K-OS Seminar

A Levine-Tristram Invariant for tori

Daniel Ruberman

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A Levine-Tristram invariant for knotted tori

Goals of talk:

① Define an invariant $\sigma_\alpha(X, T) \in \mathbb{Z}$

- X is an oriented manifold
- $H_*(X) \cong H_*(S^1 \times S^3)$ "X is a $\mathbb{Z}$HSS"
- $T^2 \subset X$ embedded torus s.t. $H_1(T) \twoheadrightarrow H_1(X)$

"T is essential"

- X has a homology orientation := gen. $\gamma \in H^1(X)$ (technical point - for fixing signs)

2 Sample calculations + relation to gauge theory

(cf. M. Echeverria (2019) + L. Ma (2021))

$\sigma_\alpha(X,T)$ analogous to signature invariant $\sigma_\alpha(K)$ of Levine (1969) + Tristram (1969)

Ex: For $(X,T) = S^1 \times (S^3, K)$, $\sigma_\alpha(X,T) = \sigma_\alpha(K)$

Motivation part I: Milnor (1970) Duality Theorem

Let X^{n+1} be a $\mathbb{Z}HS^1 \times S^n$, $X_\gamma \xrightarrow{\pi} X$ the $\mathbb{Z}$-covering corresponding to $\gamma \in H^1(X) \cong \mathbb{Z}$

Then (for any field $\mathbb{F}$)

(a) $H_*(X_\gamma;\mathbb{F})$ is finite dimensional

(b) $H_*(X_\gamma;\mathbb{F})$ satisfies n-dimensional Poincaré duality.
'w/ coeff. $\mathbb{F}$

((b) follows from (a))

Slogan: X_γ is 'like' an n-manifold

Question
- In what ways is this true?
- Find invariants of closed n-manifolds that apply to X_γ

Invariants
- homological (signature if $n \equiv 0\ (4)$)
- geometric (Gromov norm)
- analytic (APS η-invariants)

Relative version of this question:

Q. What is a 'knot' in X?

Answer for today: An essential torus in X

* From now on: take $n=3$

What kinds of 3-dim'l knot invariants extend to this setting?

- (presumably) Alexander poly.
- signatures

Recall some details of Levine-Tristram signatures

$\alpha \in S^1 \subset \mathbb{C}$ $\alpha \neq 1$ $K \subset S^3$ (or $\mathbb{Z}HS^3$ Y or S^{2n+1} [Levine])

$K = \partial F^2$ oriented Seifert surface $A =$ Seifert matrix

$\sigma_\alpha(Y, K) =$ signature of Hermitian matrix $(1-\alpha)A + (1-\bar{\alpha})A^T$

Other interpretations :

1. (Viro 1973; Kaufmann-Taylor 1976)

F pushed into B^4

If α is of finite order n then $\sigma_\alpha(S^3, K) =$ equivariant signature of $Z^4 =$ n-fold branched cover of $(B^4, \check{F})$

2. $Y =$ any 3-mfd , $K \subset Y$ null-homologous

$Y_0(K) =$ 0-surgery on K

$\alpha \in S^1 \rightsquigarrow \varphi_\alpha : H_1(Y_0(K)) \to U(1) \quad \varphi_\alpha(\mu_K) = \alpha$

$\Omega_3(\mathbb{Z}) = 0$ so $\exists$ 4-mfd W, $\partial W = Y_0(K)$ + extension of φ_α

$$\begin{tikzcd} H_1 W \arrow[r, "\Phi_\alpha"] & U(1) \\ H_1(\cancel{S^3_0}\, Y_0(K)) \arrow[u] \arrow[ur, "\varphi_\alpha"'] & \end{tikzcd}$$

Atiyah-Patodi-Singer (1976)

Define $\sigma_\alpha(Y,K) = \rho_\alpha^{4}(Y_0(K)) = \text{sign}(W) - \text{sign}_{\Sigma_0}(W)$

- Agrees w/ previous definition

motivation part II - third interpretation

X-S Lin (1992) Defined an invariant of (S^3, K) by "counting"
(with signs façon de Casson)
irreducible $\varphi : \pi_1(S^3 - K) \to SU(2)$ s.t. $\varphi(\mu) \sim \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ i.e. $\operatorname{tr} \varphi(\mu) = 0$

conjugate

Theorem (Lin). $\#(\text{reps}) = \frac{1}{2}\sigma_{-1}(S^3, K)$

$(-1 = i^2)$

count with signs

Other meridional holonomies ?!

Herald (1997 - general ZHS^3 Y) w/ analytic def. of signs

Heusener - Kroll (1998 - S^3) give similar result

for any α (s.t. $\Delta_K(\alpha^2) \neq 0$)

non-degeneracy condition

(Herald)

$\#$(reps w/ hol of $\mu_K \sim \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$) $= 4\lambda(Y) + \frac{1}{2}\sigma_{\alpha^2}(Y,K)$

Echeverria (2019): (X,T) essential + smooth

Count (with Donaldson-type signs) irred. reps $\varphi : \pi_1(\cancel{X}) \to SU(2)$ $\pi_1(X-T)$

$\varphi(\mu_T) \sim \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$ to get invariant $\lambda_{FO}(X,T,\alpha)$

Furuta-Ohta

- $\lambda_{FO}(X) = \frac{1}{4}$ count irred reps $\pi_1(X) \to SU(2)$
- Non-degeneracy conditions ...

Product case : $\lambda_{FO}(S'_r Y, S'_r K) = 8\lambda_{FO}(X) + \sigma_{\alpha}(Y,K)$ (*)

Problem: (Echeverria): Find $\sigma_\alpha(X,T)$ so that (*) holds in general

i.e. (*) $\lambda_{FO}(X,T) = 8\,\lambda_{FO}(X) + \sigma_{\alpha}(X,T)$ (when everything is defined)

Theorem (R. 2020) For any essential $T \subset X$, a $\mathbb{Z}$HSS

- definition of $\sigma_\alpha(X,T)$
- (*) holds for examples where $\lambda_{FO}(X,T)$ & $\lambda_{FO}(X)$ computed.

L. Ma (2021) : (*) holds in general!

Idea of construction

- Suffices to define $\sigma_\alpha(X,T)$ for α of prime power ($d = p^r$) order

· Let $T \subset X$ be essential, $\gamma \in H^1(X)$

V = 0-surgery on X along T is a $\mathbb{Z}HT^2 \times S^2$

not well-def'd indicator

γ induces $\mathbb{Z}$-cover $V_\gamma \longrightarrow V$

· want to treat V_γ "like a 3-mfd" + define "$\rho_\alpha(V_\gamma)$"

[Does a $\mathbb{Z}$-cover of a homology $S^1 \times S^1 \times S^2$ satisfy Milnor duality?

As remarked before comes to: H_* of $\mathbb{Z}$ cover finite dim?

Example? $\underbrace{S^1 \times S^3}_{\gamma} \# S^1 \times S^3 \# S^2 \times S^2$
]

Observation: $V = 0$-surgery on $T \subset X$ is in fact a cohomology $S^1 \times S^1 \times S^2$

ring

Consequence (Pajitnov 1988; Papadima - Suciu 2010)

Taubes 1989

$H_*(V_\gamma; \mathbb{C})$ $\begin{cases} \text{finite dimensional} \\ \text{satisfies Milnor duality} \end{cases}$

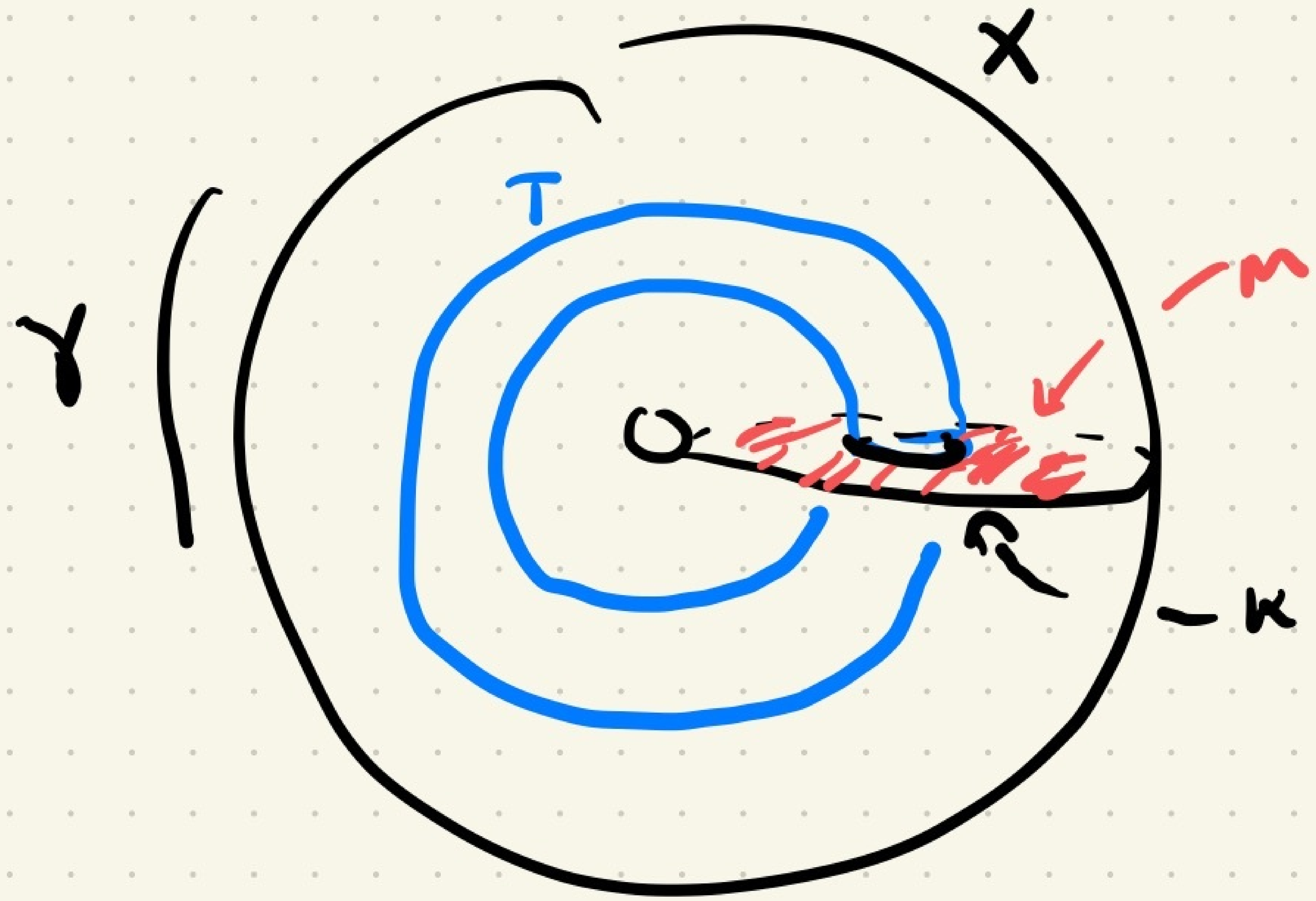
To define $\rho_\alpha(V_\gamma)$

- Analytical approach via Mrowka - R - Saveliev (2016) index theorem

  $\rightarrow$ too hard; hard to compute
- Topological approach:

Choose $M^3 \subset X$ $\quad [M] = PD(\gamma)$ s.t. $M \cap T$ = knot K

Note: $\mu_T = \mu_K$ + $\gamma[K] = 0$



Surger X along T $\rightsquigarrow V^4$

Surger M along $K \rightsquigarrow M_0(K)$

$$\begin{tikzcd}
H_1 V \arrow[r, "\varphi_\alpha"] & U(1) \\
H_1 M_0(K) \arrow[u] \arrow[ur, "\varphi_\alpha"'] &
\end{tikzcd}$$

$\alpha \in S^1$

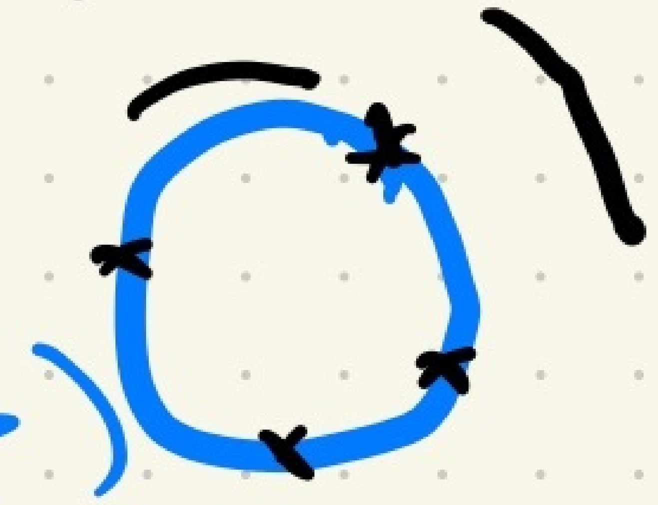
Main Theorem: ① If α has prime power order, then

$\rho_{\varphi_\alpha}(m_0(k))$ is independent of all choices !

② $\alpha \to \rho_\alpha$ extends to a piecewise constant function on S^1

Let $\bar{\rho}_\alpha$ = average of 1-sided limits, and (finally!)

Define: $\sigma_\alpha(X,T) = \bar{\rho}_{\varphi_\alpha}(M_0(k))$



Key points of theorem 1

① Suppose α has order p = a prime (p^n similar)

Then V_γ has a further p-fold cover $V_{\gamma,\alpha}$.

$$\begin{tikzcd} V_{\gamma,\alpha} \arrow[r, "\mathbb{Z}/p"] \arrow[d, "\mathbb{Z}"] & V_\gamma \arrow[d, "\mathbb{Z}"] \\ V_\alpha \arrow[r, "\mathbb{Z}/p"] & V \end{tikzcd}$$

- $V_{\gamma,\alpha}$ is itself a $\mathbb{Z}$-cover of V_α
- V_α is a $\mathbb{Z}/p$-cohomology $T^2 \times S^2$

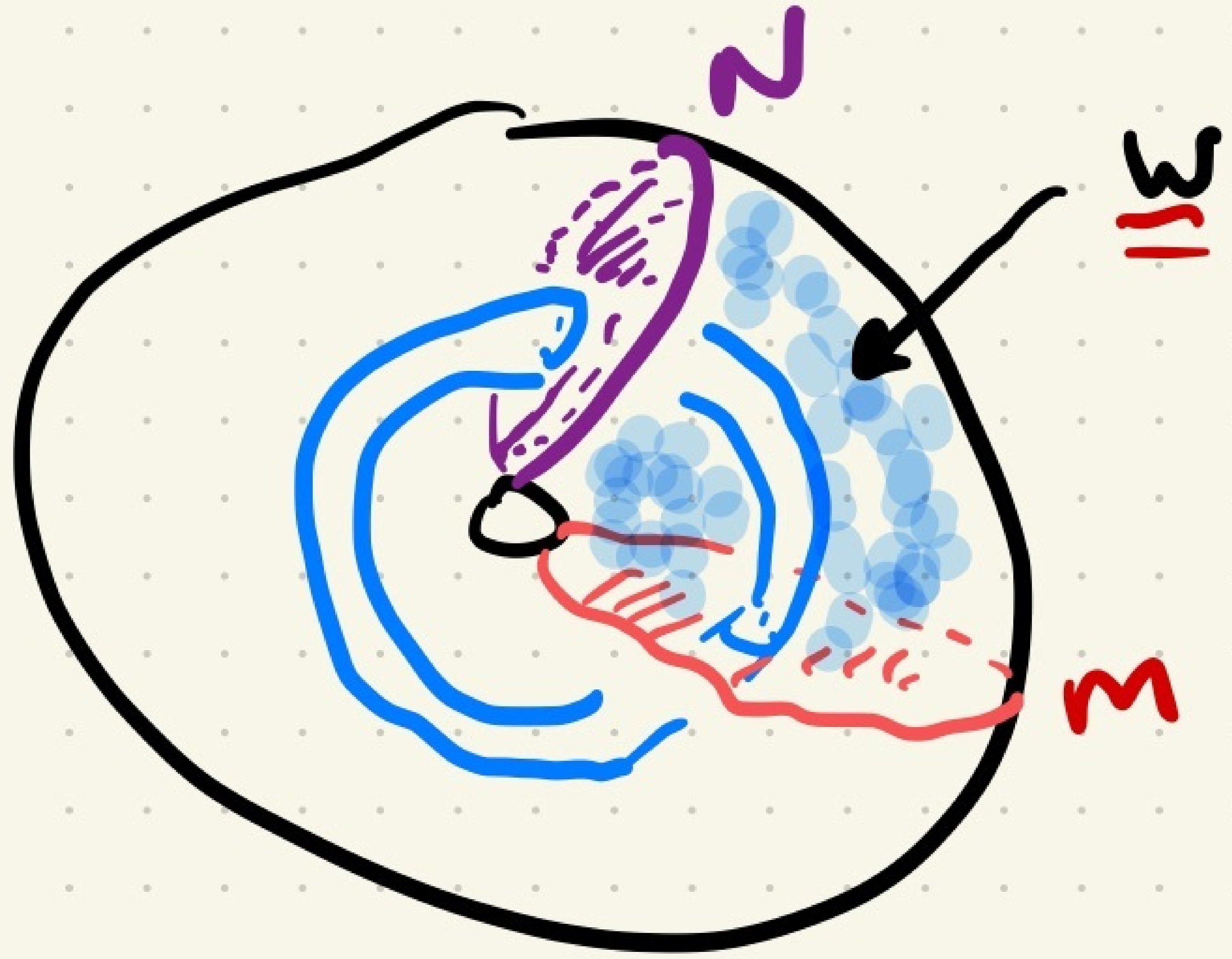
} $V_{\gamma,\alpha}$ satisfies milnor duality (with $\mathbb{C}$ coefficients)

② Independence from choice of $M \subset X$:

Suppose M, N are two choices:

If $M \cap N = \emptyset$ then have cobordism W between M & N and hence a cobordism W_0 between M_0 & N_0

Summary:



$$\rho_{\varphi_d}(N_0) - \rho_{\varphi_d}(M_0)$$

$$= \mathrm{sign}(W_0) - \mathrm{sign}_{\Phi_d}(W_0)$$

$= 0$ (using duality from (1) above)

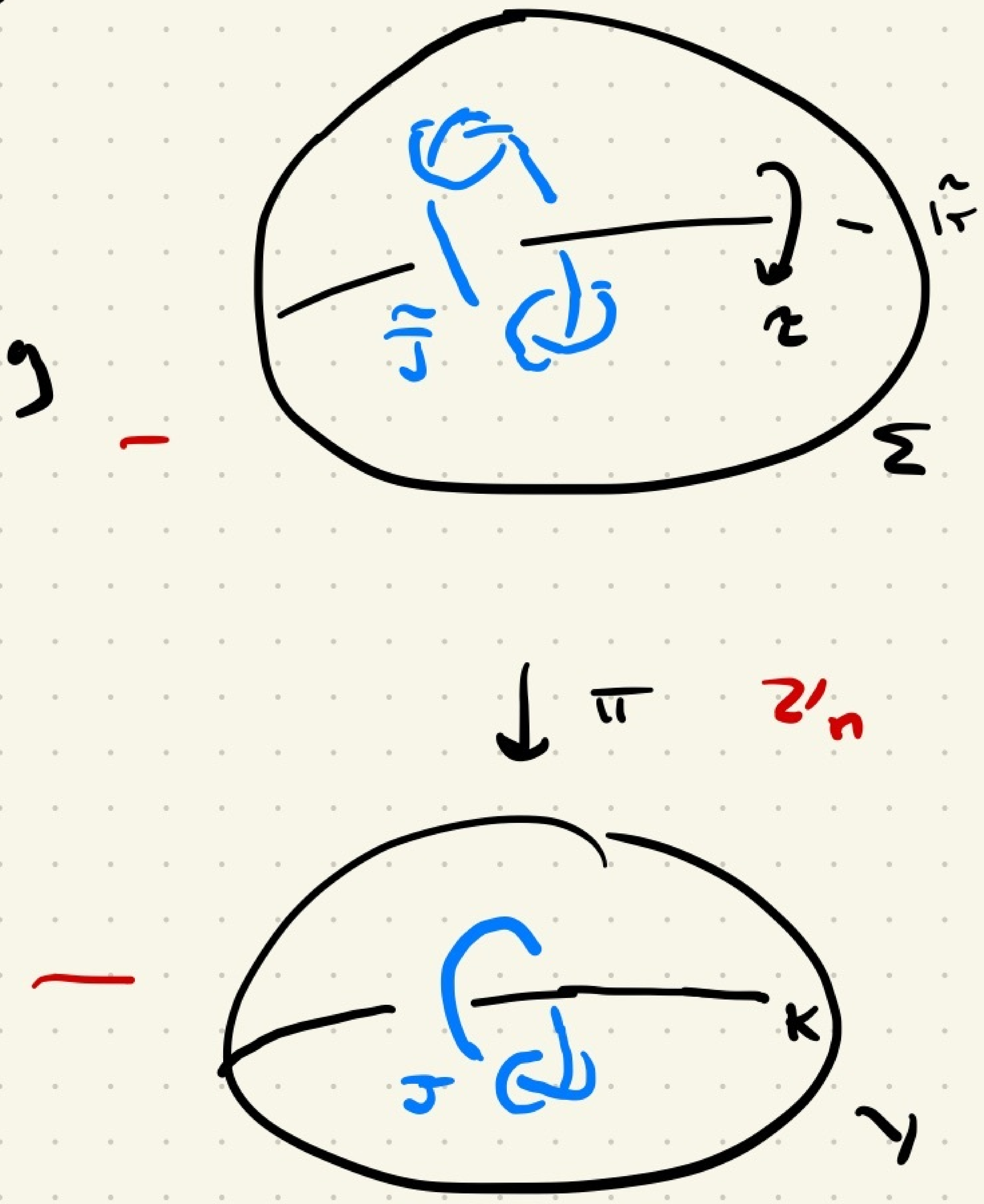
[If $M \cap N \neq \emptyset$ then a bit more work...]

Examples:

1. product
2. Let $L = (J, K)$ be a link in $Y = \mathbb{Z}HS^3$ and fix n relatively prime to $lk(J,K)$.

Let $\Sigma = \Sigma_n(K)$ n-fold br. cover.

$$\begin{array}{c} \Sigma \\ \downarrow \pi \\ Y \end{array} \qquad \tau : \Sigma \to \Sigma \text{ generating covering transformations}$$

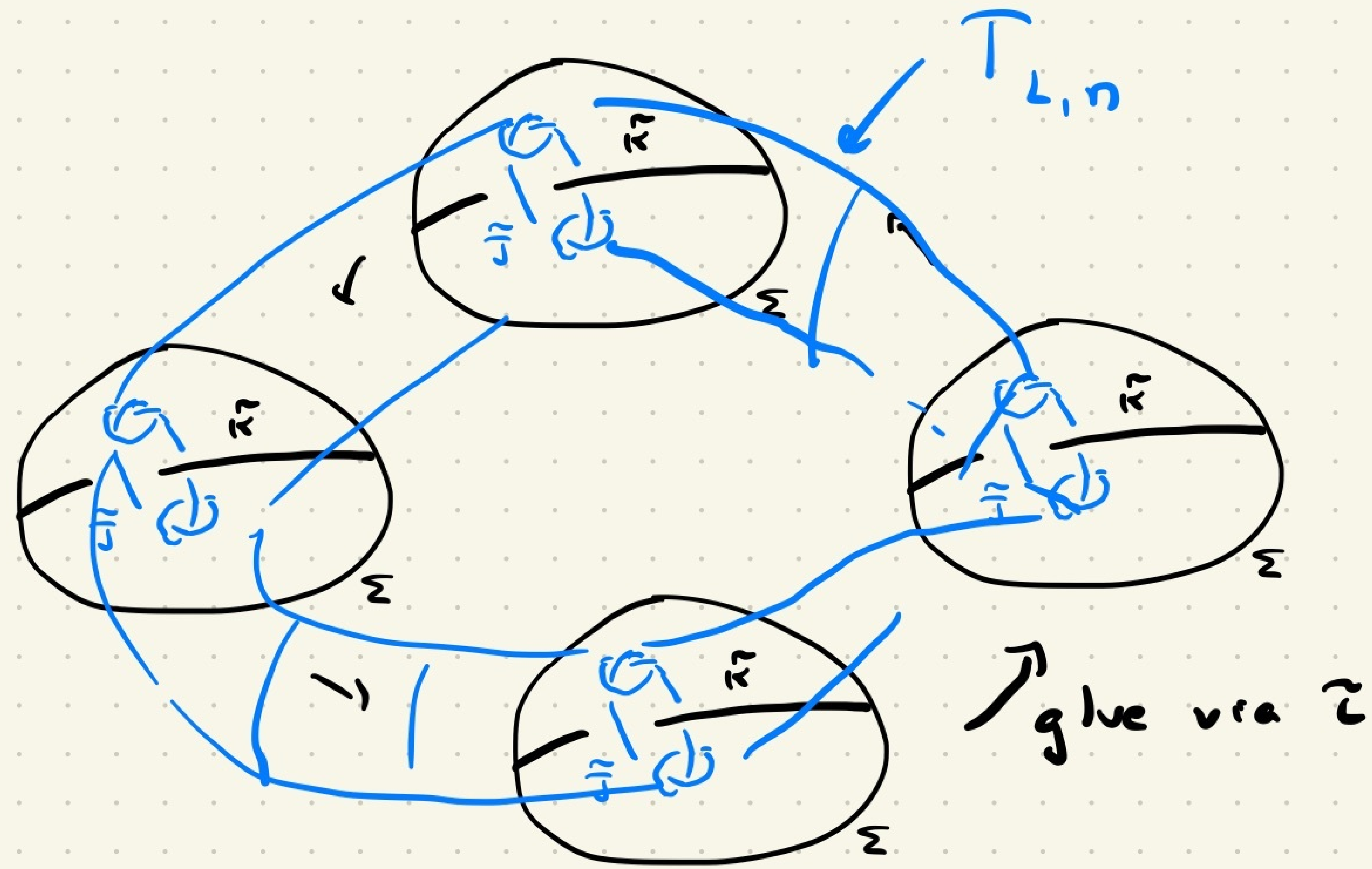


$\tilde{K} = \pi^{-1}(K)$
$\tilde{J} = \pi^{-1}(J)$
both are knots.

Build two knotted tori in $X = S^1 \times_\tau \Sigma$

$T_{K,n} = S^1 \times_\tau \tilde{K}$

$T_{L,n} = S^1 \times_\tau \tilde{J}$



Let α have order $d = p^n$ &

write $\omega = e^{2\pi i/dn}$, $\alpha = \omega^n = e^{2\pi i/d}$

Theorem: For any k,

(a) $\sigma_{\alpha^k}(X, T_{K,n}) = \sigma_{\alpha^k}(\Sigma, \tilde{K})$

$$= \sum_{j=1}^{n-1} \sigma_{\omega^j}(Y, K) + \sum_{j=0}^{n-1} \sigma_{\omega^j+k}(Y, K)$$

(b) $\sigma_{\alpha^k}(X, T_{L,n}) = \sum_{j=0}^{n-1} \sigma_{\omega^j, \alpha^k}(Y, L) - \sum_{j=1}^{n-1} \sigma_{\omega^j}(Y, K)$

L-T invariant of link L

Remark: (a) Echeverria (+ correction by Ma) calculated

R-Saveliev 2004

$$\lambda_{FO}(X, T_{K,n}) = 8\lambda_{FO}(X) + \sigma_{\alpha^2}(X, T)$$ from above.

(b) + Ma's result $\rightsquigarrow$ New calculation of $\lambda_{FO}(X, T_{L,n})$