Shake slice knots

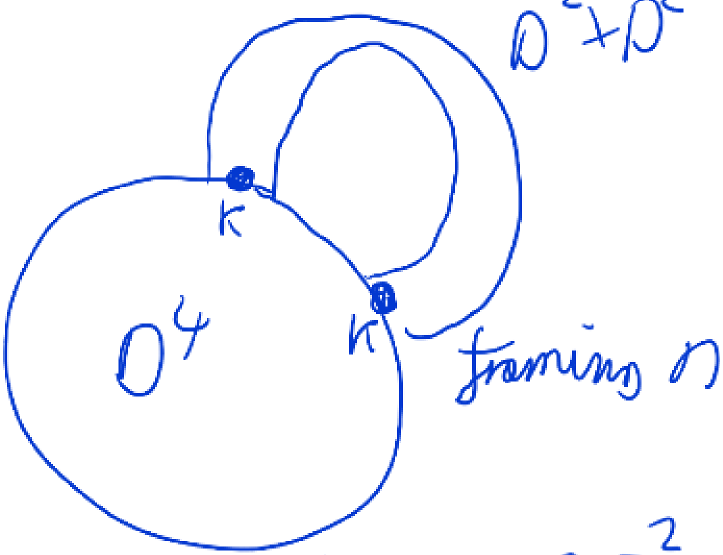
Based on joint work with: Peter Feller, Allison N. Miller, Matthias Nagel, Patrick Orson, and Arunima Ray.

Knot traces are compact 4-manifolds w. ∂.

$K \subseteq S^3$ knot

$n \in \mathbb{Z}$

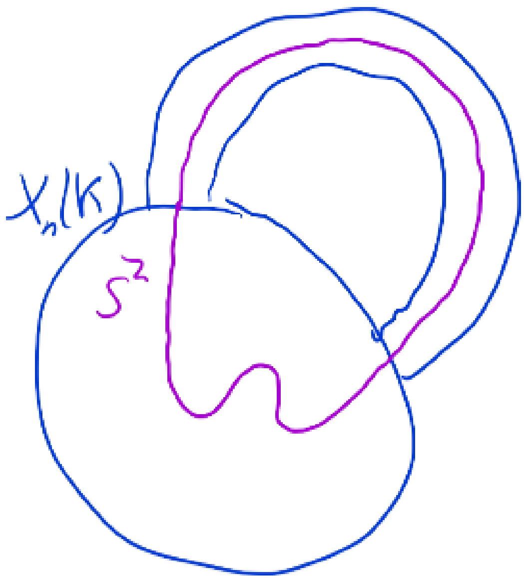
$X_n(K) :=$



$\partial X_n(K) = S^3_n(K)$ | $X_n(K) \simeq S^2$.

in handle notation

K is n-shake slice if a generator of $\pi_2(X_n(K)) \cong \mathbb{Z}$ is represented by a locally flat embedded $S^2 \subseteq X_n(K)$,



<u>Prop^n</u>

K slice $\implies$ K is n-shake slice $\forall n$.
(bounds loc flat $D^2 \subseteq D^4$.)

<u>Open Q</u>: Does 0-shake slice $\implies$ slice? (both smooth and top)

<u>Equivalent formulation</u>

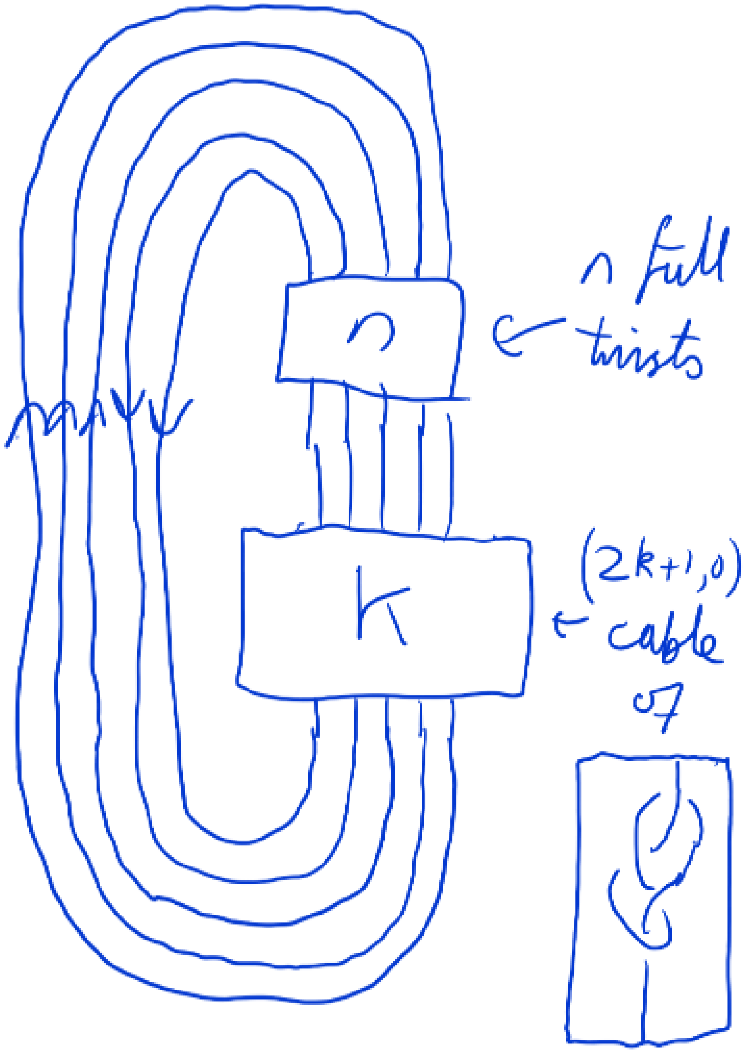
The n-shakings of K are the

$(2k+1)$ - component oriented links

$S_{2k+1,n}(K)$ $\quad k \geq 0$

built as follows:

eg
$2k+1 = 5$



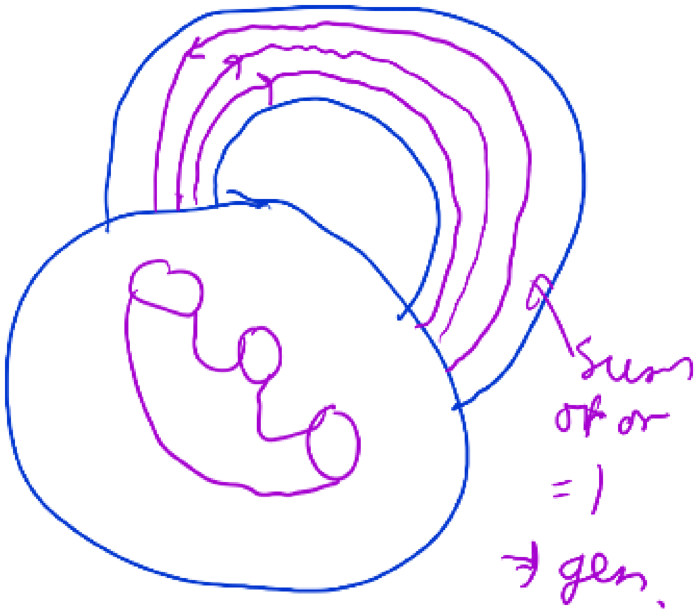
Prop^n

K is n-shake slice if and only if

$S_{2k+1,n}(K)$ bounds a locally flat connected planar surface in D^4, for some k.

Pf

$\Leftarrow$ construct an S^2.



$\Rightarrow$ Make S^2 transverse to cocore of 2h.

$\square$

In general:

n-shake slice (for some n) $\not\Rightarrow$ slice

n-shake slice $\not\Rightarrow$ m shake slice $n \neq m$.

Akbulut '77 using TL-surgs.

top n-shake slice $\not\Rightarrow$ smoothly n-shake slice

also Lickorish '79
Boyer '82

More on these disparities later.

KOS 2 obstructions

Recall some basic knot invariants:

Every knot bounds a Seifert surface, F.

Seifert form

$$A : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$

eg $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

F, A choices, but lead to invariants?

1) $\Delta_K(t) := \det(tA - A^T) \in \mathbb{Z}[t^{\pm 1}] /$ mult^n by monomials.

2) $\sigma_K(\omega) := \mathrm{sign}\left((1-\omega)A + (1-\bar{\omega})A^T\right) \in \mathbb{Z}$

   $\omega \in S^1 \subseteq \mathbb{C}$.

3) $\mathrm{Arf}(K) \in \mathbb{Z}/2$

$$\Delta_K(-1) = \begin{cases} \pm 1 \ (8) & \mathrm{Arf}\, K = 0 \\ \pm 3 \ (8) & \mathrm{Arf}\, K = 1 \end{cases}$$

Shake slice obstructions

Prop^n

1) If $\exists n$ such that K is n-shake slice, then $\mathrm{Arf}(K) = 0$

(Robertello)

2) If K is n-shake slice then $\sigma_K(\zeta) = 0$
$\forall \zeta = e^{2\pi i \frac{l}{p^a}}$ $\quad p^a | n$ prime power

(Tristram)

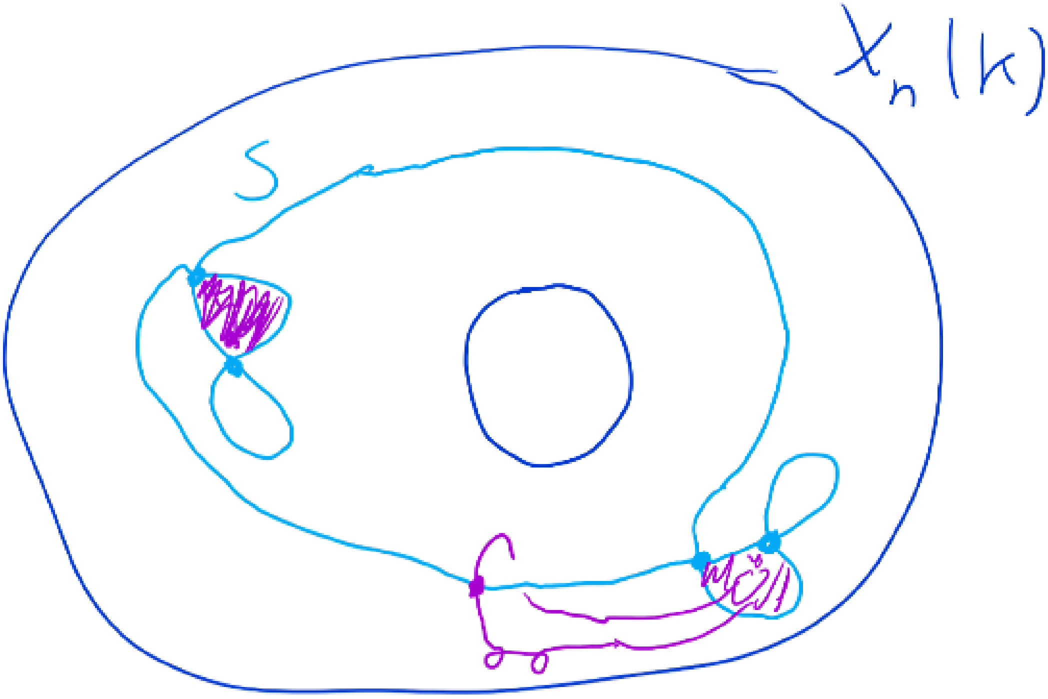
Proof of 1)
(Different from Robertello's and Tristram's).

Aim:
Define a knot invariant $\tau(S)$ (for each n)

Let $S \looparrowright X_n(K)$ generically immersed 2-sphere

$S \pitchfork S = 0$ algebraically

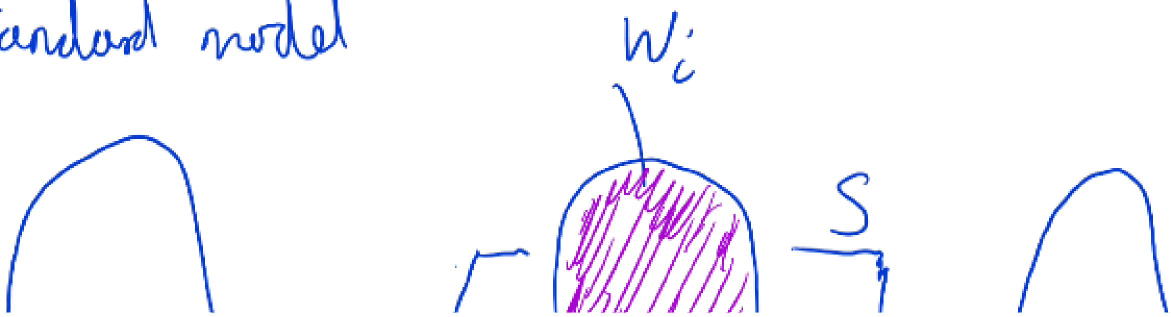
$[S] = 1 \in \mathbb{Z} \cong \pi_2(X_n(K))$



$\pi_1(X_n(k)) = 0$

$\exists$ pair up double pts with framed Whitney discs. $W_i \looparrowright X_n(k)$

Standard model



Define

$$\tau(S, \{W_i\}) := \sum_i{}' \, S \cdot W_i \quad (2)$$

$$\in \mathbb{Z}/2 .$$

<u>Lemma</u>

τ is independent of $\{W_i\}$, and

is invariant under homotopies of S

(with $S \pitchfork S = 0$ at both ends of the homotopy) $\therefore$ invariant of K.

Write $\tau(S) \in \mathbb{Z}/2$.

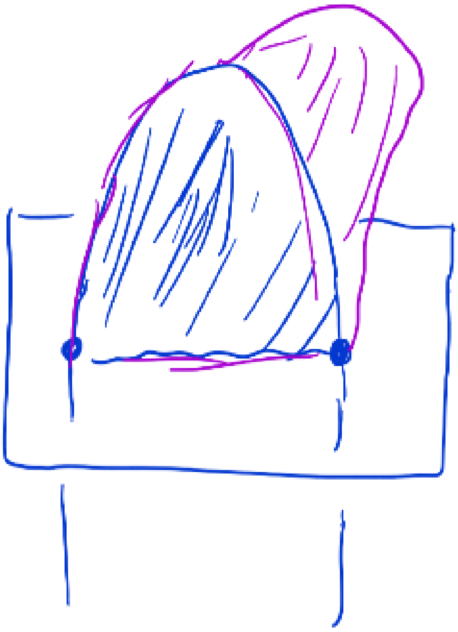
<u>Claim</u> Suffices to show that:

S is <u>characteristic</u> i.e.

$S \bullet R = R \bullet R \pmod 2 \quad \forall R \in \pi_2(X_n(K))$

<u>Idea</u> 2 Whitney discs with same ∂

union $\rightsquigarrow$ on $S^2 = R$.



$\underline{S} \cdot R + R \cdot \underline{R}$ measures the difference in the contributions of the two Wh discs to τ.

Compute homologically:

$[R] = k \in \mathbb{Z} \cong \pi_2(X_n(K))$

$[S] = 1 \quad \in$

$$S \cdot R + R \cdot R = 1 \cdot n \cdot k + k \cdot n \cdot k.$$

$$= n(k+k^2) \equiv 0 \quad (2).$$

so S is characteristic.

$\therefore \ \tau(S) \in \mathbb{Z}/2$ is well-defined.

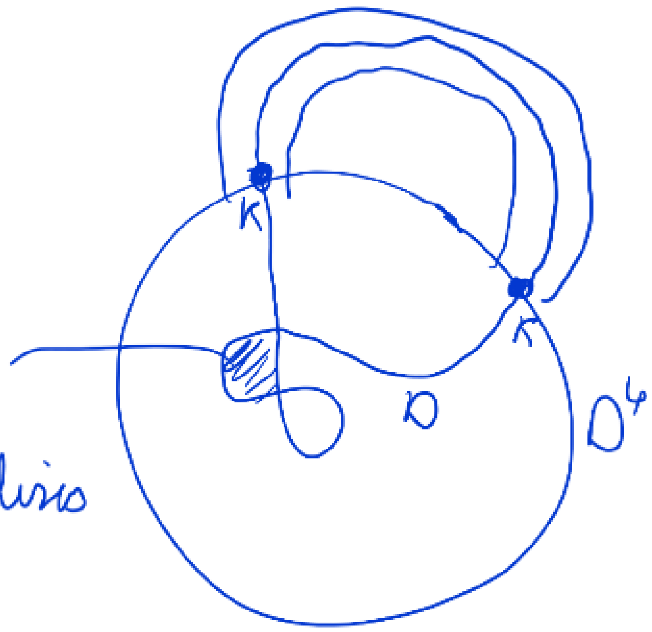
If K is n-shake slice $\Rightarrow \tau(S)=0$.

<u>But</u> $D \hookrightarrow D^2 \qquad \partial D = K$

$D \cong D^2$

$D \pitchfork D = 0$
alg

W_i
framed Whitney disc



$$\mathrm{Arf}\, K = {\textstyle\sum_i}' \, W_i \cdot D \quad (2)$$

e.g. Cha-Orr-P. (originally

Math Z. (Freedman-Kirby, Matsumoto)

Use $S = D \cup D^2 \times \{0\}$

$\tau(S) = \mathrm{Arf}(K)$.

n-shake slice $\Rightarrow \tau(S) = 0$

$\Rightarrow \mathrm{Arf}(K) = 0$.

$\square$

Recall (2): K n-shake slice

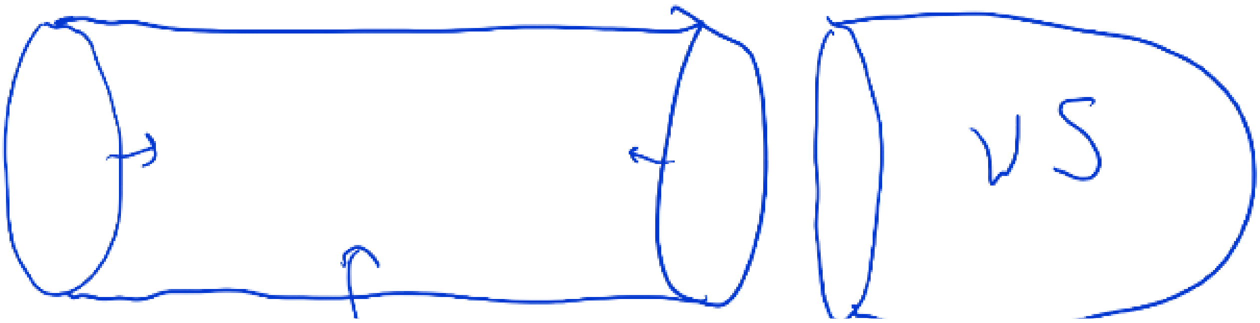
$\Rightarrow \sigma_\zeta(K) = 0 \quad \zeta = e^{2\pi i\, l/p^a}$

p^a prime power.

$p^a \mid n$.

Pf of (2)

If K n-shake slice, $X_n(K)$ decomposes: $S^{(1)}$



$S^3_n(K)$ $\quad$ $\mathbb{Z}$-hom cob. $\quad$ $L(n,1)$

$D^2 \to V \to S^2$
euler no n.

Now let W be some cobordism

$$\left(W ; S^3_n(K), L(n,1)\right)$$
$$\downarrow$$
$$B\mathbb{Z}/n$$

$\widetilde{W}$ $\mathbb{Z}/n$-cover

$\mathbb{Z}/n \curvearrowright H_2(\widetilde{W} ; \mathbb{C})$

V_k : ω^k-eigenspace $\quad \omega = e^{2\pi i/n}$.

$\lambda_k : V_k \times V_k \to \mathbb{C}$ $\quad$ int^n form restricted to V_k.

$\sigma_k(\widetilde{W}) := \mathrm{sign}(\lambda_k)$.

Lemma

$$\sigma_{\omega^k}(K) = \sigma_k(\tilde{W}) - \sigma(W)$$

Tristram-Levine signature (3D invariant)

Proof

Atiyah-Singer

Casson-Gordon

G-signature thm.

$\square$

Now, K n-shake slice

$\Rightarrow$ W $\mathbb{Z}$-hom cobordism.

$\Rightarrow$ $\sigma(W) = 0$

$V_k = 0$ $\quad$ w^k prime power root of 1.

$\Rightarrow$ $\sigma_3(K) = 0$

3 p^a-th root of 1

p^a prime power
$p^a \mid n$.

Now you have some ways to obstruct
- many knots are not n-shake slice.

Next we add an extra condition:
$\pi_1(X_n(K) \setminus S)$ is abelian.
ie $\cong \mathbb{Z}/_n$.

We say K is $\mathbb{Z}/_n$-shake slice
if this holds.

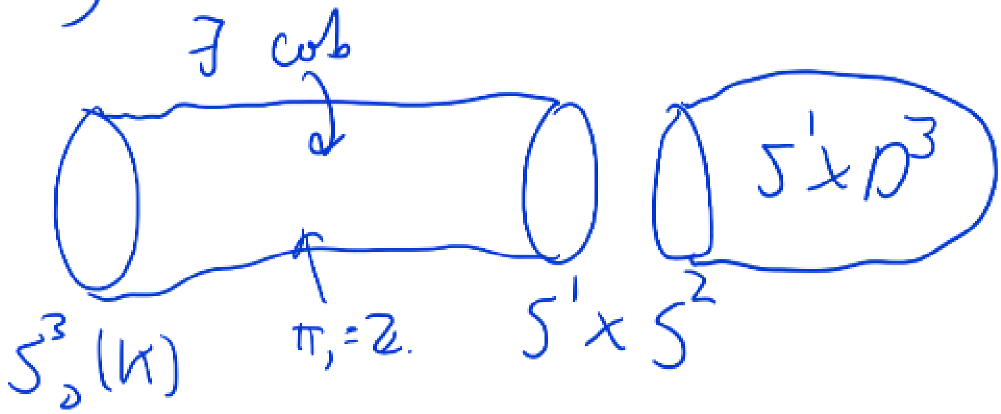
Prop ($n=0$)

K is $\mathbb{Z}$-shake slice if and only if $\Delta_K \doteq 1$.

Proof ($\Leftarrow$)

$\Delta_K = 1 \Leftrightarrow K$ slice with $\mathbb{Z}$-slice disc $\Rightarrow K$ $\mathbb{Z}$-shake slice.

($\Rightarrow$) K $\mathbb{Z}$-shake slice.



$\Rightarrow S^3_0(K)$ bounds $V \simeq S^1$.

$V \cup D^2 \times D^2 \cong D^4$

h-cob thm.

Im of cocore gives a $\mathbb{Z}$-slice disc.

$\Rightarrow \Delta_k = 1.$

$\square.$

<u>Theorem</u> (with Feller, AN Miller, Nagel, Orson, and Ray)

A knot $K \subset S^3$ is $\mathbb{Z}/_n$ - shake slice if and only if

1) $\operatorname{Arf} K = 0$

2) $\sigma_K(\zeta) = 0 \quad \forall \, \zeta \in S^1$ with $\zeta^n = 1$.

(not just prime power roots of 1.)

3) $\displaystyle\prod_{\{\zeta \,|\, \zeta^n = 1\}} \Delta_K(\zeta) = 1$

$$\left(= \left| H_1(\Sigma_{|n|}(K)) \right| \right)$$

---

i.e. obstructions from before

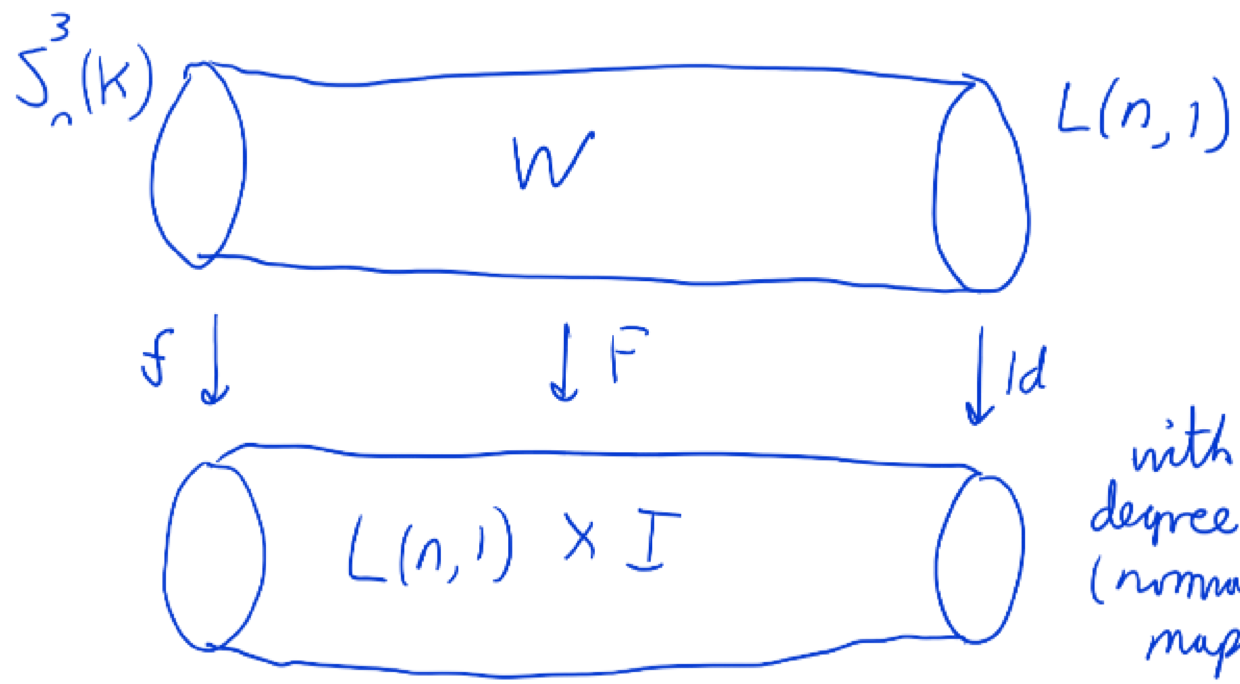
+ natural modification of $\Delta_K = 1$

condition.

Almost did only if proof, so leave it there.

<u>Proof of "if"</u>

Construct a cobordism over $L(n,1) \times I$



with degree one (normal) maps.

There is a surgery obstruction in

$L_4^s(\mathbb{Z}[\mathbb{Z}/n])$ the Witt group

of non-singular, hermitian, sesquilinear

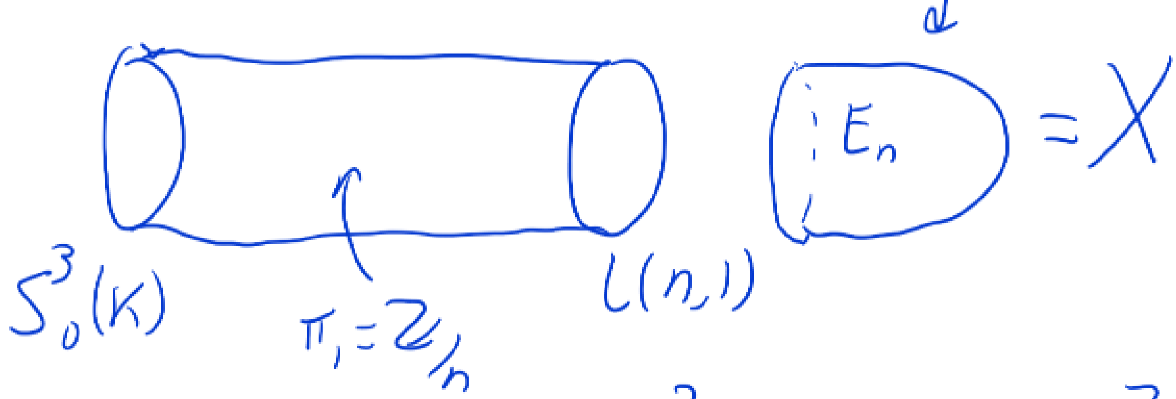
forms

(int form) $\prod_{\{\zeta \mid \zeta^n = 1\}} \Delta_K(\zeta) = 1$

$\exists$ 5-dim bordism of (W, F) rel ∂ to (W', F') with F' a $\simeq$ if surgery obstruction $= 0$.

But $L_4^s(\mathbb{Z}[\mathbb{Z}/n]) \cong \mathbb{Z}^m$ given by
$$\left[m = \begin{cases} \lceil \frac{n}{2} \rceil & n \text{ odd} \\ \frac{n}{2} + 1 & n \text{ even} \end{cases} \right] \quad \sigma_K(\zeta) \text{ with } \zeta^n = 1.$$

Obtain $\mathbb{Z}$-hom cobordism, glue on



$$D^2 \longrightarrow E_n \longrightarrow S^2$$
disc bundle with euler no. n.

Theorem (Boyer)

(special case)

$X \cong X_n(K)$

if and only if 1) n even

or 2) n odd and $ks(X) = 0$

If $X \cong X_n(K)$ then image of 0-section of $\begin{matrix} E_n \\ \downarrow \\ S^2 \end{matrix}$ gives a locally flat embedded S^2.

Lemma

$$\mathrm{Arf}(K) = \tau(S) = ks(X)$$

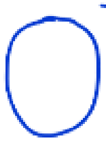
$$(S \hookrightarrow X_n(K))$$

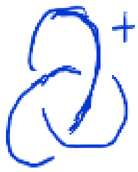
Then $\mathrm{Arf}(K) = 0 \Rightarrow ks(X) = 0$

$\Rightarrow X = X_n(K)$

$\Rightarrow K$ $\mathbb{Z}/n$-shake slice $\square$

Sketch pf of lemma

 $^{+1}$ $\mathbb{C}P^2$ $ks = 0$

 $^{+1}$ $*\mathbb{C}P^2$ $ks = 1$

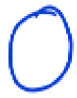
$$\underset{\substack{Z\\ \|}}{} ks\left(X \underset{S^3_n(K)}{\cup} -X_n(K)\right) = ks(X) + ks(X_n(K))$$

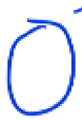
$$= ks(X)$$

$\cong ?$

$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

so $Z \cong \begin{cases} \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & ks(X) = 0 \\ \mathbb{C}P^2 \# *\overline{\mathbb{C}P^2} & ks(X) = 1 \end{cases}$

$= \Big($  $^{+1}$  $^{-1}$ $\cup D^4$

 $^{+}$  $^{-1}$ $\cup$ (contractible)

Detect with Arf invariant

via τ of suitable immersed

spheres

Corollaries to main theorem

(all FMNOPR)

1) There are ∞ top concordance classes of knots that are n-shake slice $\forall$ prime powers n.

(In particular not slice.)

Pf/ Take J with $\Delta_J = \Phi_m(t)$

m div by 3 distinct primes.

(cyclotomic poly)

Consider $J \# -J$.

$\exists$ ∞ many top conc classes with same Seifert form (Livingston + Taehee Kim)

All are n-shake slice for n prime power

2) $\forall n$ $\exists$ (∞ top conc classes of)

knots that are n-shake slice but not smoothly n-shake slice.

Pf/ $C_{n,1}(T_{p,q})$ for suitable p, q

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3) If $m \nmid n$, $\exists$ (∞ top conc classes of) knots that are n shake slice but not m-shake slice.

Pf/ $C_{n,1}(T_{p,q})$

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4) $m | n$ K $\mathbb{Z}/n$ shake slice

$\Rightarrow K$ $\mathbb{Z}/m$ shake slice

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5) K n shake slice for some n, then K is 1-shake slice ($\Leftarrow$ Arf K)

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6) K $\mathbb{Z}/n$-shake $\Rightarrow$ K is $\mathbb{Z}/(-n)$ shake slice.

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7) K, J $\mathbb{Z}/n$-shake slice
$\Rightarrow$ $K \# J$ is too.

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8) $\exists$ infinitely many knots $\{K_i\}$ distinct in concordance and an infinite family of integers $\{n_j\}$ such that $S^3_{n_j}(K_i)$ homology cobordant to $S^3_{n_j}(U) = L(n_j, 1)$ $\forall i, j$.

<u>Some open questions</u>

- Does K n shake slice $\Rightarrow$ K $(-n)$-shake slice ?

- Does K, J n shake slice $\Rightarrow$ $K \# J$ n shake slice ?

- Does 0-shake slice $\Rightarrow$ slice ?

- Does ($S^3_n(K)$ hom cobordant to $S^3_n(J)$ $\forall n$) $\Rightarrow$ K concordant to J ?

- Higher order obstructions: (topologically) does algebraically slice $\Rightarrow$ n shake slice $\forall n \neq 0$?

- Is every topologically n-shake slice knot topologically concordant to a smoothly n shake slice knot ?