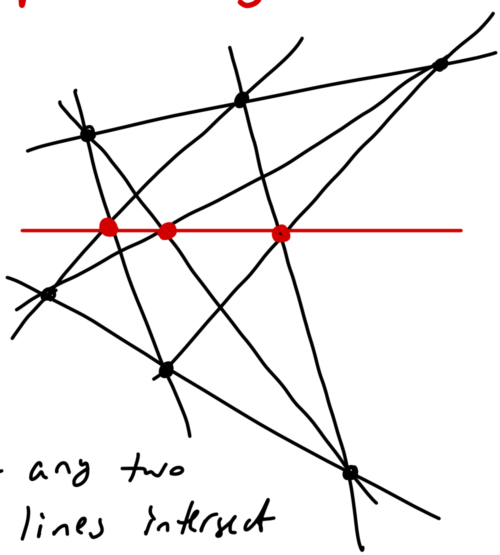
$\neq$

Symp fillings
smooth graphical homotopies

the difference is encoded by properties of curve deformation

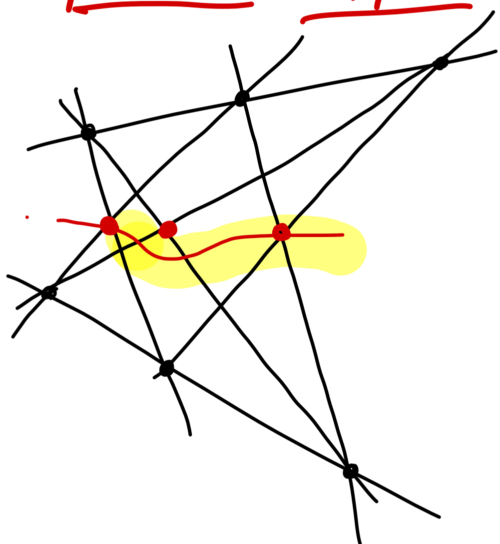
(not properties of arrangement)

Pappus arrangement

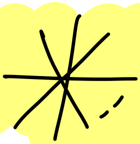


+ any two lines intersect

pseudo-Pappus



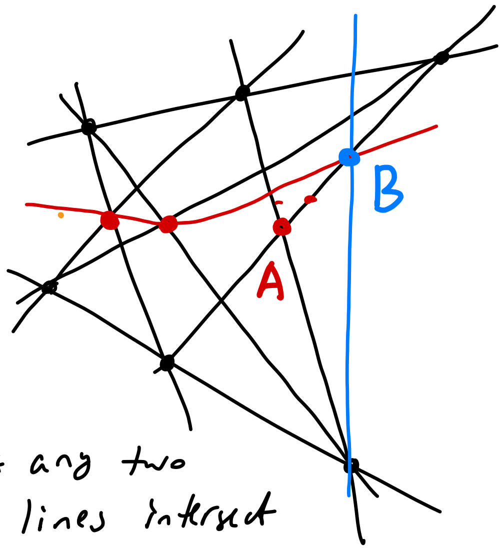
complexifications of red pseudolines



germ
9 lines
large weights

also gives a smoothing!

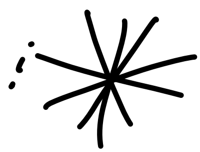
BUT:



+ any two lines intersect

this arrangement cannot appear via an algebraic deformation!

10 lines



intuition: red "line" is anchored at pt B but a cpx line would go thru A [PAPPUS]

Thm Corresp. Stein filling is not diffeomorphic (rel boundary) to any Milnor filling!

algebraic deform	smooth graphical homotopies
• topology of arrangement is the same! (for time $\neq 0$)	• topology may change
• positive intersections, weight restrictions throughout	• weights & positivity only holds for the final arrangement (may be violated during homotopy)

Thm "Unexpected" pseudoline arrangements cannot arise from algebraic deformations

$L_1, \ldots L_n$ unexpected if the only complex line arrangement w/ all incidences of $L_1, \ldots L_n$ is the pencil!