An $SL_2(\mathbb{R})$ Casson-Lin Invariant

w/ Nathan Dunfield

Character Varieties:

Y a space

$G \subset GL_n(\mathbb{C})$ alg. group

$R_G(Y) = \{\rho : \pi_1(Y) \to G\}$

$X_G(Y) = R_G(Y)/\sim$

$\rho \sim \rho'$ if $\operatorname{tr}\rho(\gamma) = \operatorname{tr}\rho'(\gamma)$, $\forall \gamma \in \pi_1(Y)$

$G = SL_2(\mathbb{C})$

- hyperbolic geometry
- exceptional fillings
- cyclic surgery thm

$G = SU(2)$

- Casson invariant
- Instanton HF (Floer, Donaldson, Kronheimer-Mrowka...)

$G = SL_2(\mathbb{R})$

- real parabolic reps
- real ideal pts
- left orders
- L-space conjecture

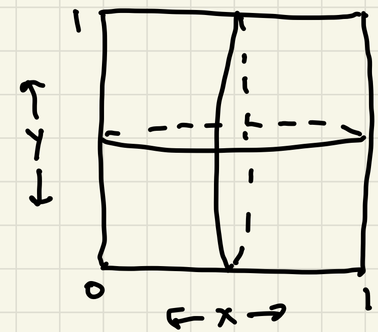
SU_2 character variety:

$X_{SU_2}(Y) = \{\rho : \pi_1(Y) \to SU_2\}/\sim$ $\quad$ $\rho \sim \rho'$ if $\exists A \in SU_2$ s.t. $\rho'(x) = A\rho(x)A^{-1}$

Ex: $Y = T^2$, $\pi_1(T^2) = \mathbb{Z}^2 = \langle m, \ell \rangle$

$\rho : \mathbb{Z}^2 \to SU_2 \Rightarrow \rho(m), \rho(\ell)$ commute

$\Rightarrow \rho(m), \rho(\ell) \in$ a maximal torus T



wlog $\rho(m) = \begin{pmatrix} e^{i\pi x} & 0 \\ 0 & e^{-i\pi x} \end{pmatrix}$

$\rho(\ell) = \begin{pmatrix} e^{i\pi y} & 0 \\ 0 & e^{-i\pi y} \end{pmatrix}$

$\Rightarrow X_{SU_2}(T^2) = T \times T / W$ Weyl group

$= S^1 \times S^1/\sim \quad (x,y) \sim (-x,-y)$

$=$ pillowcase orbifold

$Y = S^3 - \nu(K)$

$i_*: \pi_1(\partial Y) \to \pi_1(Y)$ induces

$i^*: X_{SU_2}(Y) \to X_{SU_2}(\partial Y)$

$\qquad\qquad\qquad\qquad \cong X_{SU_2}(T^2)$

General Properties:

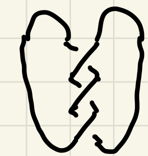
1) $X_{SU_2}(K) = X^{red}_{SU_2}(K) \cup X^{irr}_{SU_2}(K)$

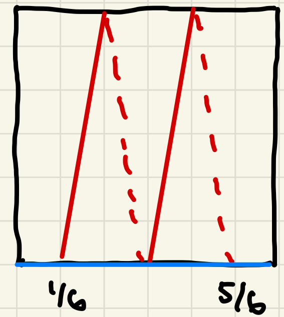
$X^{red} = \{[\rho] \mid \rho: \pi_1(S^3-K) \to H_1(S^3-K) \to SU_2\}$, $H_1(S^3-K) \cong \mathbb{Z}$

2) expected dim'n of $X^{irr} = 1$

3) X^{irr} limits to X^{red} at points of the form $(x,0)$ $\Delta_K(e^{2\pi i x}) = 0$

Ex: $K = T(2,3)$





$i^*(X_{SU_2}(K))$

$\Delta_T \sim \frac{t^3-1}{t-1}$

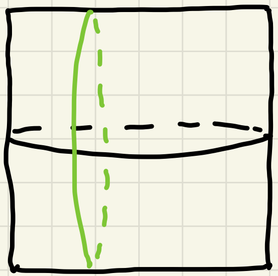
$S^3 - T$ Seifert fibred

$\pi_1(\partial Y) \to \pi_1(Y)$

$h \in$ center of $\pi_1(Y)$

$\Rightarrow \rho(h) = \pm I$ if $[\rho] \in X^{irr}$

Casson-Lin Invariant:

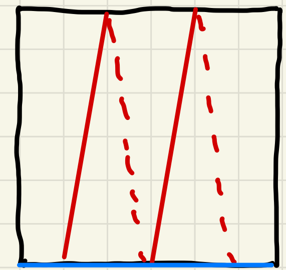


V_θ

$$V_\theta = \{(\theta, y)\} \subset X_{SU_2}(T^2)$$

$$h^c_{SU_2}(K) \text{ "="} \ X^{irr}_{SU_2}(K) \cdot V_\theta$$

jumps at roots of Δ_K



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Herald / Heusener-Kroll: $h^c_{SU_2}(K) = -\frac{1}{2}\,\sigma_K(\omega)$ $\Rightarrow$ $h^0_{SU_2}(K) = -\frac{1}{2}\sigma(K)$

$\omega = e^{2\pi i \theta}$

Levine-Tristram signature

$$X^c_{SU_2}(K) = \{[\rho] \in X_{SU_2}(Y) \mid \operatorname{tr} \rho(\mu) = c\}$$

$$h^c_{SU_2}(K) = \text{"signed count of points in } X^{c,irr}_{SU_2}(K)\text{"}$$

$$X^{2,irr}_{SU(2)}(K) = \emptyset$$

since $\operatorname{tr} \rho(\mu) = 2$

What if $G = SL_2(\mathbb{R})$?

$SL_2(\mathbb{R})$ not compact $\Rightarrow$ compactness issues w/ $X^c_{SL_2(\mathbb{R})}(K)$

Prop: If K is small, (no closed incompressible surfaces in Y)

can define $h^c_{SL_2}(K)$ which "counts points in $X^{c,\text{irr}}_{SL_2}(K)$"

Thm (Dunfield-R): If K is small, $\exists\, h(K) \in \mathbb{Z}$

s.t $\quad h^c_{SU_2}(K) + h^c_{SL_2}(K) \equiv h(K)$ for

all $\quad c \in [-2,2]$ with $\Delta_K(e^{2i\theta}) \neq 0$

$h(K)$ counts equivalence classes of

$$\rho : \pi_1(Y) \to \begin{cases} \widetilde{\mathrm{Isom}}^+(S^2) \\ \widetilde{\mathrm{Isom}}^+(\mathbb{E}^2) \\ \widetilde{\mathrm{Isom}}^+(\mathbb{H}^2) \end{cases} \quad \text{w/ } \operatorname{tr}_\mu = c$$

Geometric View:

$$h^c_{SL_2} = X_{SL_2}(K) \cdot V_\theta$$

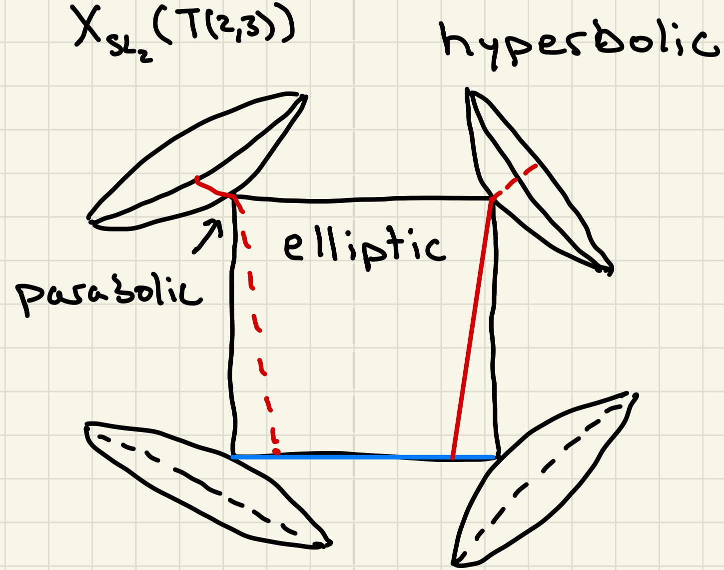
1-parameter subgroups of $SL_2(\mathbb{R})$ are not all conjugate.

3 conjugacy classes:

1) $T_e = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ elliptic

2) $T_p = \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix}$ parabolic

3) $T_h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ hyperbolic



$c=2$: $h(K)$ counts equivalence classes of real parabolic

$$\rho: \pi_1(Y) \to SL_2(\mathbb{R})$$

Properties: 1) $h(\overline{K}) = -h(K)$

2) $h(K_1 \# K_2) = h(K_1) + h(K_2)$

3) $h(K) \equiv -\frac{1}{2}\sigma(K) \mod 2$

Computations:

1) $h(K) = -\frac{1}{2}\sigma(K)$ if
- K is small alternating
- K is small Montesinos
- All K w/ ≤ 10 crossings except maybe 10_{161}

2) $h(T(p,q)) = g(T(p,q))$

Conj: If $K_p = L(p,q)$ (Dehn surgery)

$h(K) = \frac{1}{2}\#$ roots of Δ_K on S^1

Why Care?

$\pi_1(SL_2(\mathbb{R})) = \pi_1(SO_2) = \mathbb{Z}$ universal cover $\widetilde{SL}_2$

$$\begin{tikzcd} & \widetilde{SL}_2 \arrow[d] \\ \pi_1(Y) \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"'] & SL_2(\mathbb{R}) \end{tikzcd}$$

Obstruction to lifting is Euler class $e(\rho) \in H^2(\pi_1(Y))$

Def: G is left orderable (LO) if $\exists$ a total order on G with $x < y \Rightarrow gx < gy$.

Ex: G finite $\Rightarrow$ G not LO

> **Thm:** (Boyer-Rolfsen-Wiest): If Y^3 is prime and there is a nontrivial homomorphism $\pi_1(Y) \to \widetilde{SL}_2$, then $\pi_1(Y)$ is LO.

L-space Conjecture: If Y is a prime orientable 3-manifold, the following are equivalent:

Boyer-Gordon-Watson

Juhasz

1) $\pi_1(Y)$ is LO
2) Y admits a coorientable taut foliation
3) Y is not an L-space

Application: Branched covers

$\Sigma_n(K)$ = n-fold cyclic branched cover

$\varphi_n : \pi_1(S^3 - K) \to \mathbb{Z}/n$ $\qquad$ $\pi_1(\Sigma_n(K)) = \ker \varphi_n / \langle \mu^n \rangle$

$\rho : \pi_1(S^3 - K) \to SL_2$ with $\rho(\mu) \sim \begin{pmatrix} e^{ik\pi/n} & \\ & e^{-ik\pi/n} \end{pmatrix}$

$\hat{\rho} : \pi_1(\Sigma_n(K)) \to PSL_2\mathbb{R}$ with $e(\hat{\rho}) = 0$ $\qquad$ ρ irreducible $\Rightarrow \hat{\rho}$ nontrivial

<u>Thm</u> (Dunfield-R): If $\sigma_\omega(K) \not\equiv 0$, then $\Sigma_n(K)$ is LO for all $n \gg 0$

<u>Proof</u>:

<u>Ex</u>: $\sigma_\omega(4_1) \equiv 0$

$\Sigma_n(4_1)$ is L-space, not LO for all n.

(Gordon - Lidman)

Application: Real Parabolics

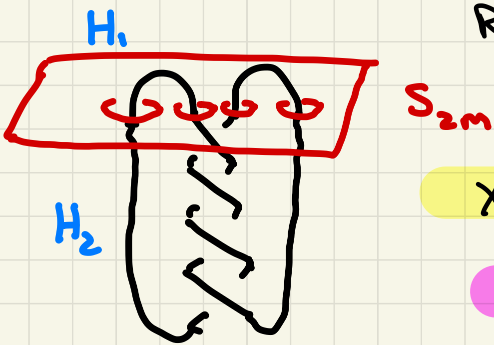
Riley Conj: (Gordon) If $K_{p/q}$ is 2-bridge with $\sigma(K_{p/q})=0$, then K has a real parabolic representation.

Thm (Dunfield-R): If K is small alternating or small Montesinos with $\sigma(K)\neq 0$, then K has a real parabolic representation.

Thm (D+R) If K is small $\sigma(K)\equiv 2\ (4)$, then $\hat{X}_{PSL_2\mathbb{C}}(K)$ has a real ideal pt.

Lin's Construction:

Plat closure: $S^3 - \nu(K) = H_1 \cup_{S_{2m}} H_2$ $\quad H_i$ = handlebody



$$\pi_1(S_{2m}) = \langle s_1, s_2, \dots s_{2m} \mid s_1 s_2 \cdots s_{2m} = 1 \rangle = F_{2m} / \langle \pi \rangle$$

$$R^c(F_{2m}) = \{\rho : F_{2m} \to SU_2 \mid \operatorname{tr} \rho(s_i) = c\}$$
$$= (S^2)^{2m}$$

$X^c(F_{2m}) = R^c(F_{2m}) / SU_2$ has dim'n $4m-3$

Singular at reducibles: locally modelled on cone on $\mathbb{CP}^{2m-2}$

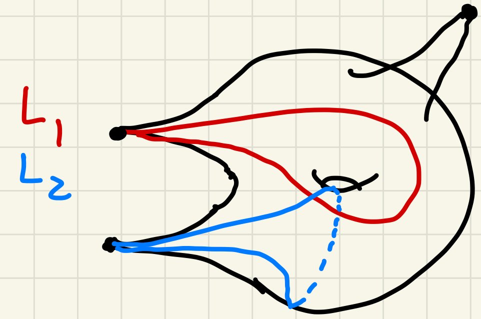
$F : R^c(F_{2m}) \to SU_2$, $F(\rho) = \rho(s_1)\rho(s_2)\cdots\rho(s_{2m})$

$R^c(S_{2m}) = F^{-1}(I)$ submanifold of dim'n $4n-3$

$X^c(S_{2m}) = R^c(S_{2m})$
- has dim'n $4m-6$
- singular at reducibles.

$X^c(H_j) \simeq X^c(F_m)$ has dim'n $2m-3$

$c_j^*: X^c(H_j) \hookrightarrow X^c(S_{2m})$ image $= L_j$



$X^c(S_{2m})$

$X^c(S^3 - K) = L_1 \cap L_2$

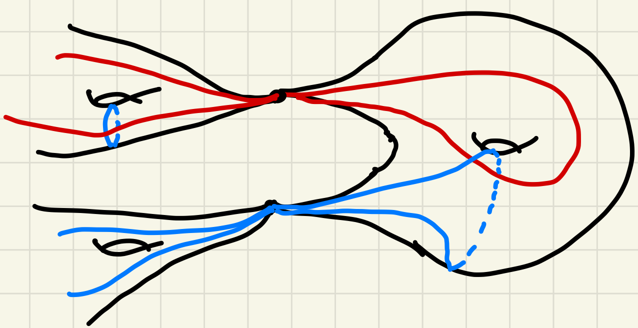
$h^c_{SU_2}(K) = \langle L_1, L_2 \rangle_{X^{c,irr}_{SU_2}(S_{2m})}$

$c = 2\cos\theta$

$X^c(S_{2m}) = X^c_{SL_2}(S_{2m}) \cup X^c_{SU_2}(S_{2m})$

$=$ real part of $X^c_{SL_2(\mathbb{C})}(S_{2m})$

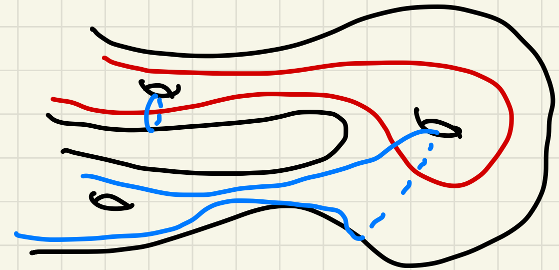
$X_{SL_2}(S_{2m})$



$X_{SU_2}(S_{2m})$

To define $h(K)$:

Resolve singularities by producing smooth manifolds



$$\mathcal{X}^c(F_{2m}) \longrightarrow \mathcal{X}^c(F_{2m})$$

$$\mathcal{X}^c(S_{2m}) \longrightarrow \mathcal{X}^c(S_{2m})$$

$$U_t = \left\{ \begin{pmatrix} a & b \\ -t\bar{b} & \bar{a} \end{pmatrix} \right\} \subset SL_2\mathbb{C}$$

$t>0 \quad U_t \simeq \mathrm{Isom}^+(S^2) \simeq SU_2$

$t=0 \quad U_t = \mathrm{Isom}^+(\mathbb{E}^2)$

$t<0 \quad U_t \simeq \mathrm{Isom}^+(\mathbb{H}^2) \simeq SL_2(\mathbb{R})$

$B_t =$ Killing form $\quad B_t(x,y,z) = t(x^2+y^2)+z^2$

$$Q_t = \{ v \in \mathbb{R}^3 \mid B_t(v) = 1 \}$$

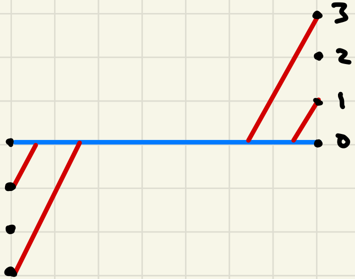
$$s\cdot(x,y,z) = (sx, sy, z) \qquad s\cdot Q_t = Q_{t/s^2}$$

$$\mathcal{R}^c(F_n) = \{ (t, v_1, \dots v_n) \mid \text{all } v_i \in Q_t,\ \exists\, i,j \text{ w/ } v_i \neq \pm v_j \}$$

$\mathcal{X}^c(F_n) = \mathcal{R}^c(F_n) / \mathbb{R}_{>0}$ and U_t.

Bonus Slide:

$X_{\widetilde{SL}}(T(2,5))$



$\tilde{h}(T(2,5)) = t + t^3$

Refined Lin Invt:

$\tilde{h}(K) = \sum n_i t^i$

n_i = signed # of arcs exiting at parabolic of height i

$\tilde{h}(K)|_{t=1} = h(K)$

$\deg \tilde{h}(K) \leq 2g(K) - 1$

Milnor-Wood

<u>Def</u>: $p(t) = \sum a_i t^i$ is good if all $a_i = 0, 1$.

If $K_p = L(p,q)$, $\tau_M(K) \sim \frac{\Delta_K(t)}{1-t}$ is good

<u>Ex</u>: $\tau_M(T(2,5)) =$

<u>Conj</u>: If $K_p = L(p,q)$, $\tau_M(K) + \tilde{h}(K)$ is good.

<u>Haydys</u>: $SL_2\mathbb{R}$ connections $\updownarrow$ solutions to SW w/ 2 spinors.