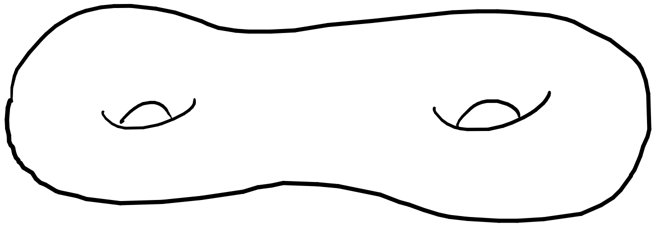
K-OS, 2021-09-30, 60 min talk

Handlebodies, Trivial tangles and Group Trisections for Knotted Surfaces

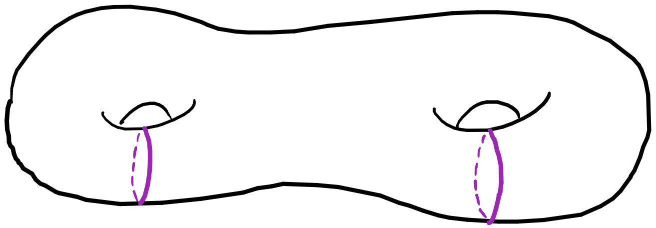
with Sarah Blackwell, Rob Kirby, Michael Klug and Vincent Longo

Benjamin Matthias Ruppik, 3rd year PhD student at the Max-Planck-Institute for Mathematics, Bonn

Handlebodies:

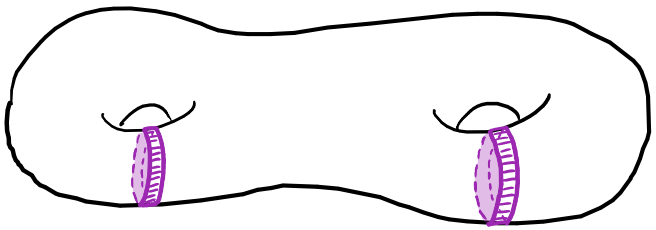


surface Σ_g



cut system of a handlebody:

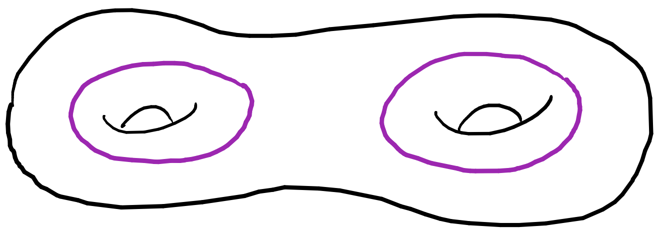
curves on Σ_g

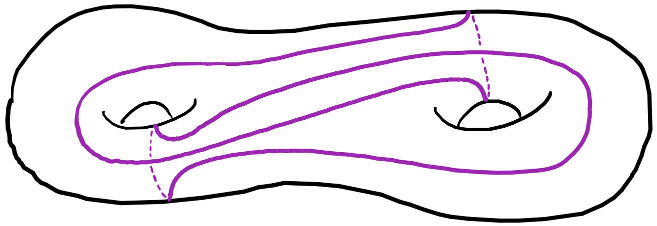


attach 2-handles along the curves

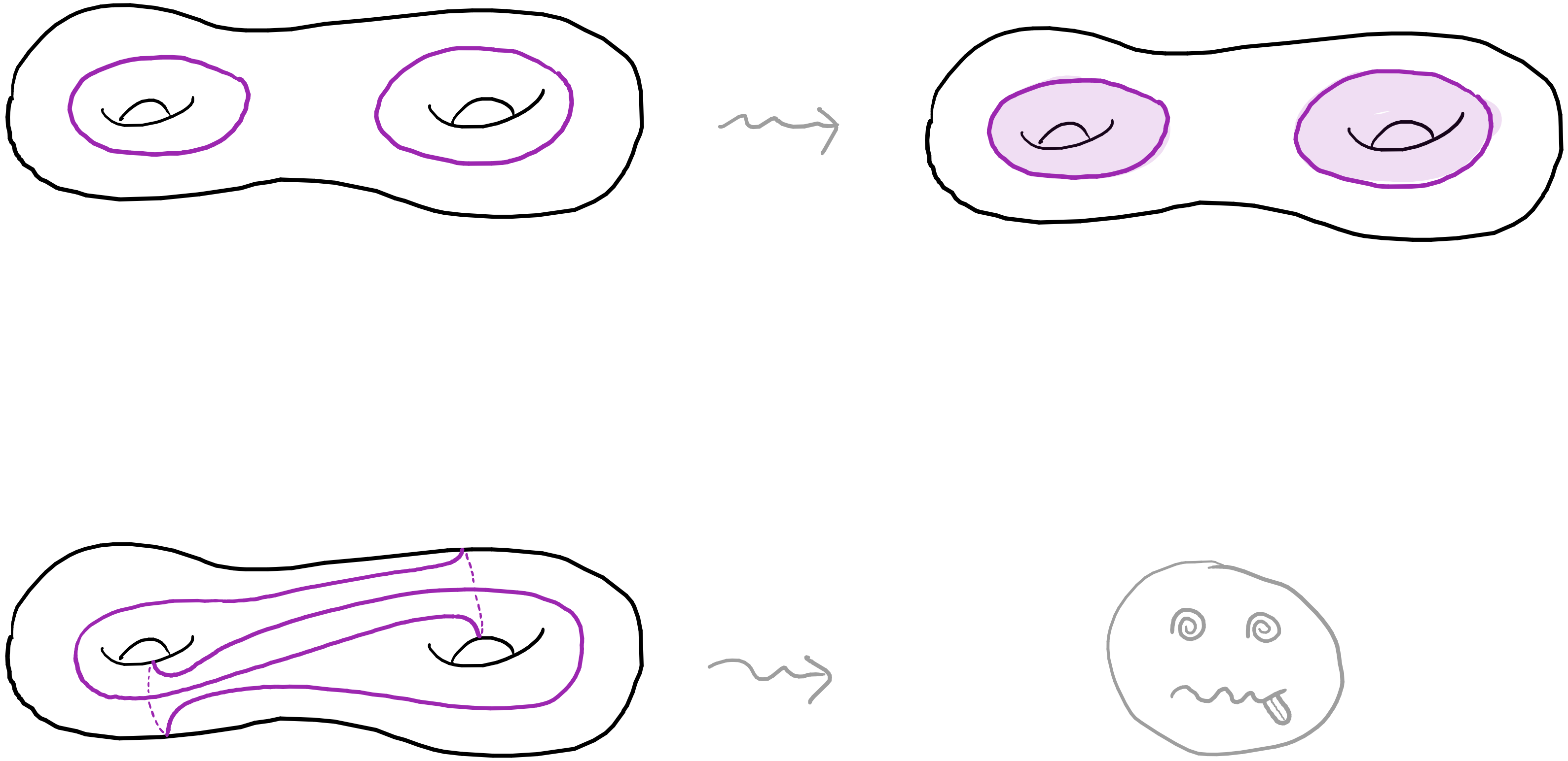
fill 2-sphere boundaries with 3-balls

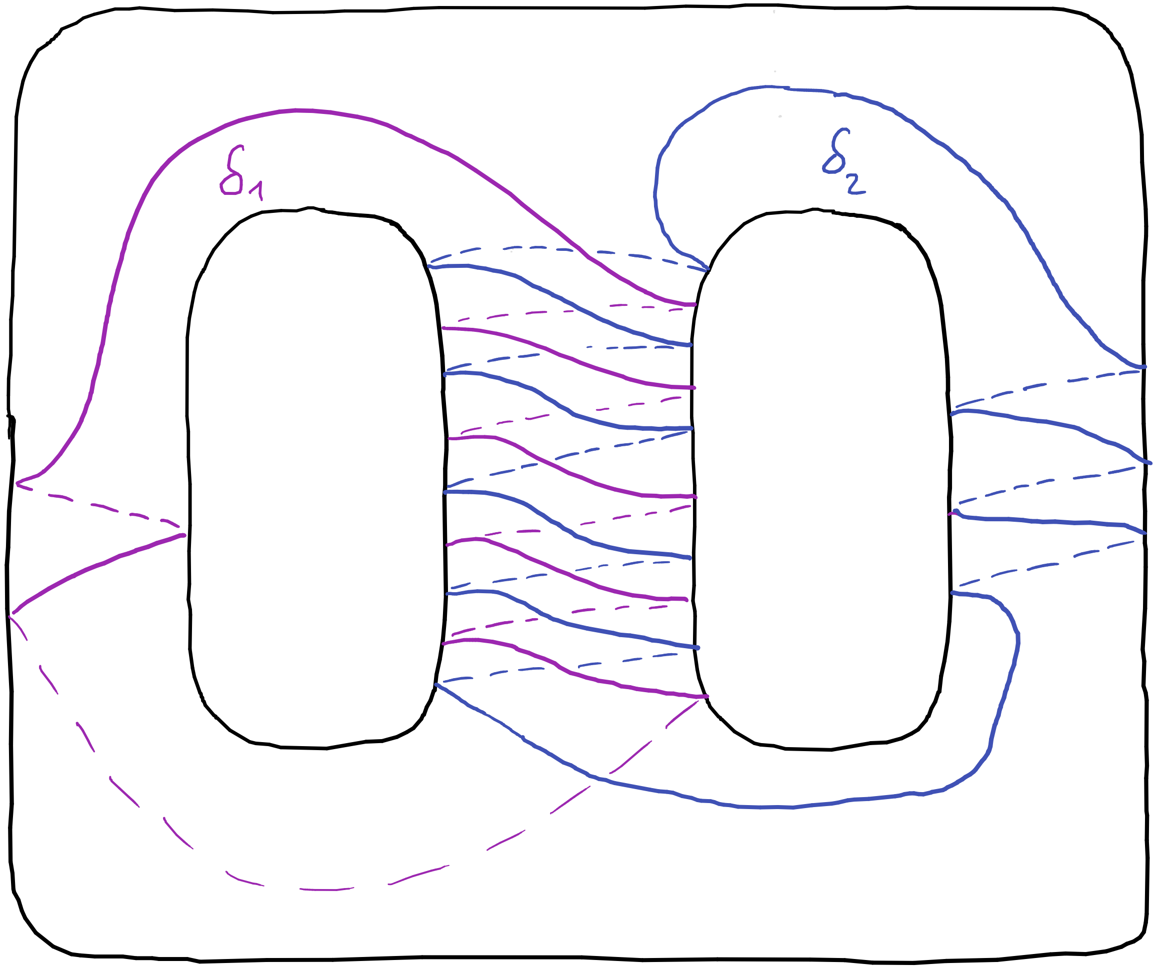
Can you see the handlebodies?





Can you see the handlebodies?

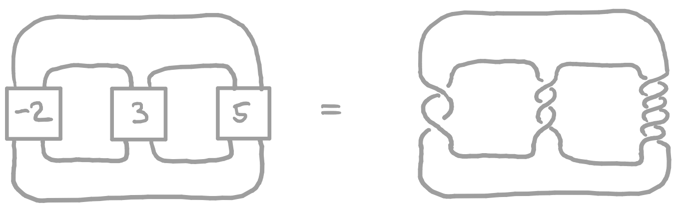


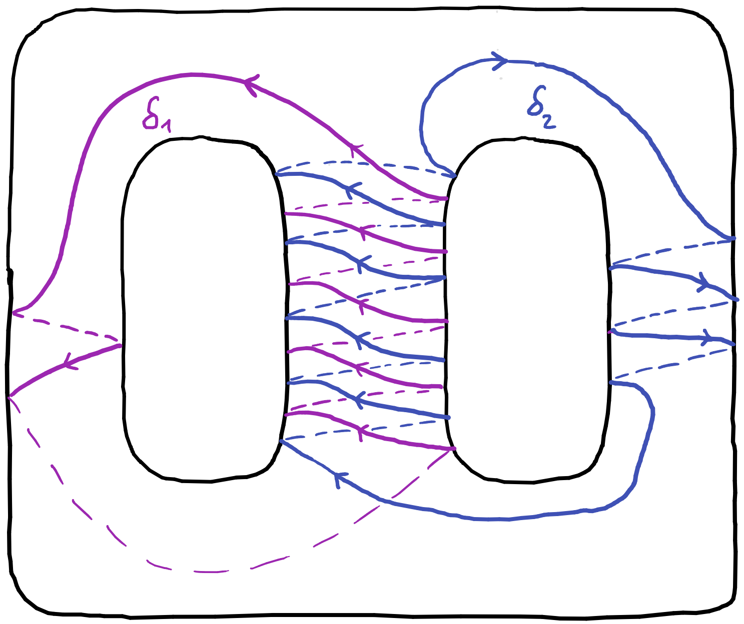


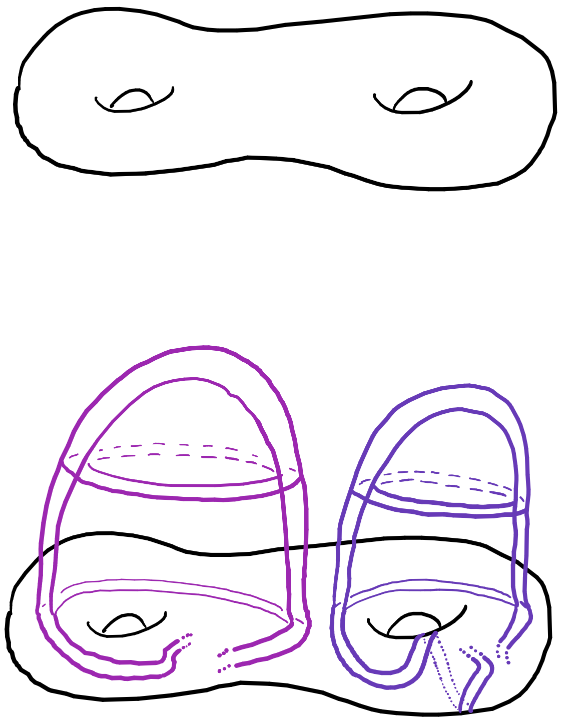
<u>Side remark:</u> This is one of the handlebodies in a genus 2 Heegaard diagram for the 3-mfld. P = Poincaré homology sphere

P = double branched cover $\Sigma_2(K)$ of S^3 branched over

$K = (-2,3,5)$ Pretzel knot



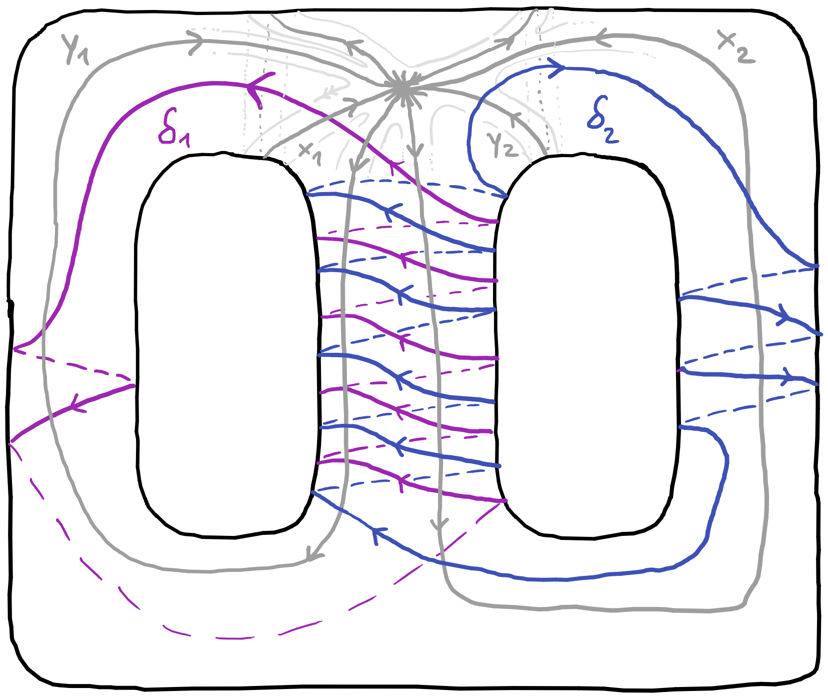




Σ_2

$\Sigma_2 \cup$ 2-handle $\cup$ 2-handle

Topology



Algebra

$$\pi_1(\text{surface}) \longrightarrow\!\!\!\!\rightarrow \pi_1(\text{handlebody})$$

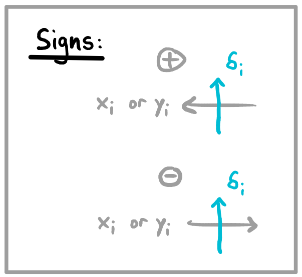
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \longrightarrow\!\!\!\!\rightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^{3}$$

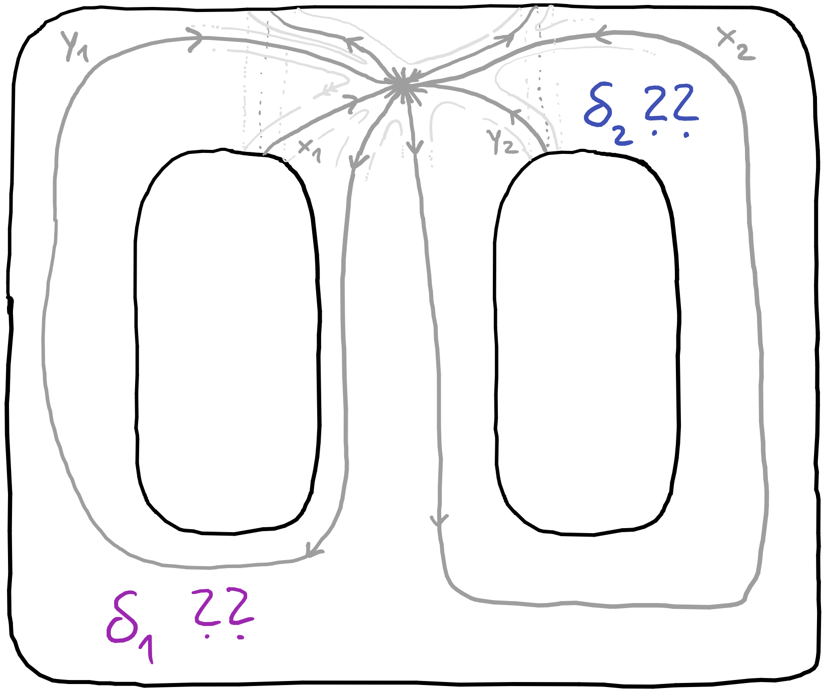
$$y_2 \longmapsto d_2$$



Topology

↑

Algebra



$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

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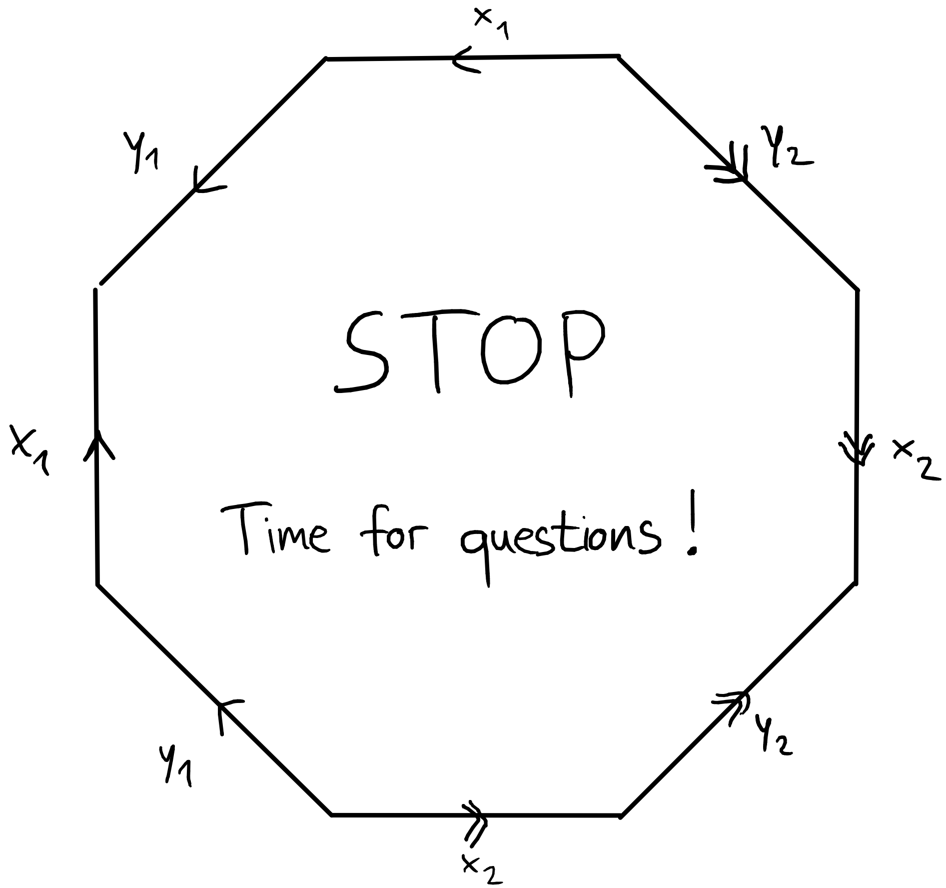

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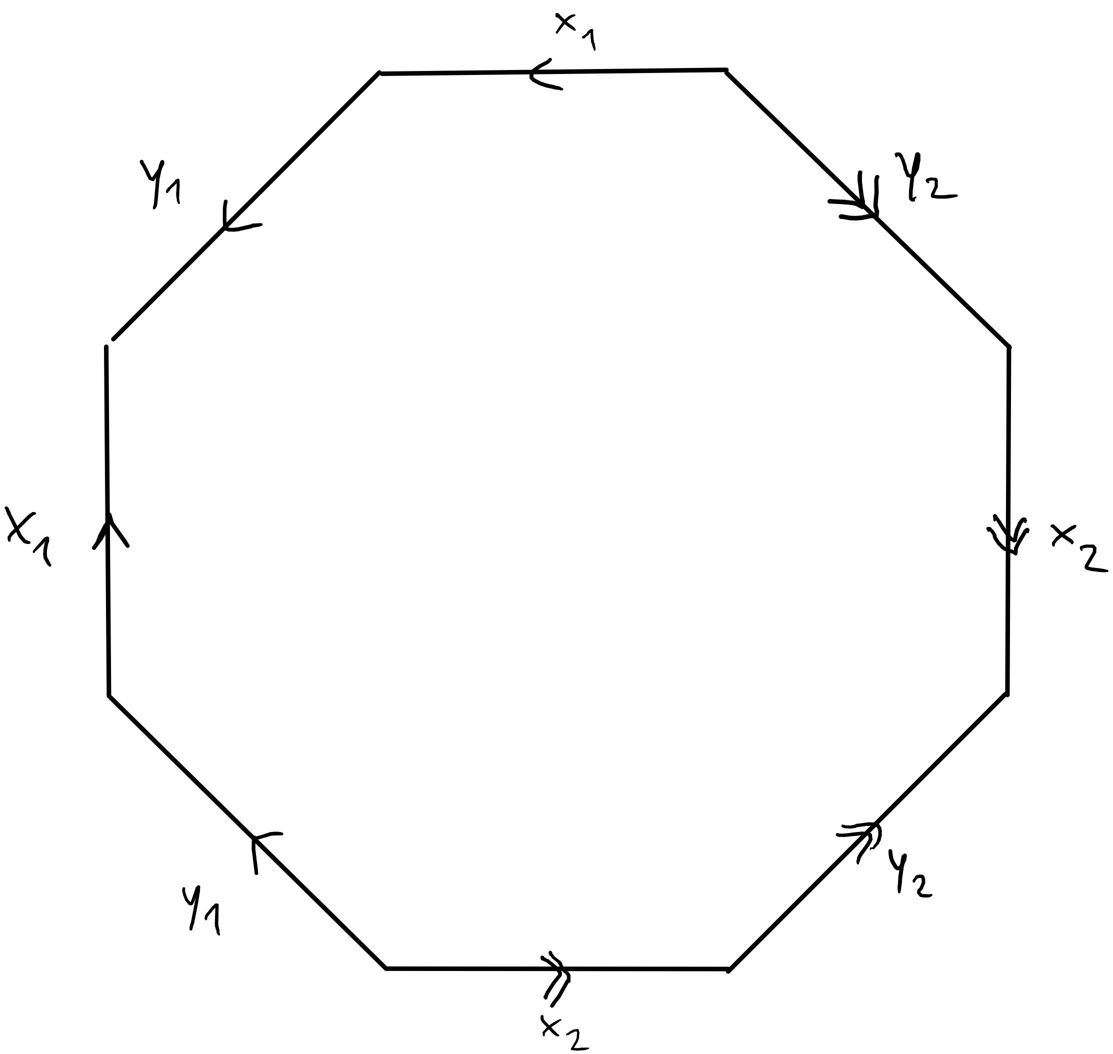
Surface relation:

$$x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$$

$$\mapsto$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2} (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3} (d_1 d_2)^{-5}][d_2^{-1}]$$



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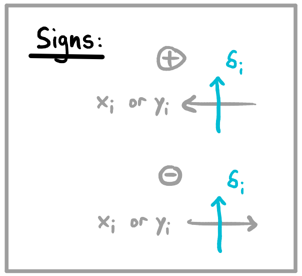
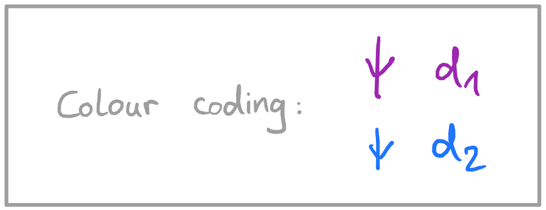
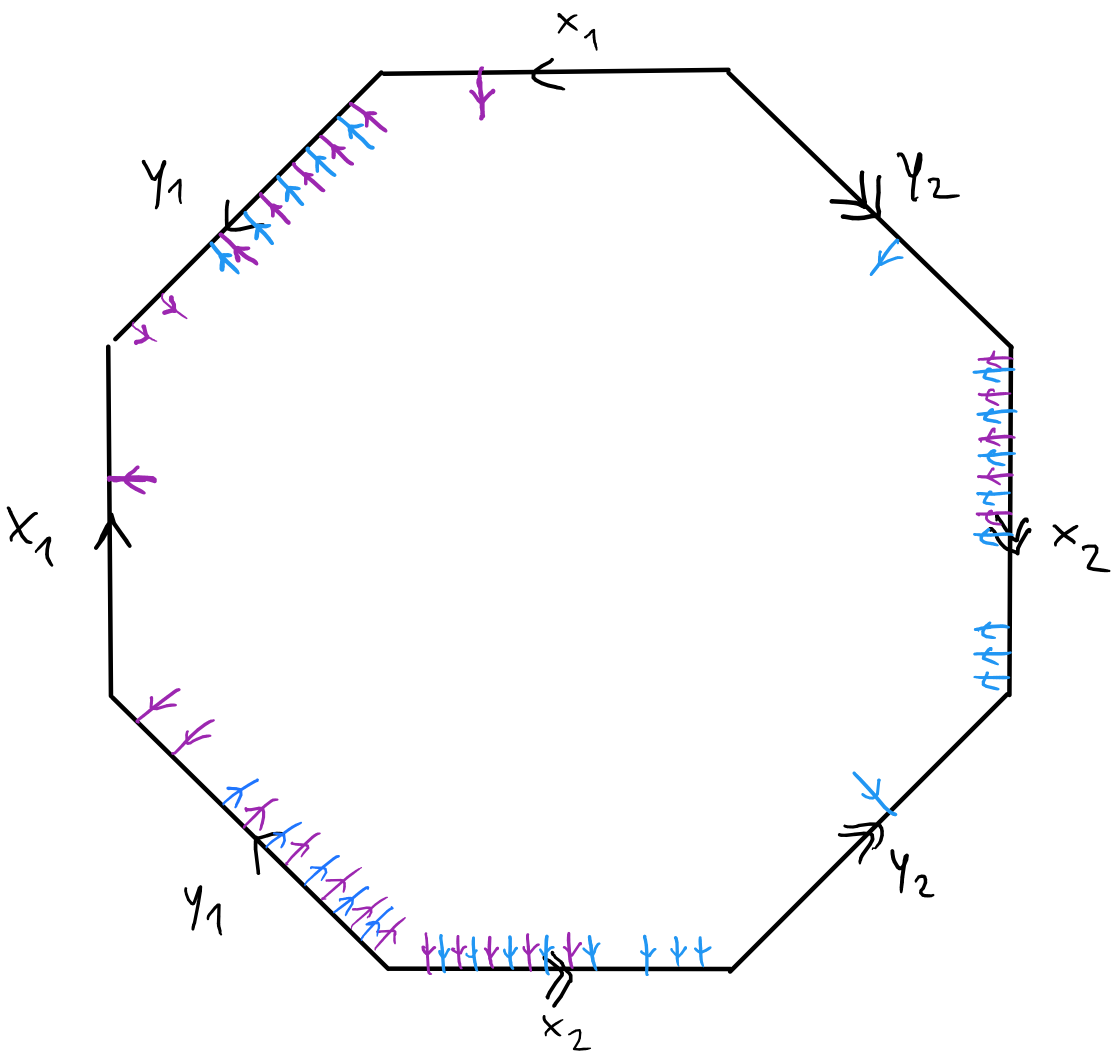




Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2} (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3} (d_1 d_2)^{-5}][d_2^{-1}]$$



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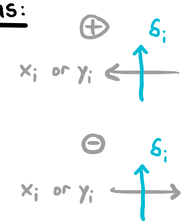
Signs:

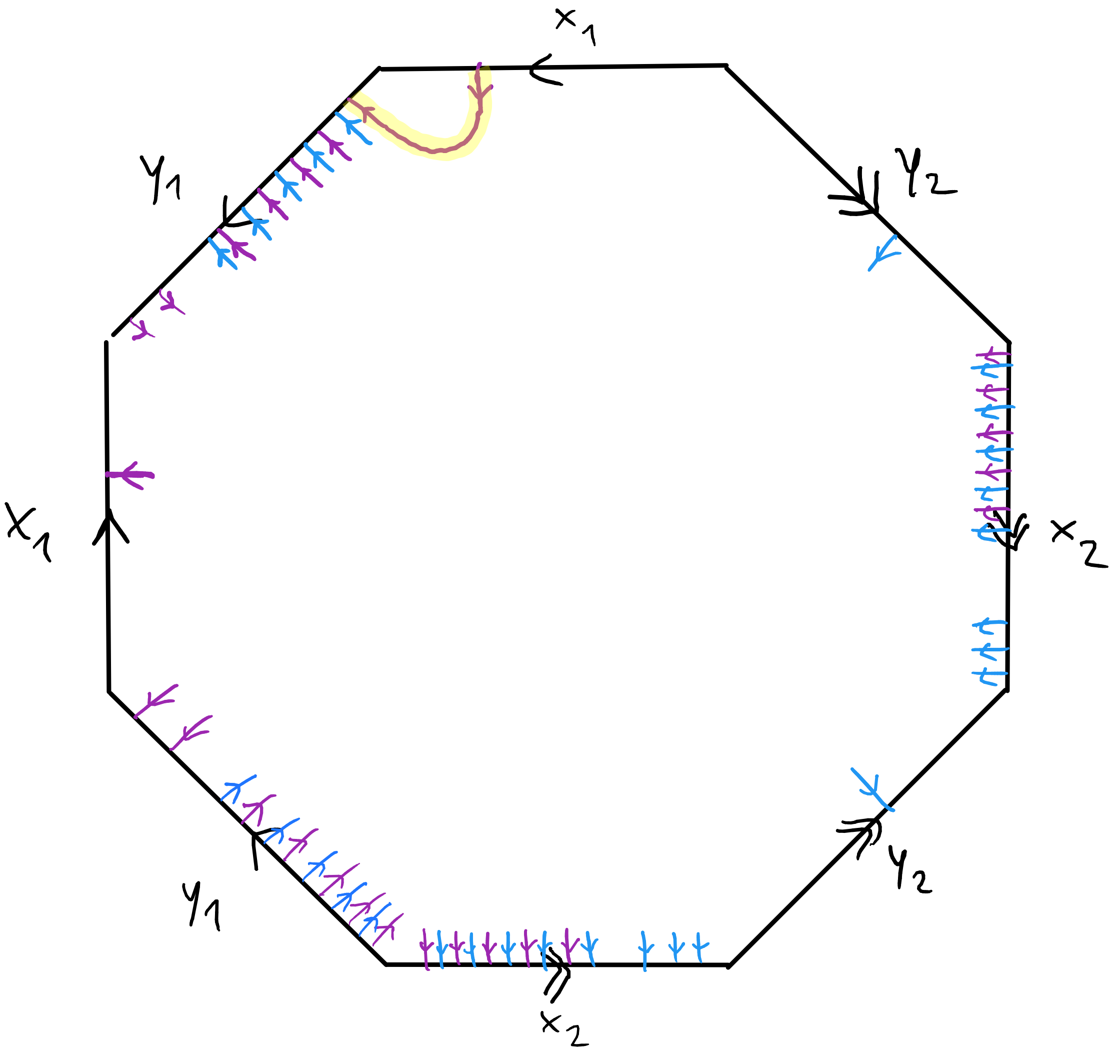


Colour coding: purple arrow d_1, blue arrow d_2

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



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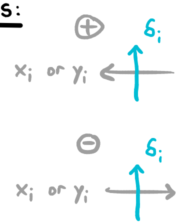
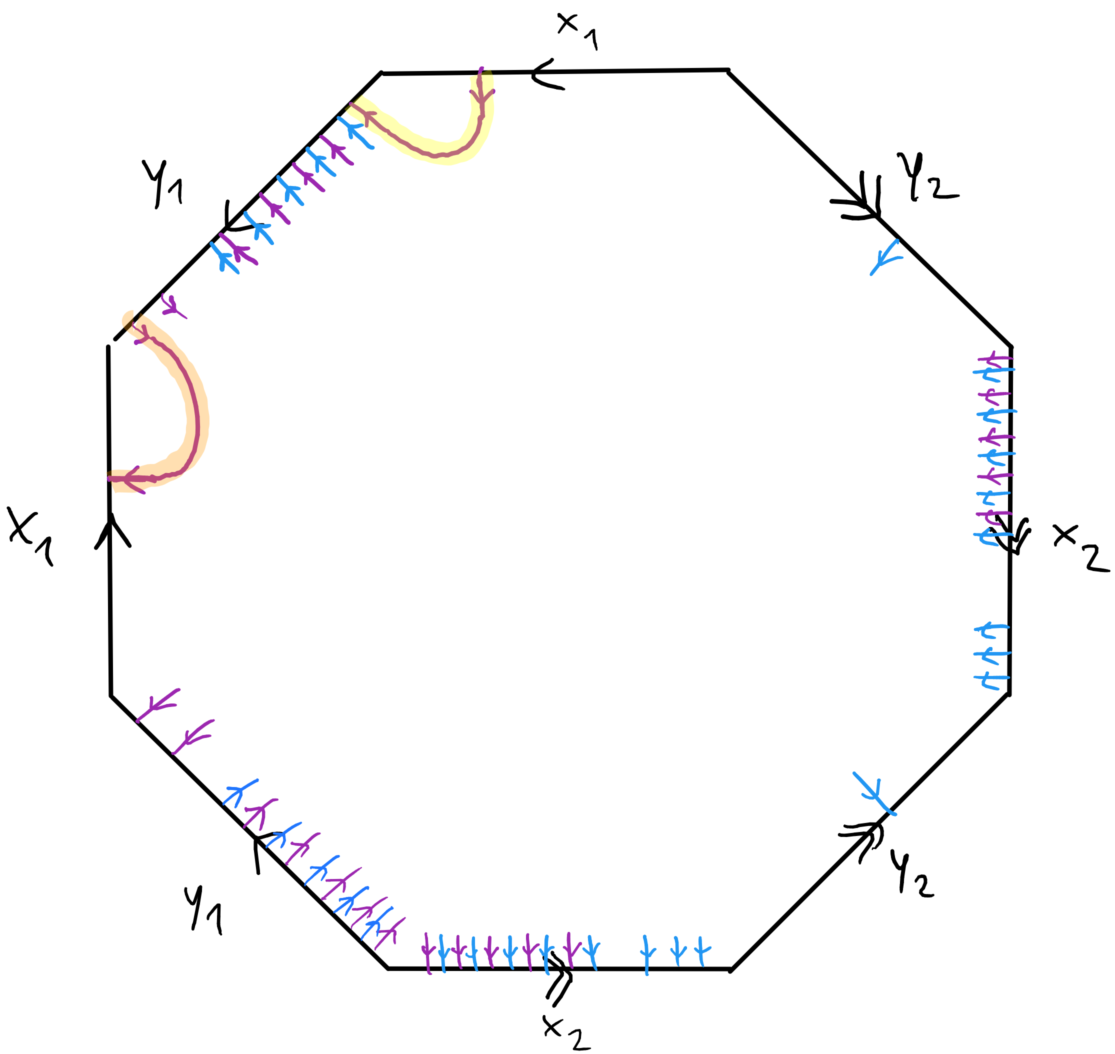


Colour coding: d_1 (purple), d_2 (blue)

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2} (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3} (d_1 d_2)^{-5}][d_2^{-1}]$$



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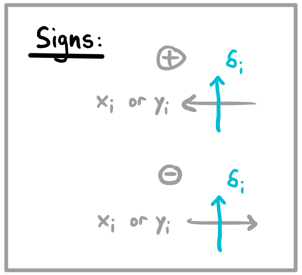

Colour coding: ↓ d_1 ↓ d_2

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

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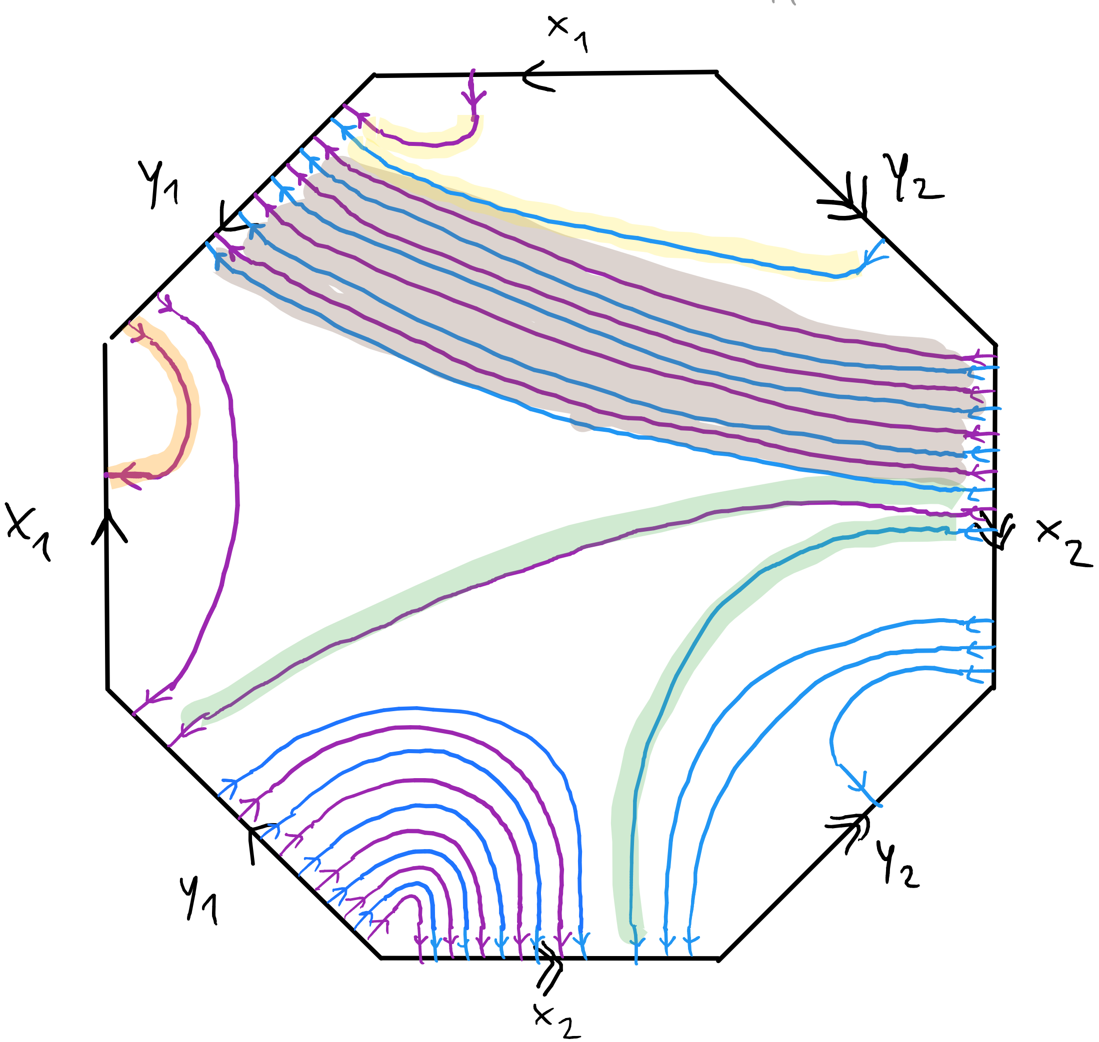
$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$

$x_1 \quad y_1 \quad x_1^{-1} \quad y_1^{-1} \quad x_2 \quad y_2 \quad x_2^{-1} \quad y_2^{-1}$



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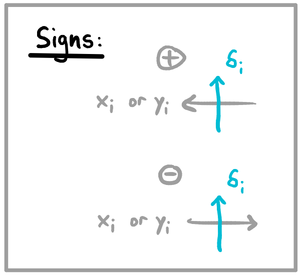
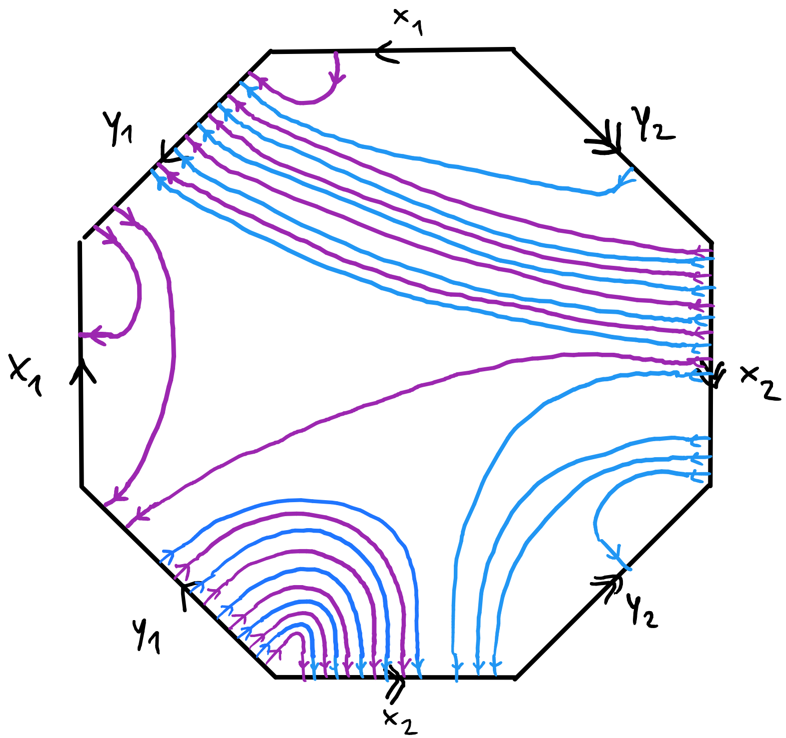


Colour coding: d_1 d_2

Topology

Algebra



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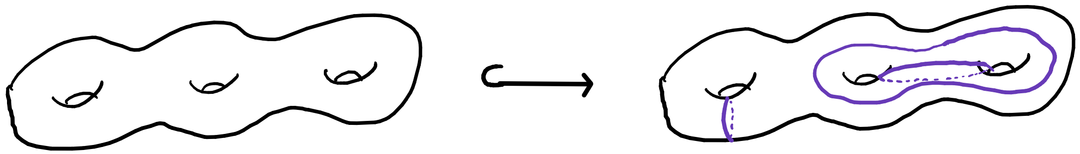
From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} F_g$

surface group $\twoheadrightarrow$ free group

is (uniquely) realized geometrically by a handlebody.

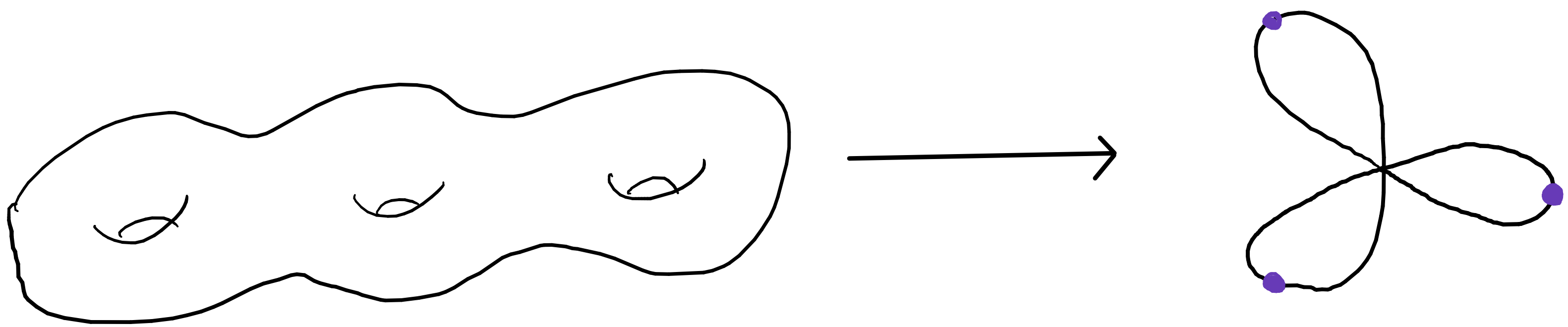


Folklore proof sketch:

Homomorphism $\pi_1(\Sigma_g) \overset{\varphi}{\twoheadrightarrow} F_g$

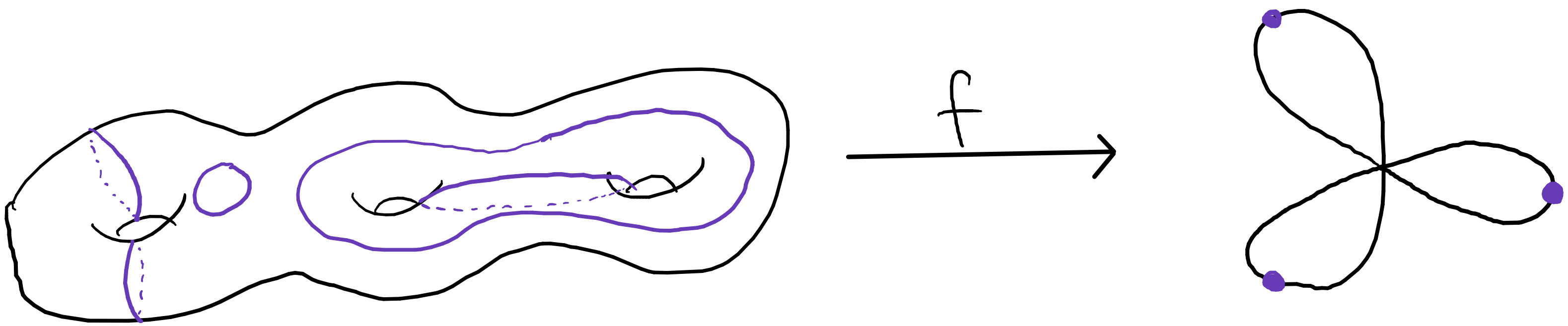
determines a unique map up to homotopy $\Sigma_g \xrightarrow{f} \bigvee^g S^1$

$\Sigma_g \simeq K(\pi_1(\Sigma_g), 1)$, $\bigvee^g S^1 \simeq K(F_g, 1)$



make map transverse to north poles

$$\Sigma_g \xrightarrow{\ f\ } \bigvee^{g} S^1$$



look at preimage

f^{-1}(North poles)

make map transverse to north poles

Collection of simple closed curves in Σ_g which contains a cut system

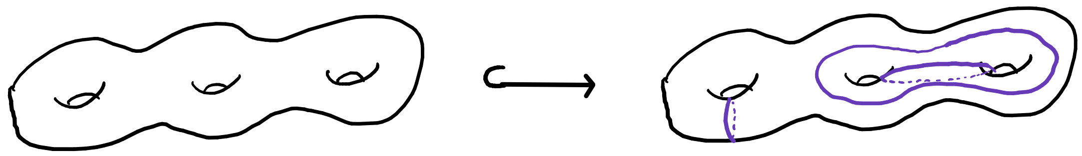
$\square$ (Folklore)

From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} F r_g$

surface group $\twoheadrightarrow$ free group

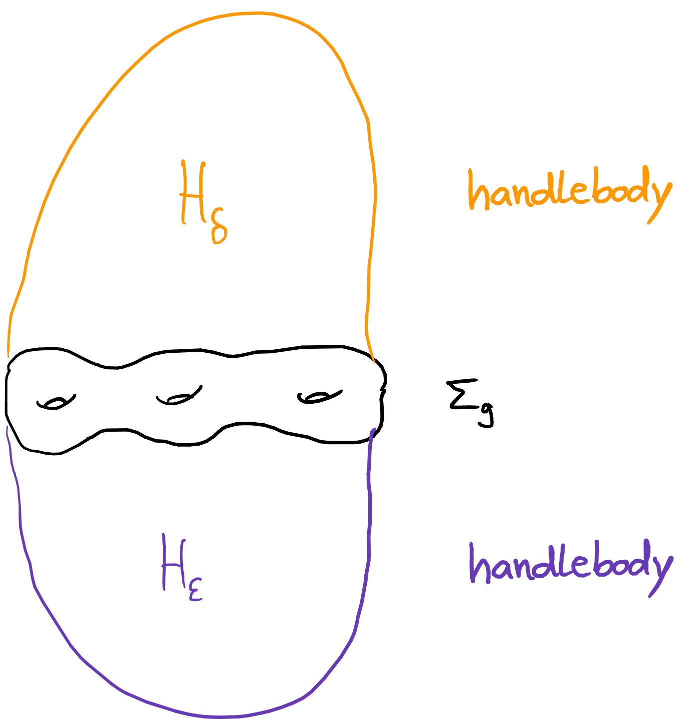
is realized geometrically by a handlebody (uniquely) ...

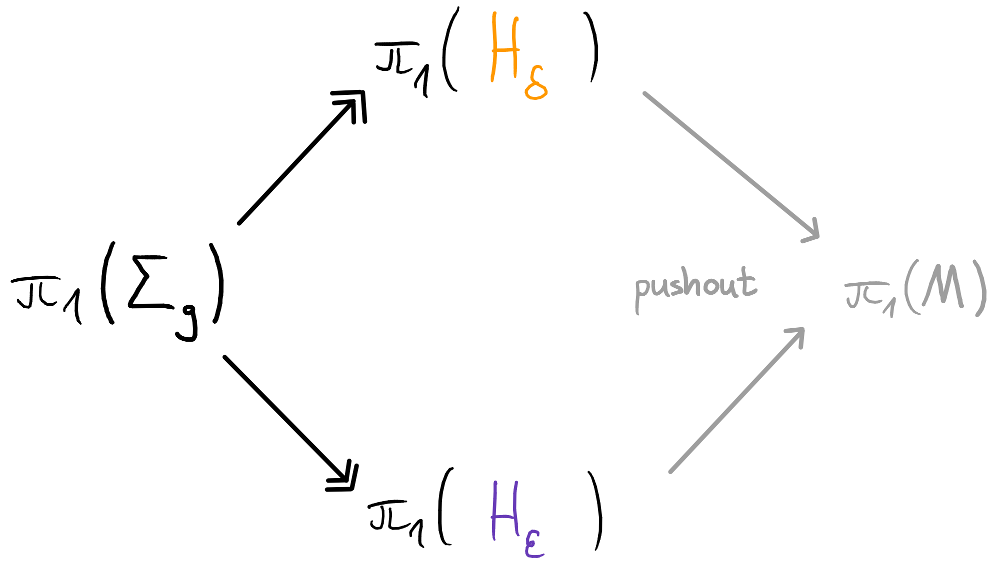


[Blackwell-Kirby-Klug-Longo-R, 2021]

... which can be computed algorithmically.

Heegaard splitting of a 3-manifold M^3



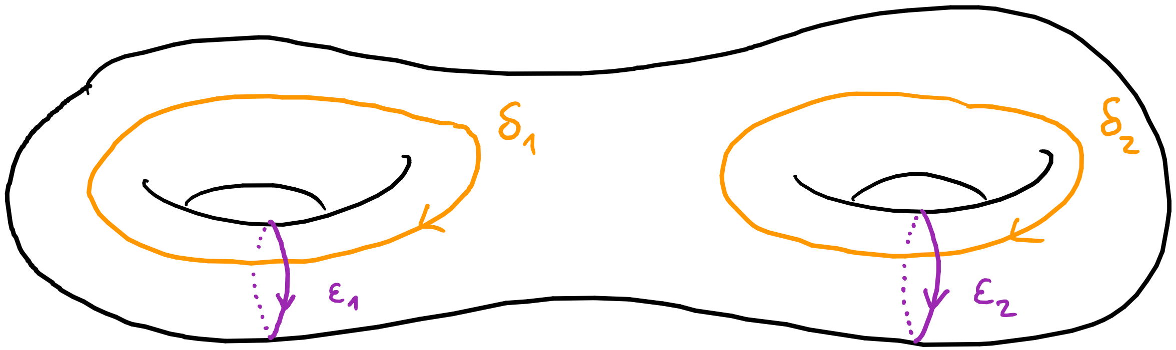


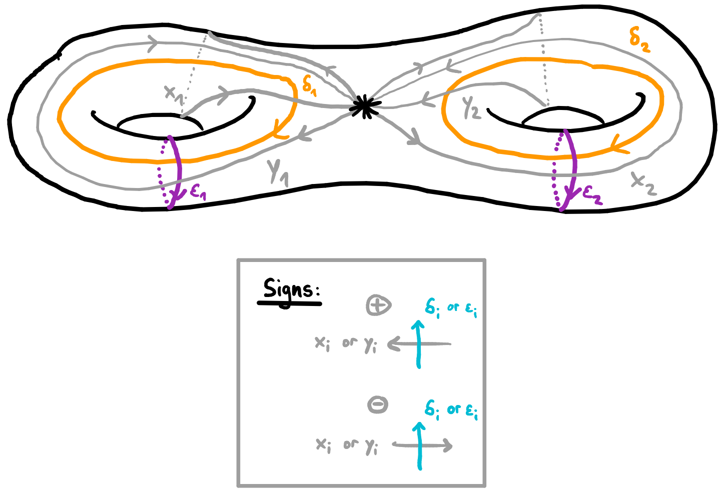
Splitting homomorphism

[Jaco : Heegaard splittings and splitting homomorphisms (1969)]

[Stallings : How not to prove the Poincaré conjecture (1966)]

Ex.: Splitting homomorphism for genus 2 splitting of S^3





$x_1 \mapsto d_1$

$y_1 \mapsto 1$ (trivial)

$x_2 \mapsto 1$

$y_2 \mapsto d_2$

$$\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1]\cdot[x_2, y_2] \rangle \to \langle d_1, d_2 \rangle$$

$$\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1]\cdot[x_2, y_2] \rangle \to \langle e_1, e_2 \rangle$$

$$\langle d_1, d_2 \rangle \to \langle d_1, d_2 \rangle *_{\pi_1(\Sigma)} \langle e_1, e_2 \rangle \cong \langle 1 \rangle$$

$$\langle e_1, e_2 \rangle \to \langle d_1, d_2 \rangle *_{\pi_1(\Sigma)} \langle e_1, e_2 \rangle \cong \langle 1 \rangle$$

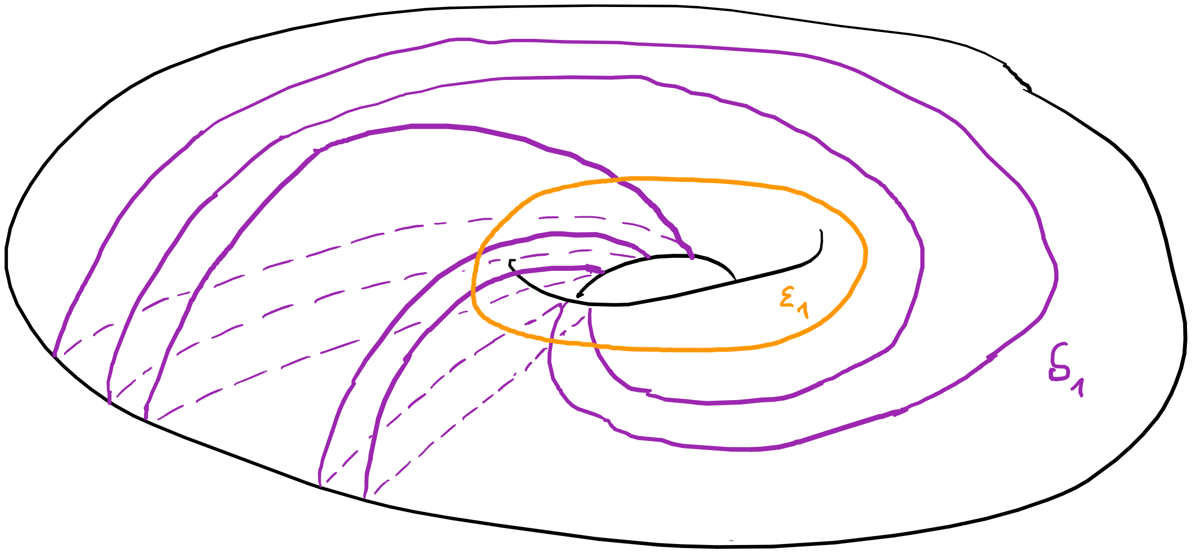
$x_1 \mapsto 1$ (trivial)

$y_1 \mapsto e_1^{-1}$

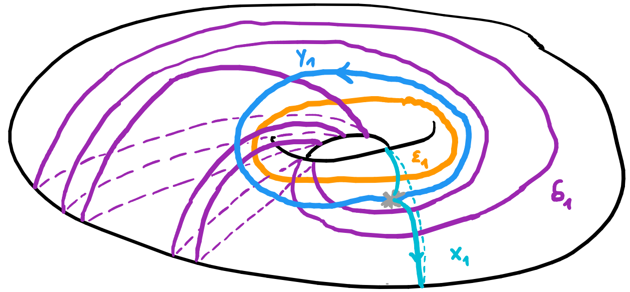
$x_2 \mapsto e_2$

$y_2 \mapsto 1$

Ex.: Splitting homomorphism for genus 1 splitting of $L(5,2)$



x_1, y_1 generators of $\pi_1(\Sigma)$



$$\langle x_1, y_1 \mid [x_1, y_1] \rangle \to \langle d_1 \rangle: \quad x_1 \mapsto d_1^2, \quad y_1 \mapsto d_1^5$$

$$\langle x_1, y_1 \mid [x_1, y_1] \rangle \to \langle e_1 \rangle: \quad x_1 \mapsto e_1, \quad y_1 \mapsto 1 \text{ (trivial)}$$

$$\langle d_1 \rangle \to \langle d_1 \rangle *_{\pi_1(\Sigma)} \langle e_1 \rangle \leftarrow \langle e_1 \rangle$$

$$\langle d_1 \rangle *_{\pi_1(\Sigma)} \langle e_1 \rangle \cong \mathbb{Z}/5$$

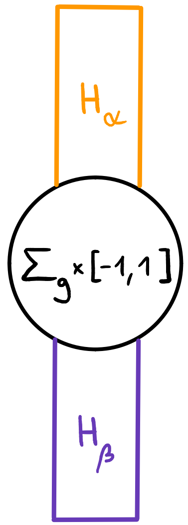
$$\left(\begin{array}{l} x_1 = d_1^2 \\ y_1 = d_1^5 \\ x_1 = e_1 \\ y_1 = 1 \end{array} \right)$$

$d_1^5 = 1 \qquad e_1 = d_1^2$

$\Bigg\{$ (based, parameterized)

Heegaard splittings

of a 3-manifold Y^3



$\Bigg\}$

take π_1 of pieces $\downarrow$

1 : 1

[Stallings, Jaco]

$\uparrow$ glue the handlebodies corresponding to the epimorphisms to $\Sigma_g \times \{-1\}$ and $\Sigma_g \times \{1\}$ respectively

$\Bigg\{$ group bisections of $\pi_1(Y,*)$

$$\begin{array}{ccccc} & & \pi_1(H_\alpha) & \twoheadrightarrow & \pi_1(Y) \\ & \nearrow\!\!\!\!\nearrow & & \nearrow\!\!\!\!\nearrow & \\ \pi_1(\Sigma_g) & \twoheadrightarrow & \pi_1(H_\beta) & & \end{array}$$

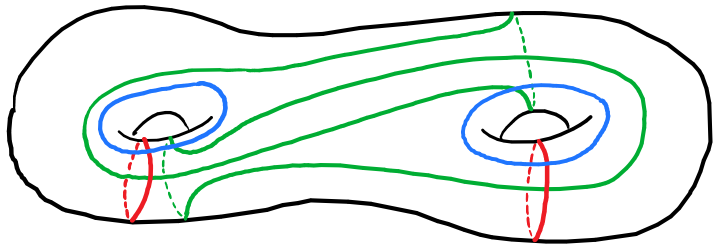
$\Bigg\}$

Plan:

- Recall the 4-dim. closed case where triples of handlebodies determine 4-manifolds

  ⇝ group trisection

  [Abrams, Gay, Kirby]

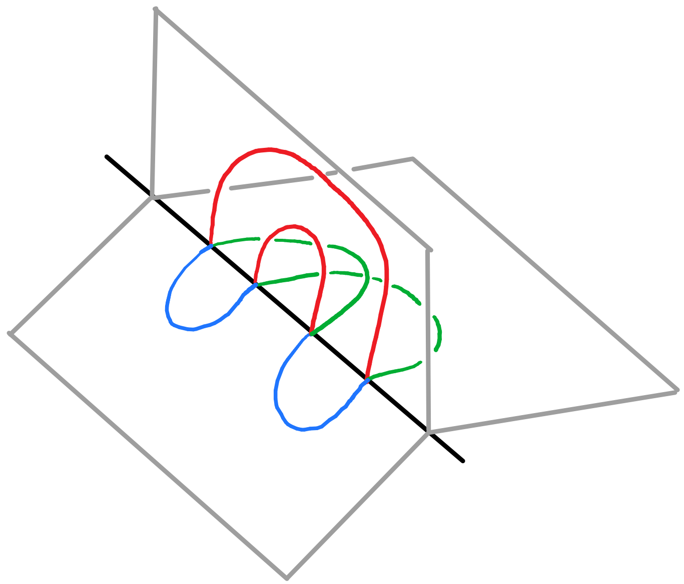


- Relative case:

  bridge-trisected surface F^2

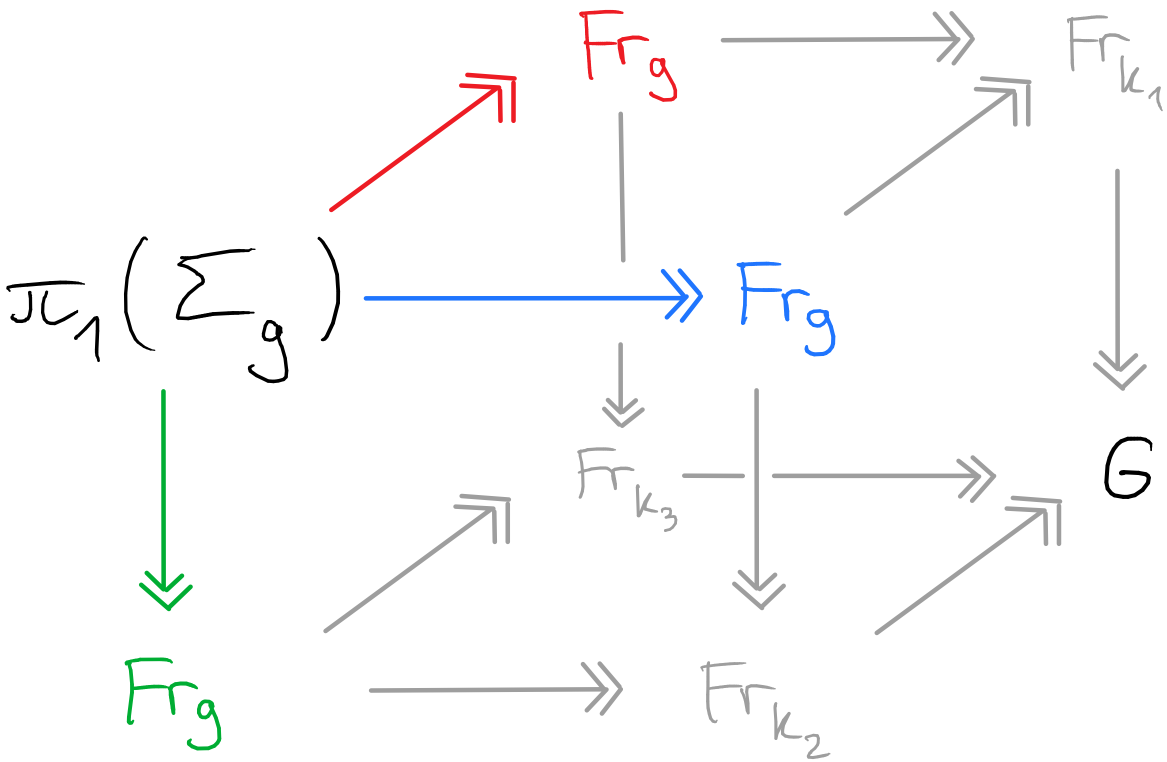
  $\cap$

  trisected 4-manifold X^4



Group trisections of a finitely presented group G:

Commutative cube



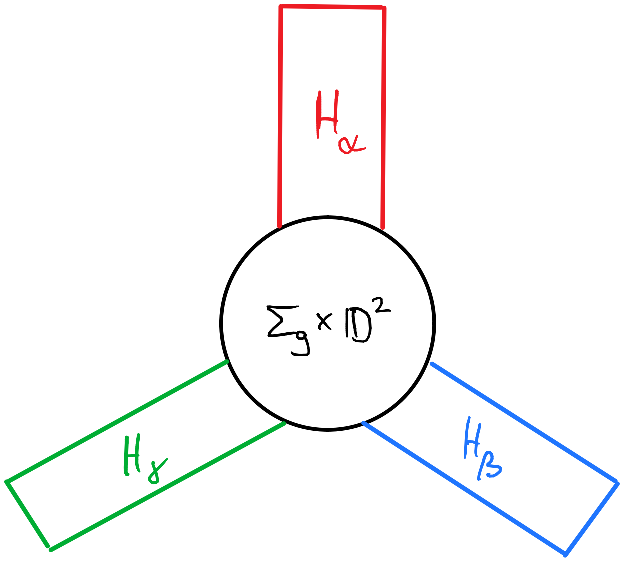
s.th. all maps are surjective

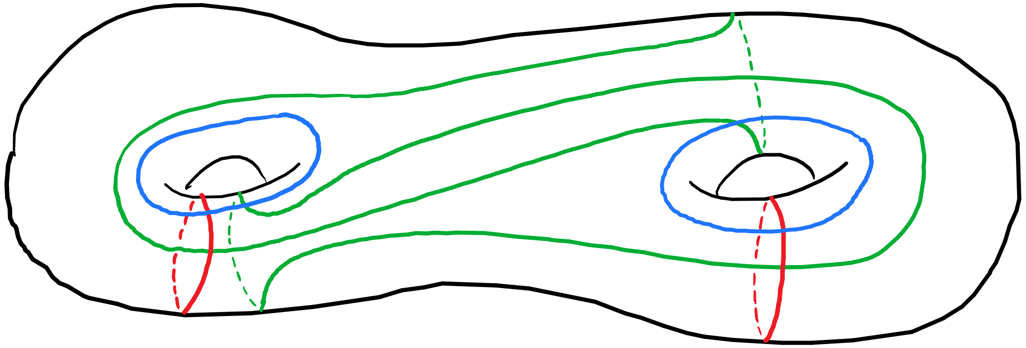
and all faces are push-outs

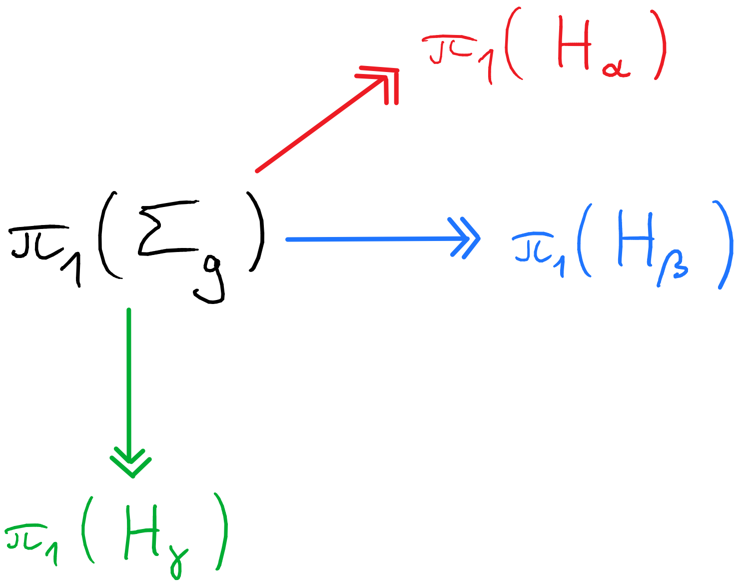
[Abrams, Gay, Kirby]

Group trisections of closed 4-manifolds:

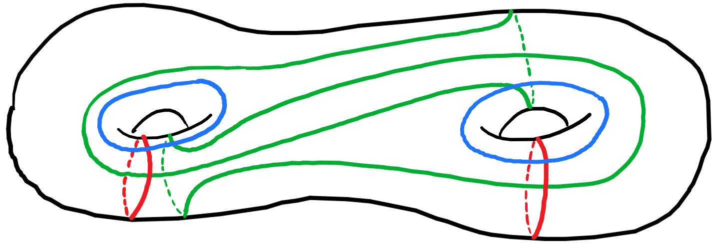
The handlebody-story three times

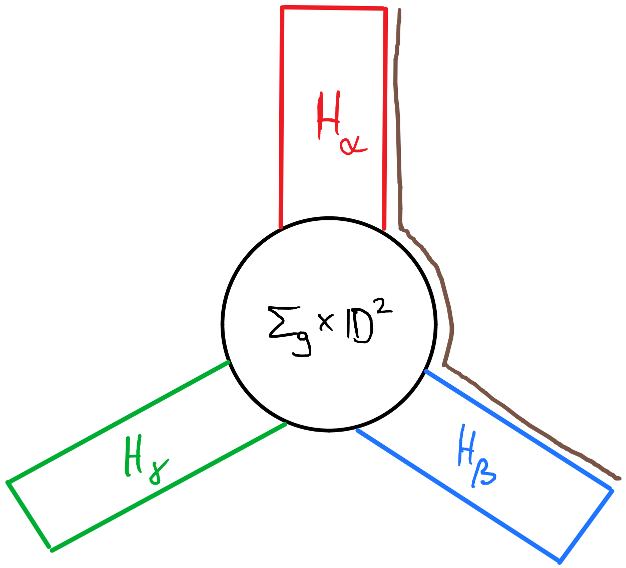






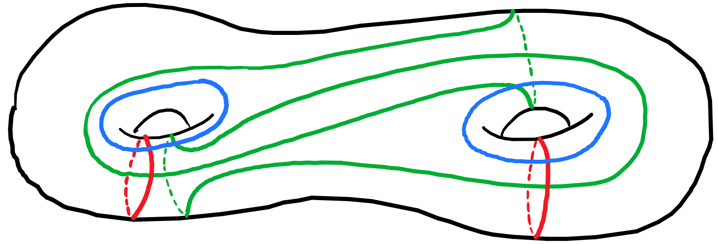
[Abrams, Gay, Kirby]

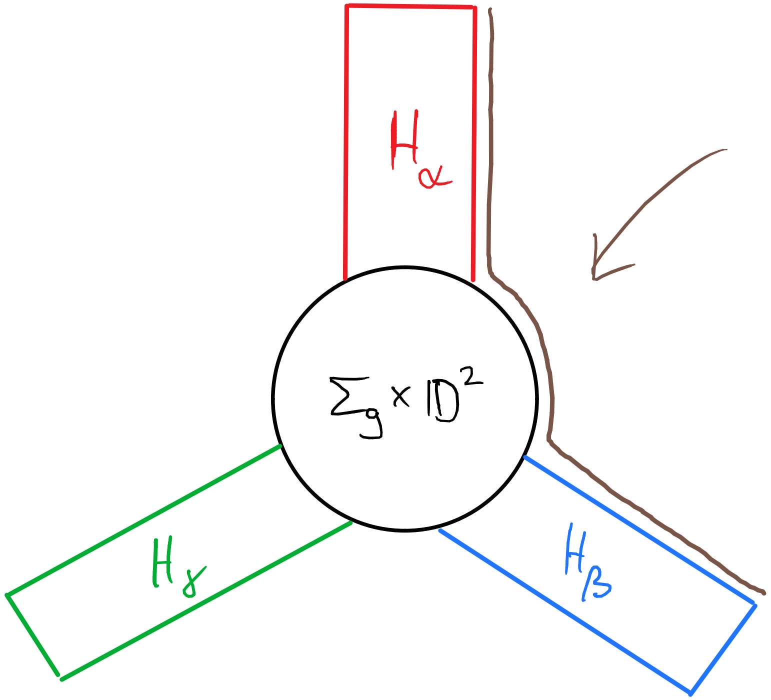




$$
\begin{array}{ccccc}
 & & \pi_1(H_\alpha) & \twoheadrightarrow & \pi_1(H_\alpha \cup_\Sigma H_\beta) \\
 & \nearrow & & \nearrow & \\
\pi_1(\Sigma_g) & \twoheadrightarrow & \pi_1(H_\beta) & & \\
\downarrow & & & & \\
\pi_1(H_\gamma) & & & &
\end{array}
$$

[Abrams, Gay, Kirby]





from our algebra assumption:

this is a closed 3-manifold M

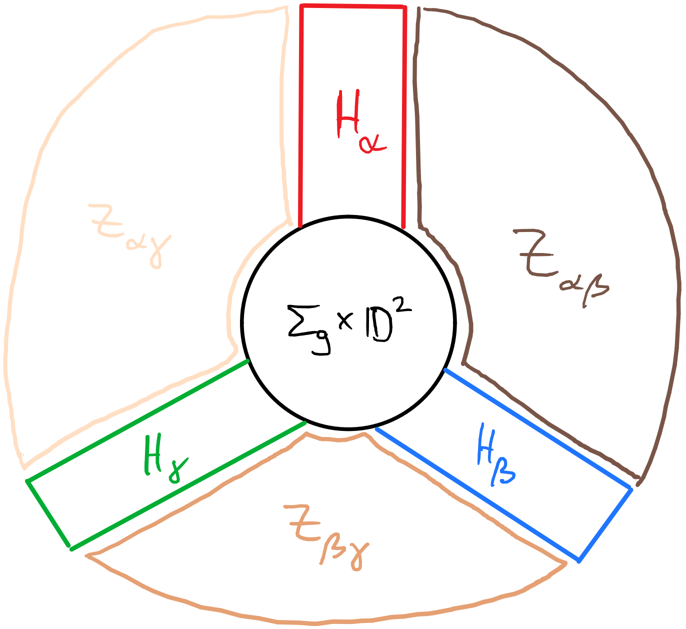
with $\pi_1(M) \cong Fr_k$ free

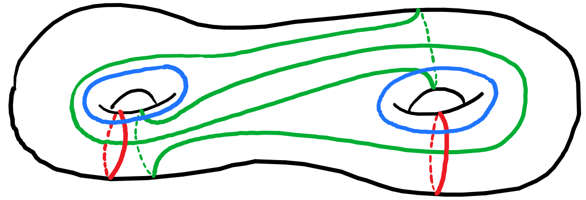
Kneser's thm. + 3D Poincaré conj.

$$\Longrightarrow \qquad M \cong \#^k S^1 \times S^2$$

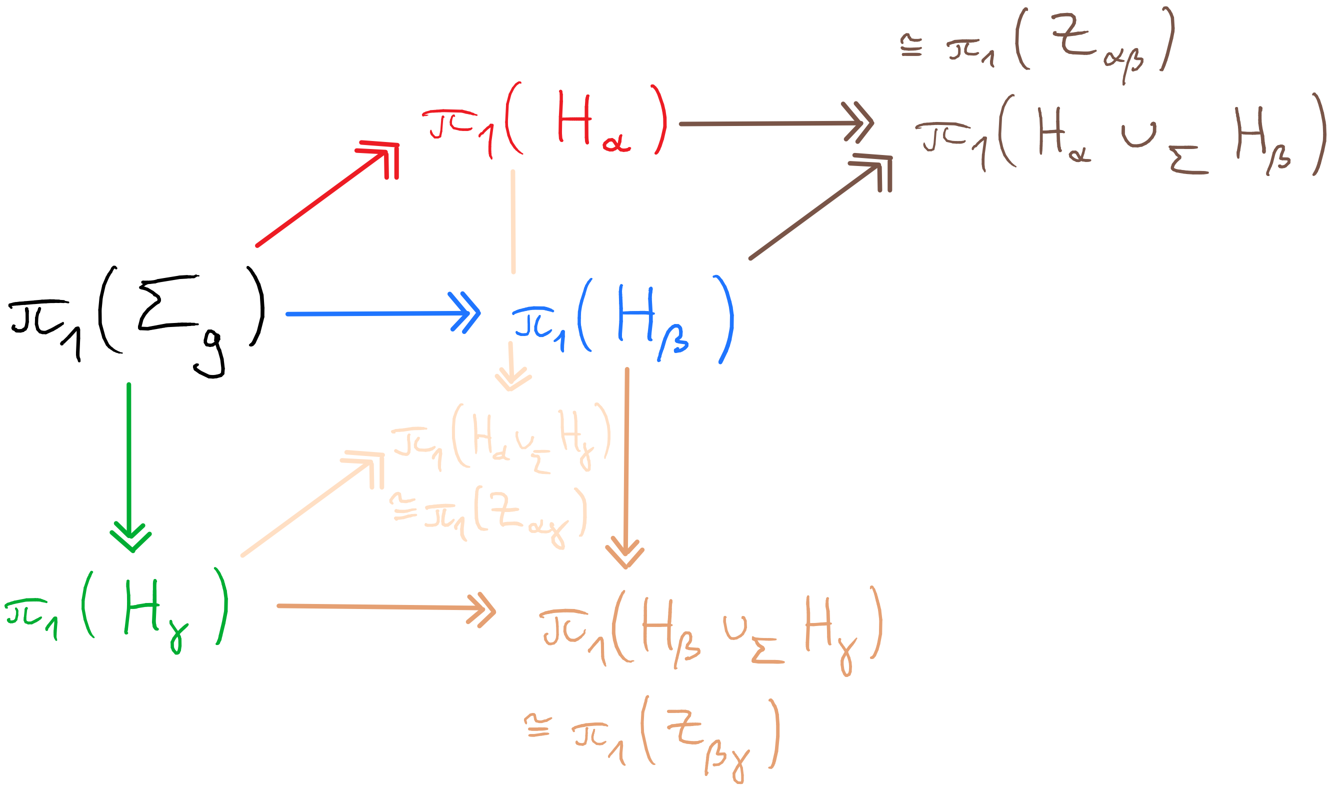
[Abrams, Gay, Kirby]

[Laudenbach-Poenaru] allows us to fill the sectors uniquely with $\natural^{k_i} S^1 \times D^3$

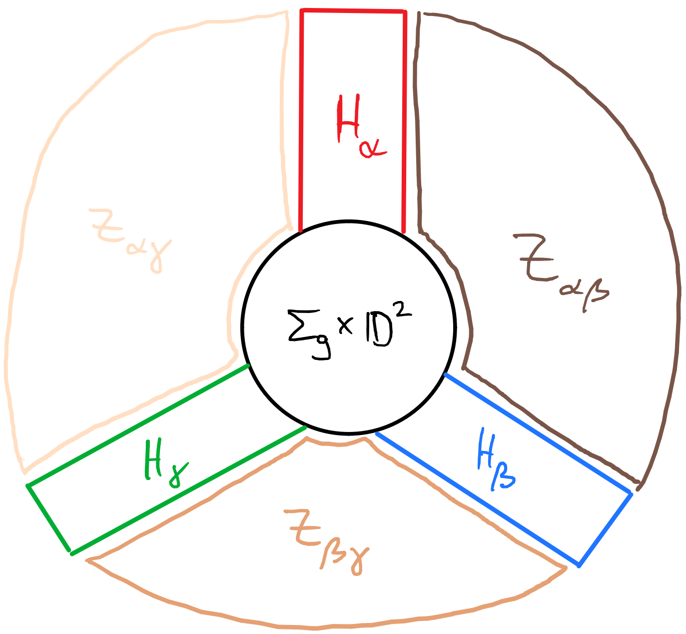




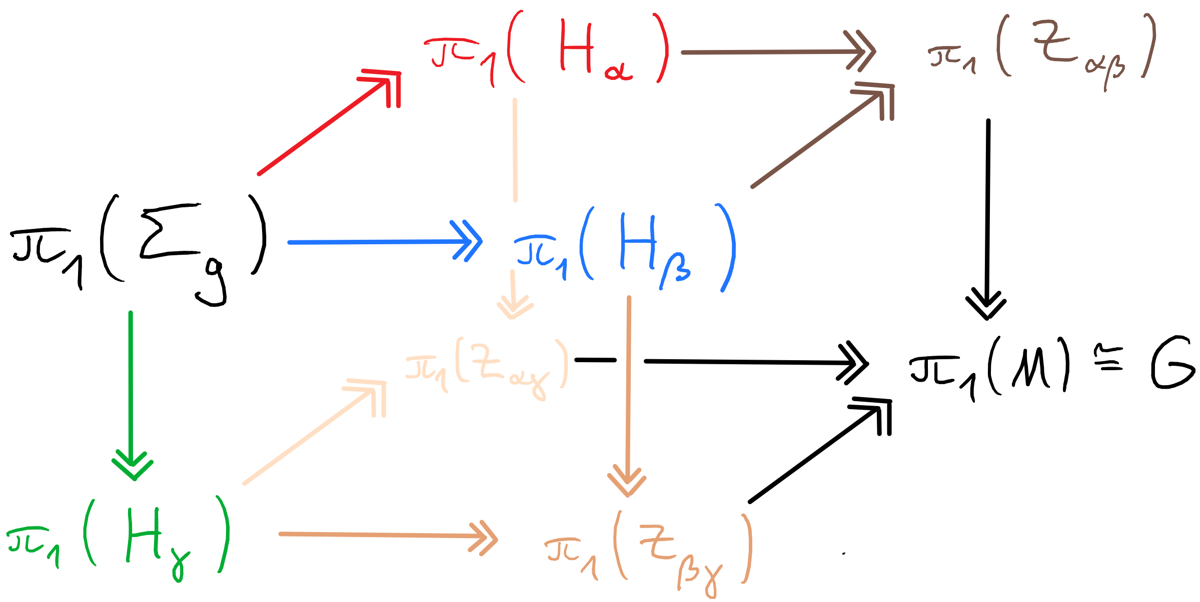
We can do this for all pairs of handlebodies



[Abrams, Gay, Kirby]



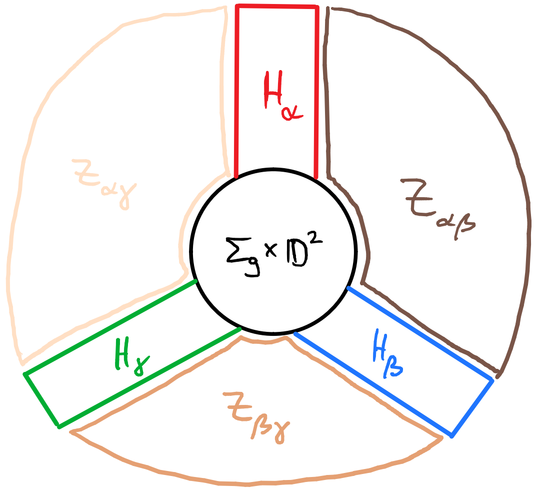
4 – manifold M^4 with $\pi_1(M^4) \cong G$ and group trisection corresponding to the cube below



[Abrams, Gay, Kirby]

(based, parameterized)

trisections of a 4-manifold X^4



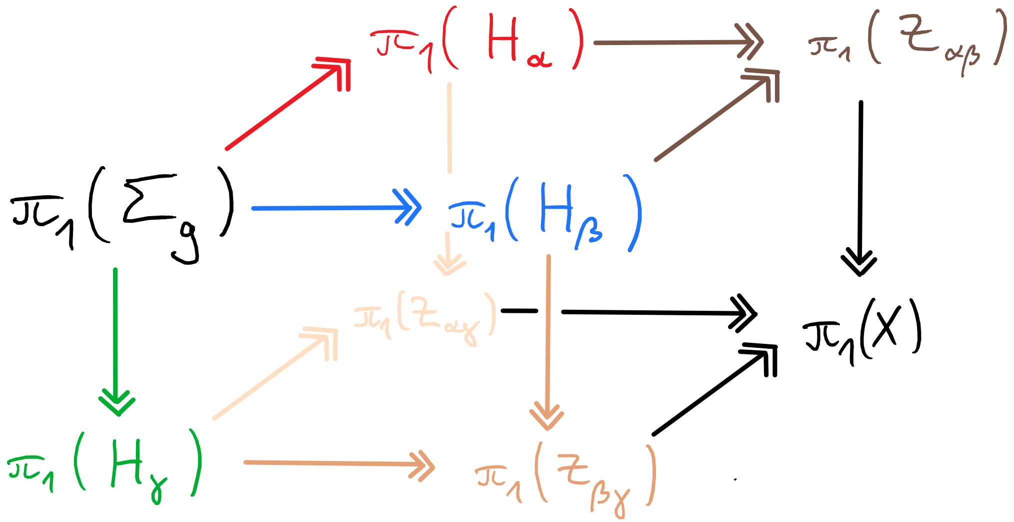
take π_1 of pieces

1 : 1

[Abrams, Gay, Kirby]

the previously explained construction

group trisections of $\pi_1(X,*)$

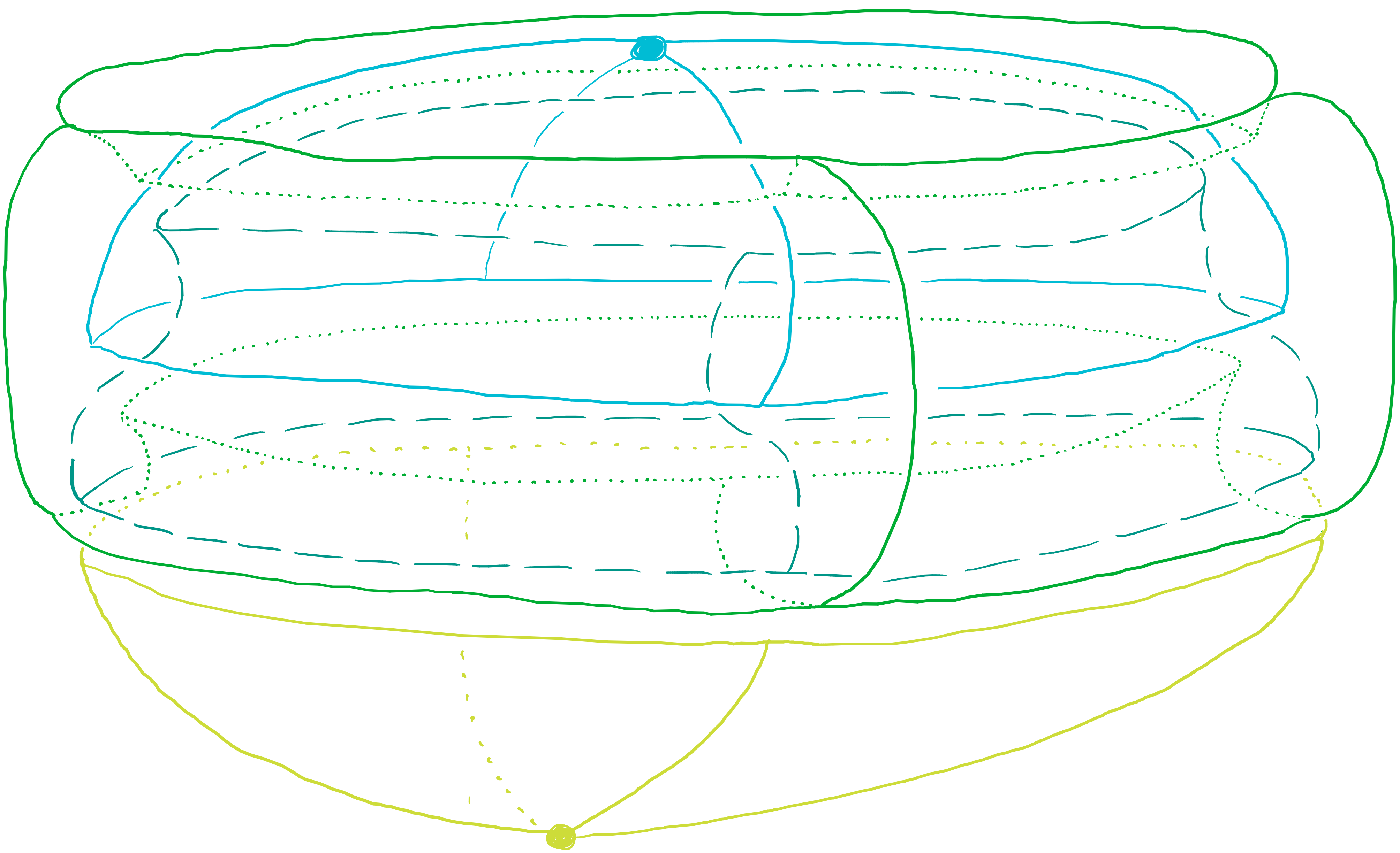


Now: bridge-trisected surface F^2

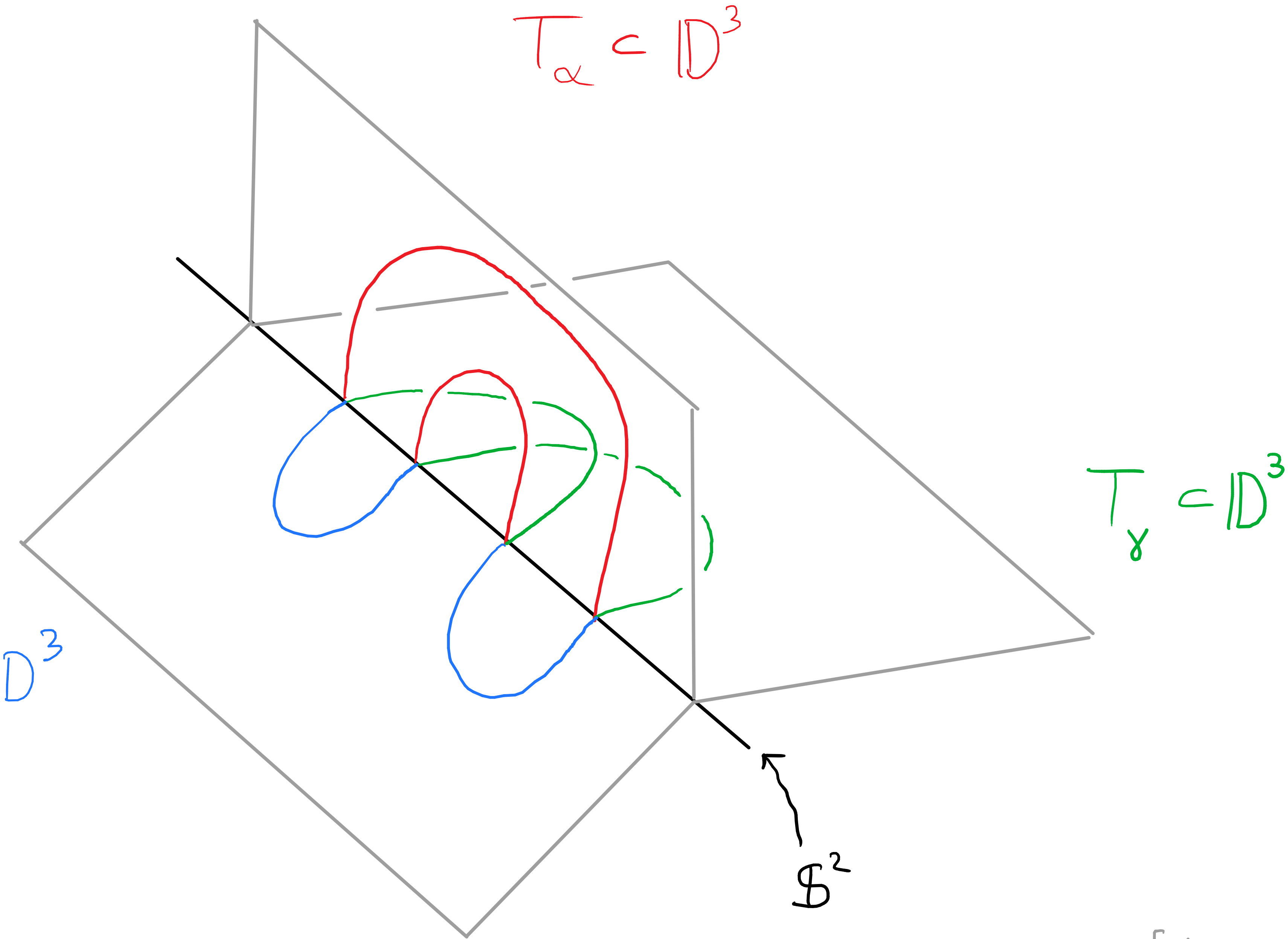
contained in

trisected 4-manifold X^4

Spun trefoil - a knotted surface in S^4



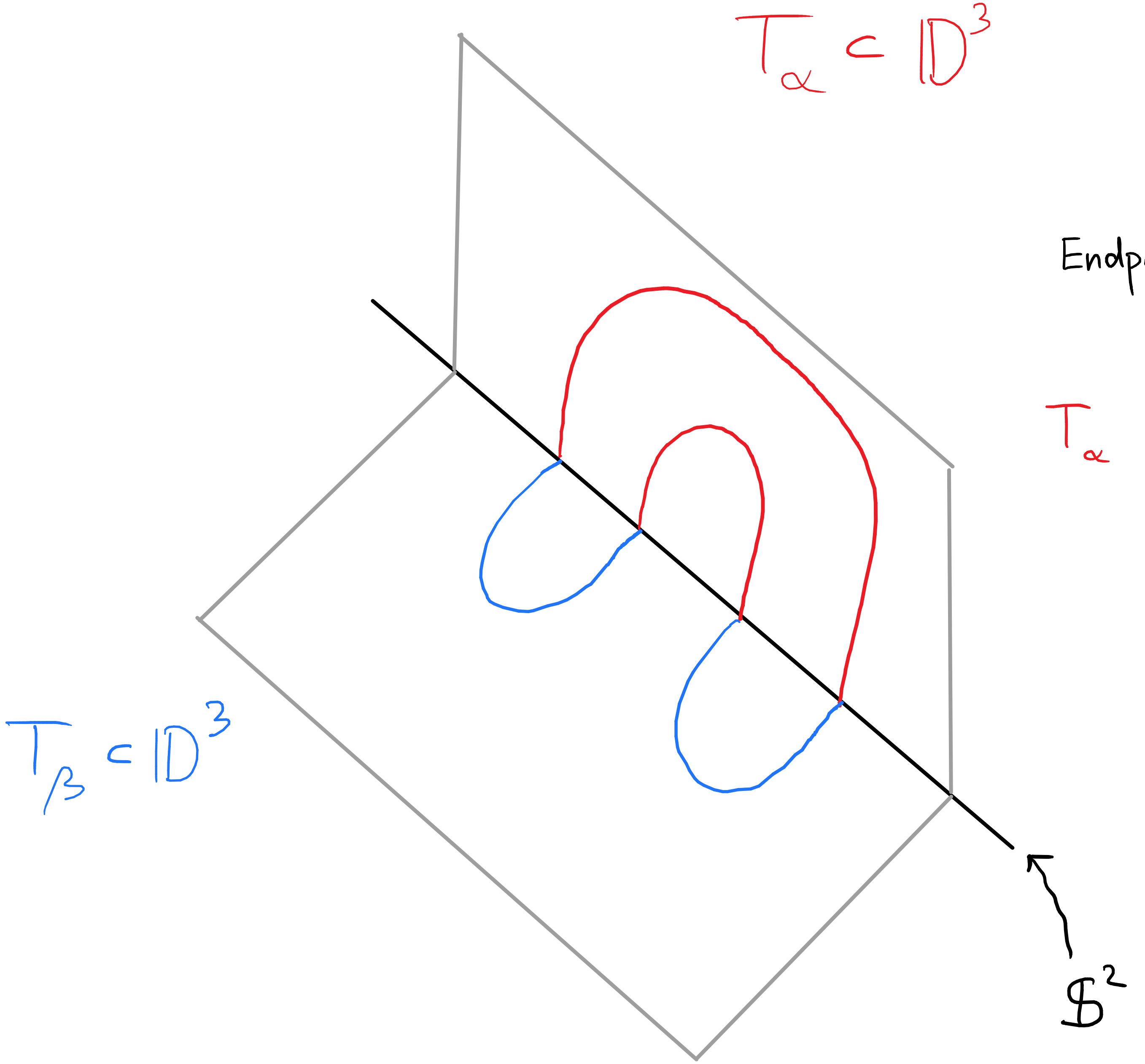
Bridge-trisected surfaces in the 4-sphere



[Meier-Zupan]

Endpoint-unions of trivial tangles form unlinks

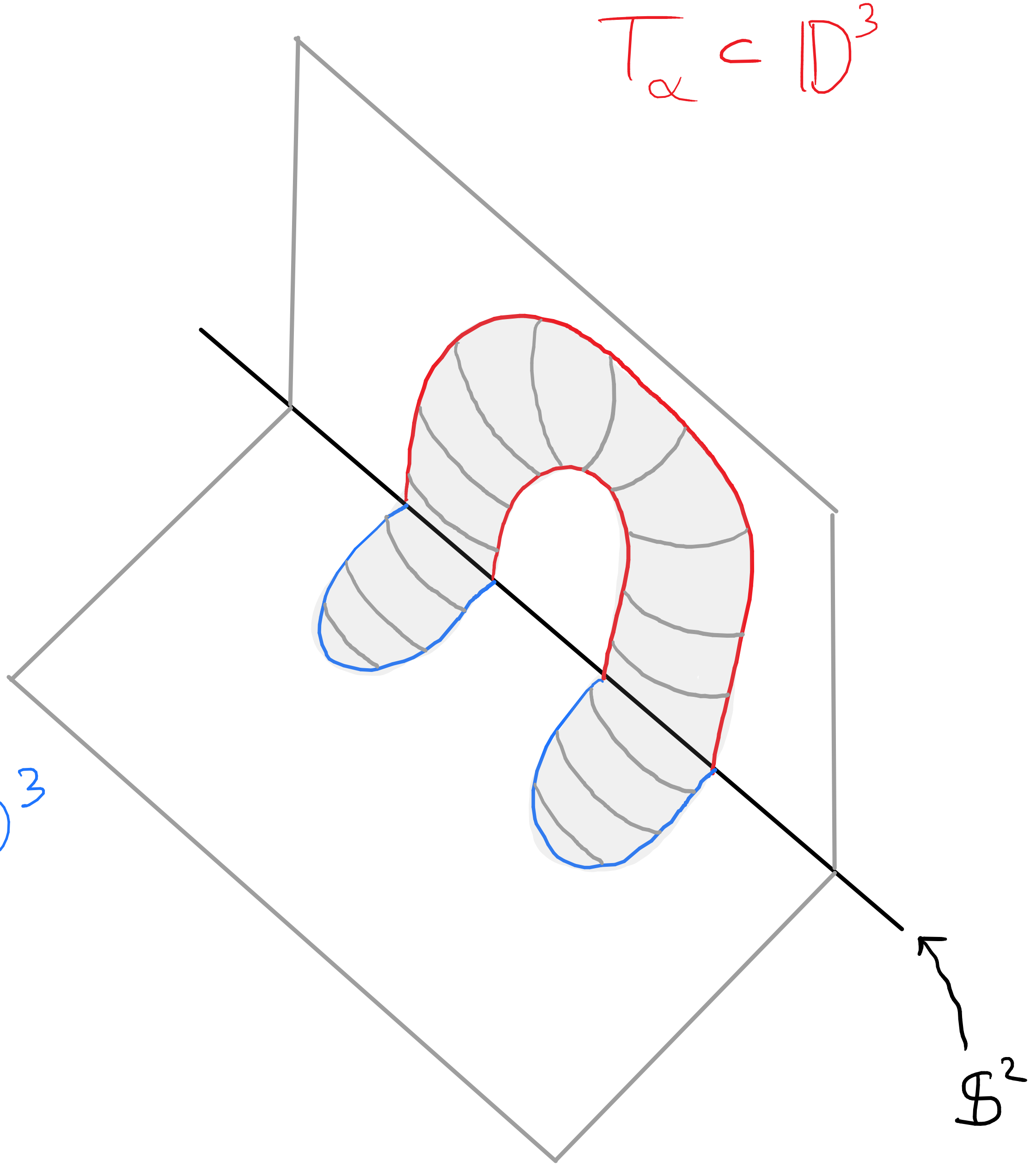
$T_\alpha \cup_\partial T_\beta \subset \mathbb{D}^3 \cup_\partial \mathbb{D}^3 \cong \mathbb{S}^3$



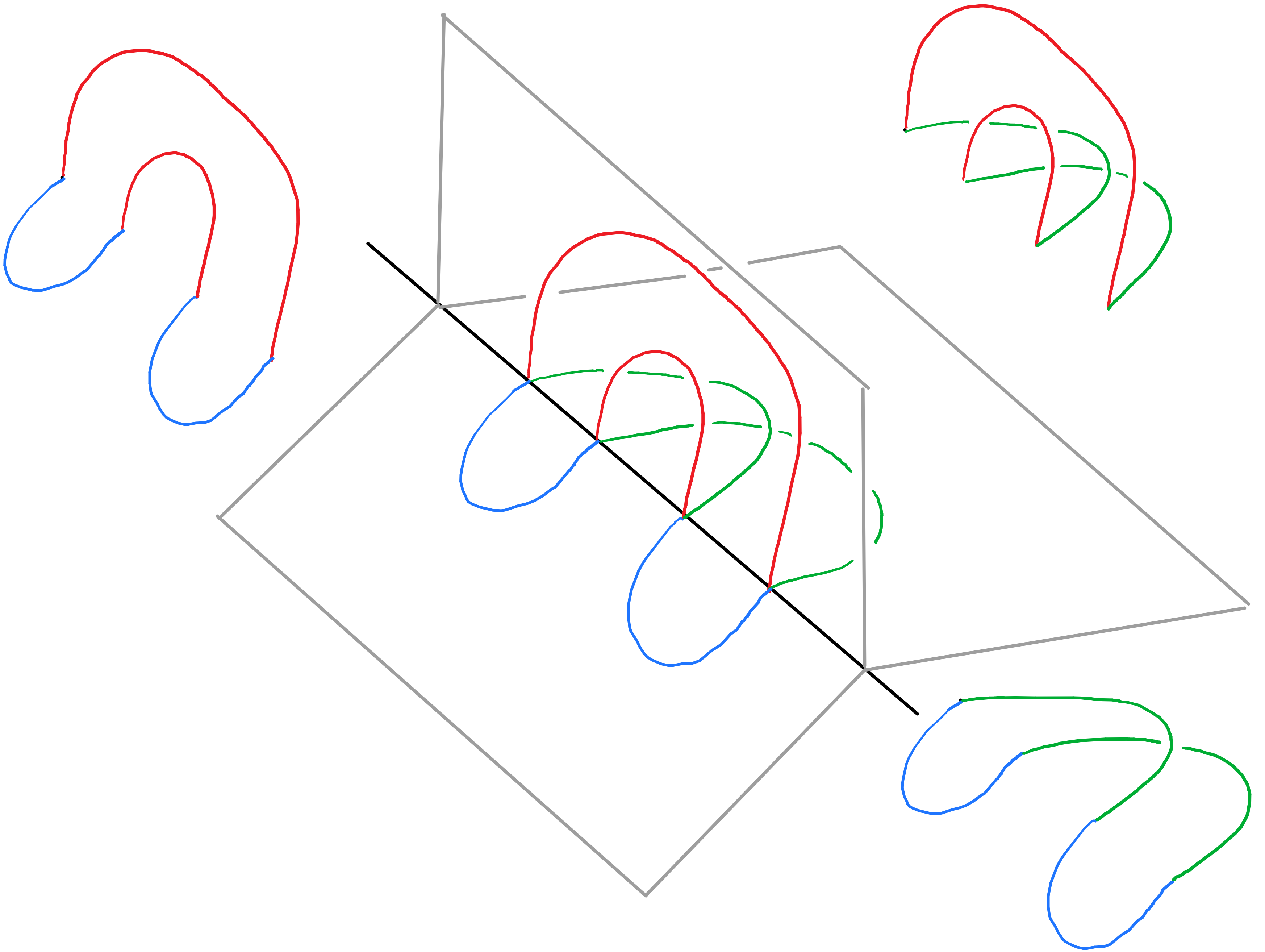
[Meier-Zupan]

Unlinks in S^3 bound (uniquely)

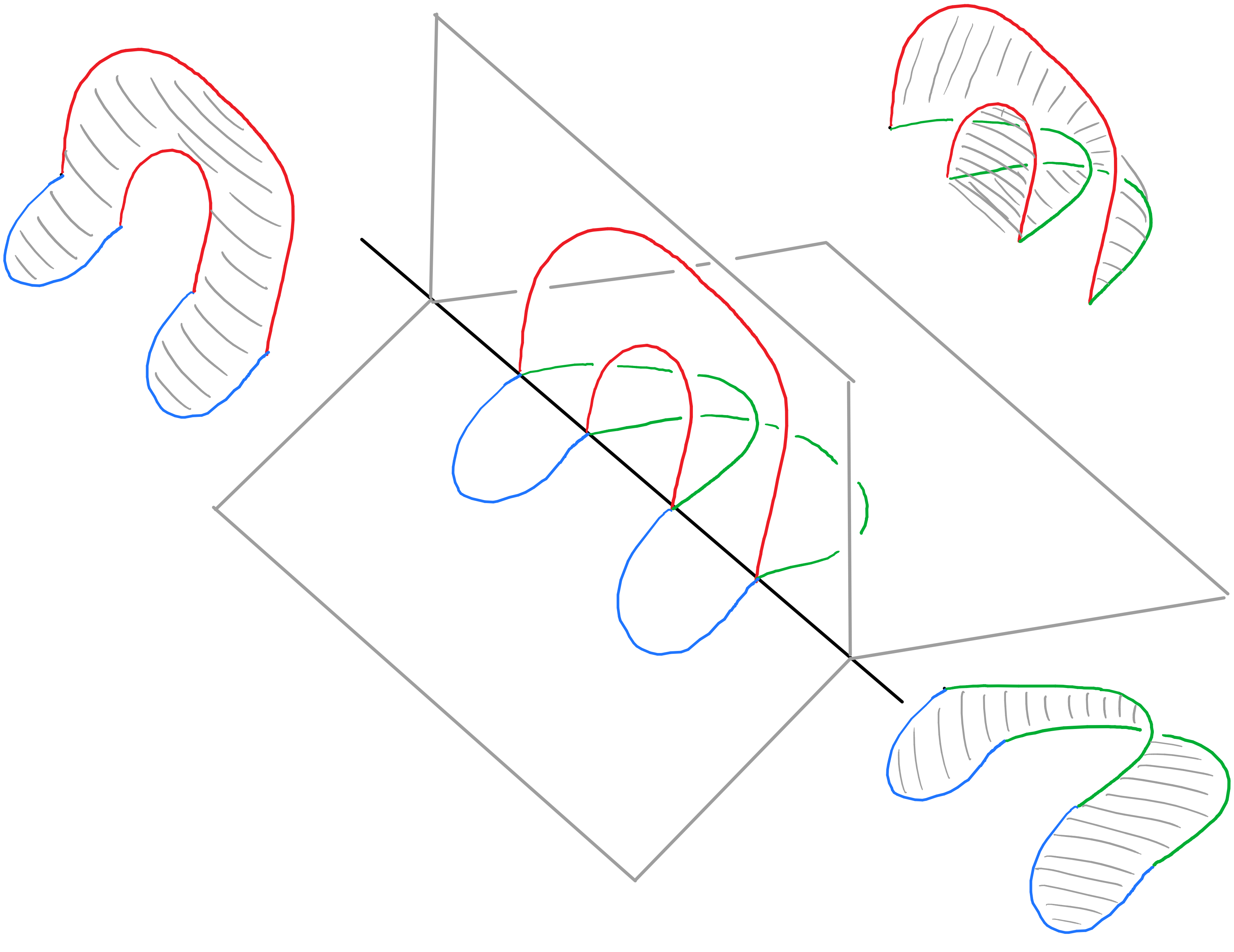
"undisks" in D^4



[Meier-Zupan]

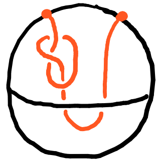


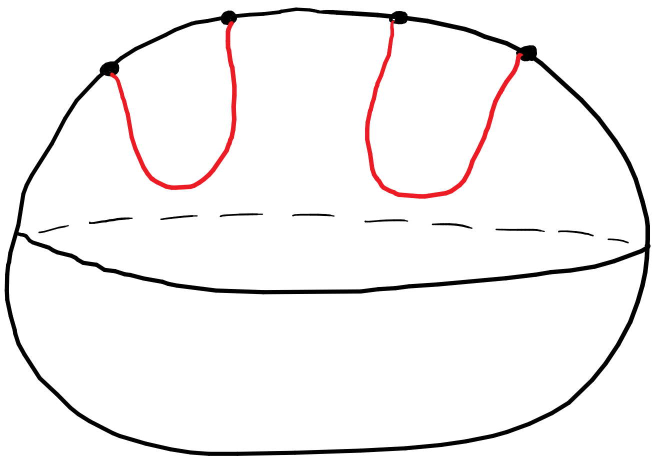
[Meier - Zupan]

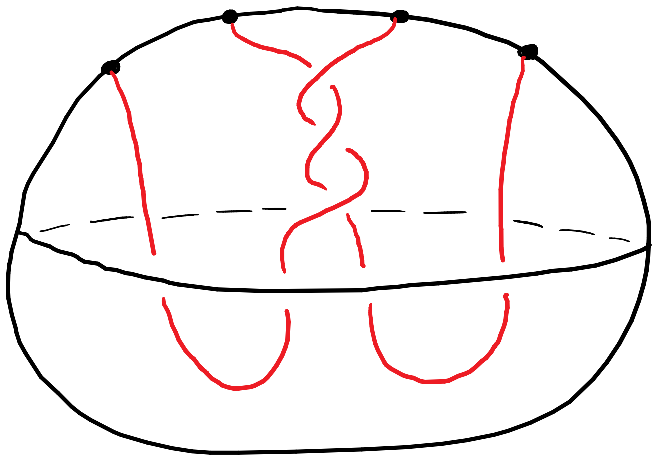


[Meier-Zupan]

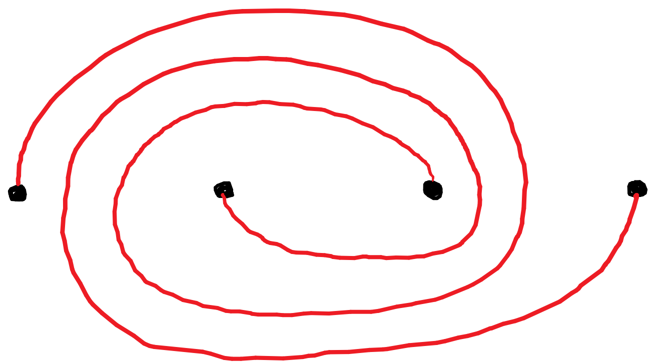
Trivial tangles in 3-balls (and in handlebodies)



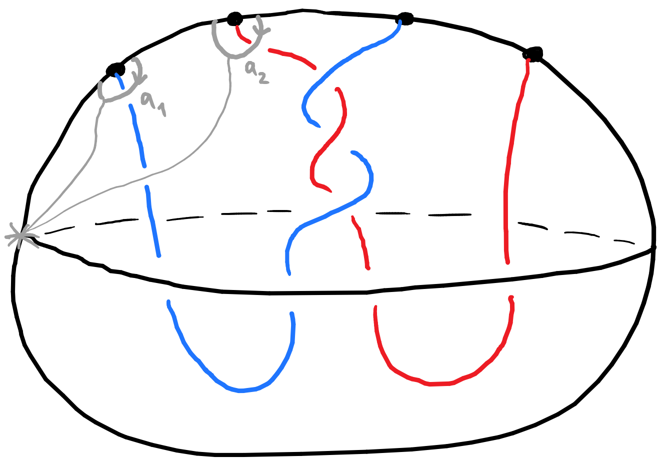


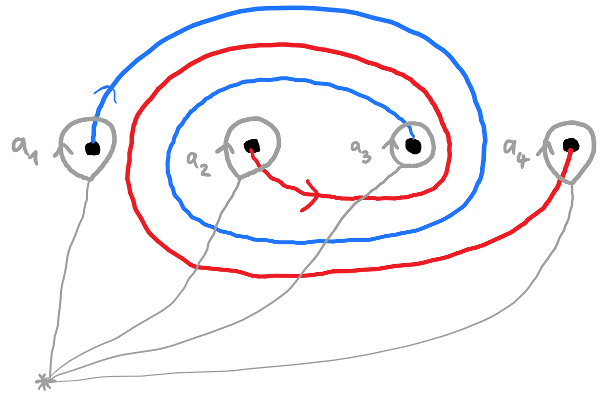


We like to draw the "shadows" of the tangles on a punctured plane:

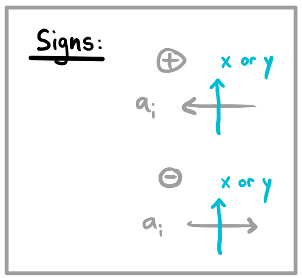


Topology





Algebra



$$\pi_1\left(S^2 - \left\{\begin{matrix}4 \text{ bridge}\\ \text{points}\end{matrix}\right\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

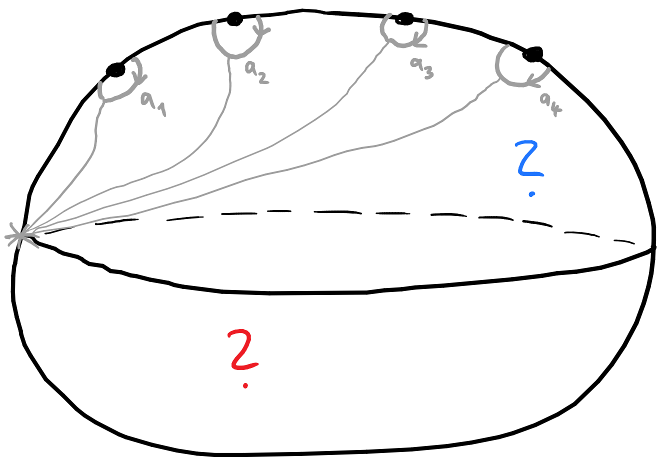
$$a_1 \longmapsto x^{-1}$$

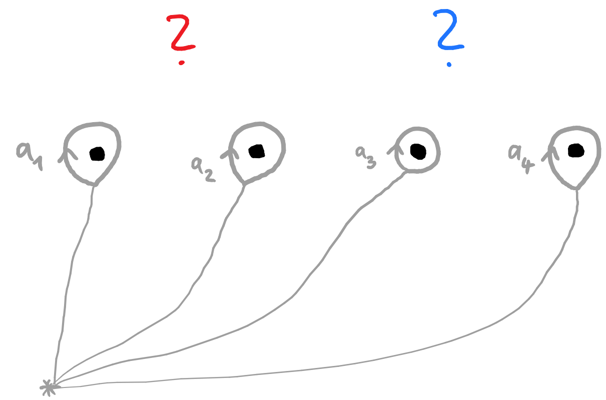
$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

$$a_4 \longmapsto y$$

Topology





Algebra

$$\pi_1\left(S^2 - \{4 \text{ bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

$$a_1 \longmapsto x^{-1}$$

$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

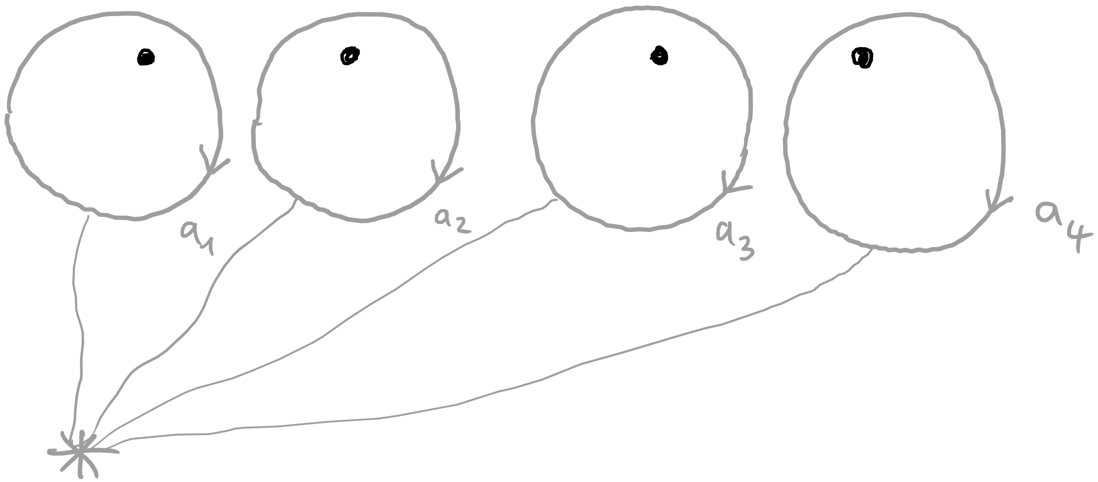
$$a_4 \longmapsto y$$

Punctured Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

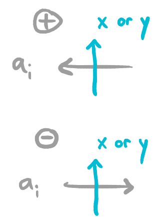
$$a_1 \longmapsto yxy^{-1}$$

$$a_2 \longmapsto yx^{-1}y^{-1}$$

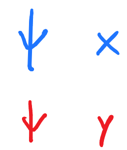
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



Colour coding:

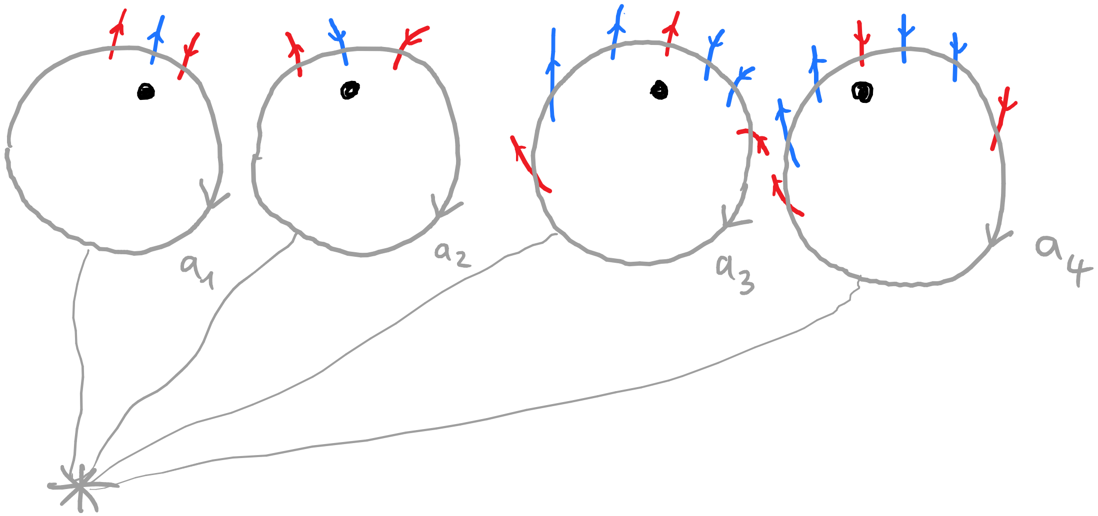


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



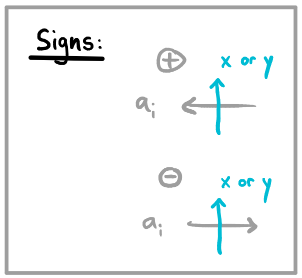
$$\pi_1\left(S^2 - \{\text{4 bridge points}\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

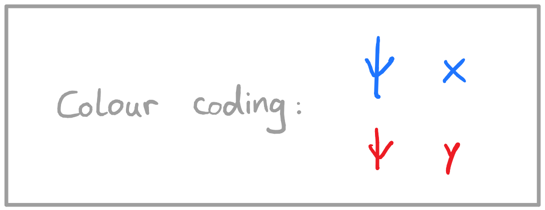
$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$



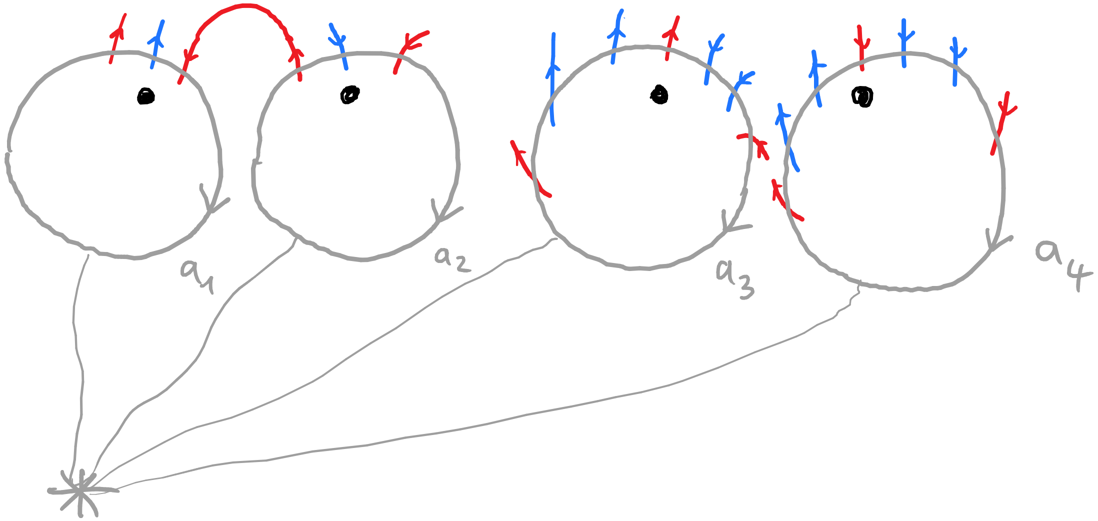


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[yxy^{-1}]\,[yx^{-1}y^{-1}]\,[yxxyx^{-1}x^{-1}y^{-1}]\,[yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \{4 \text{ bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

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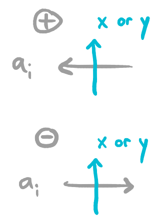
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



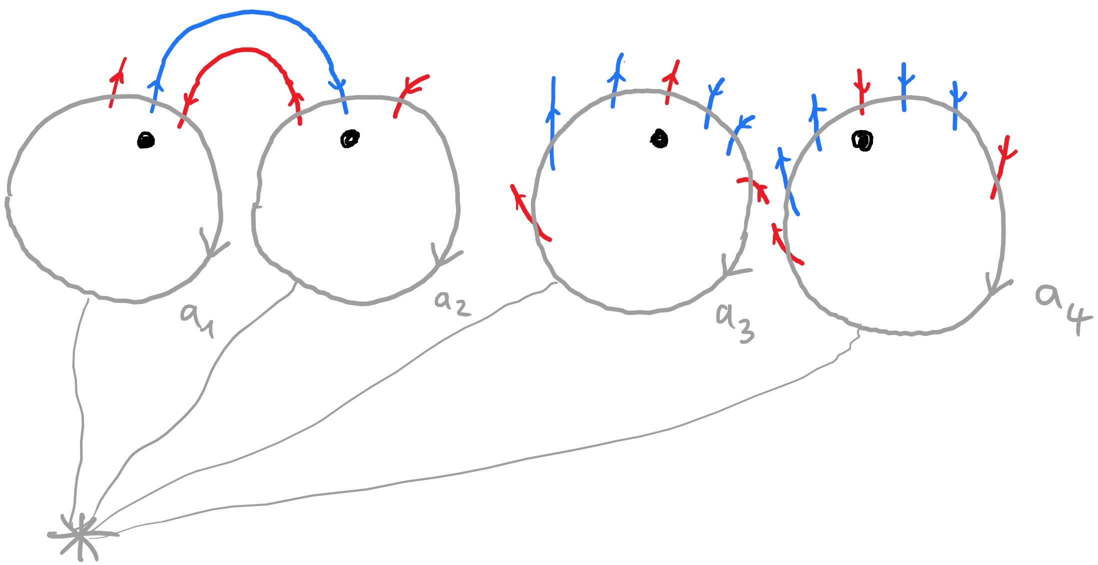
Colour coding: blue arrow = x, red arrow = y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[y\,x\,y^{-1}]\,[y\,x^{-1}y^{-1}]\,[y\,x\,x\,y\,x^{-1}x^{-1}y^{-1}]\,[y\,x\,x\,y^{-1}\,x^{-1}x^{-1}y^{-1}]$$



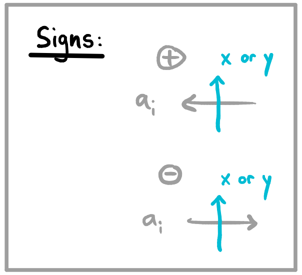
$$\pi_1\left(S^2 - \{\text{4 bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

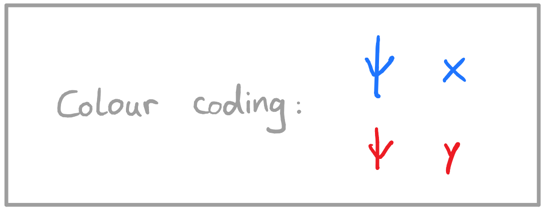
$$a_1 \longmapsto y\,x\,y^{-1}$$

$$a_2 \longmapsto y\,x^{-1}y^{-1}$$

$$a_3 \longmapsto y\,x\,x\,y\,x^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto y\,x\,x\,y^{-1}\,x^{-1}x^{-1}y^{-1}$$



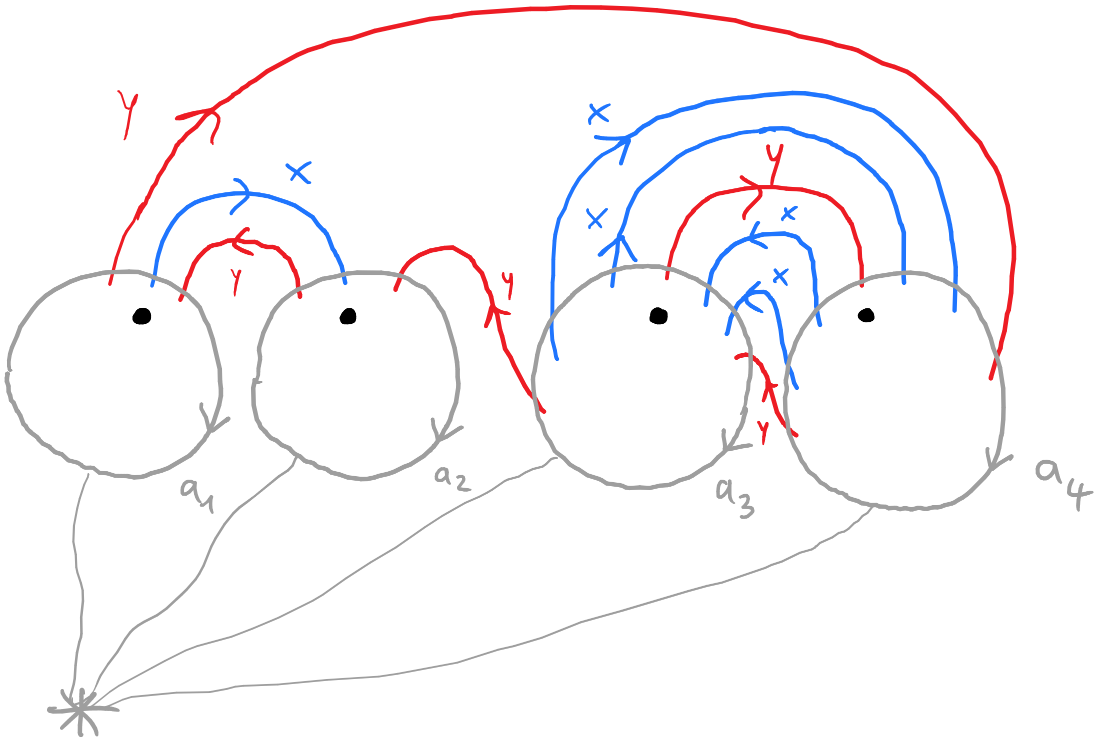


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \{4 \text{ bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$a_1 \longmapsto yxy^{-1}$$

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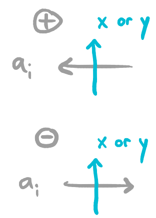
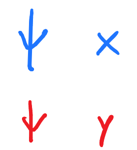
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



Colour coding:

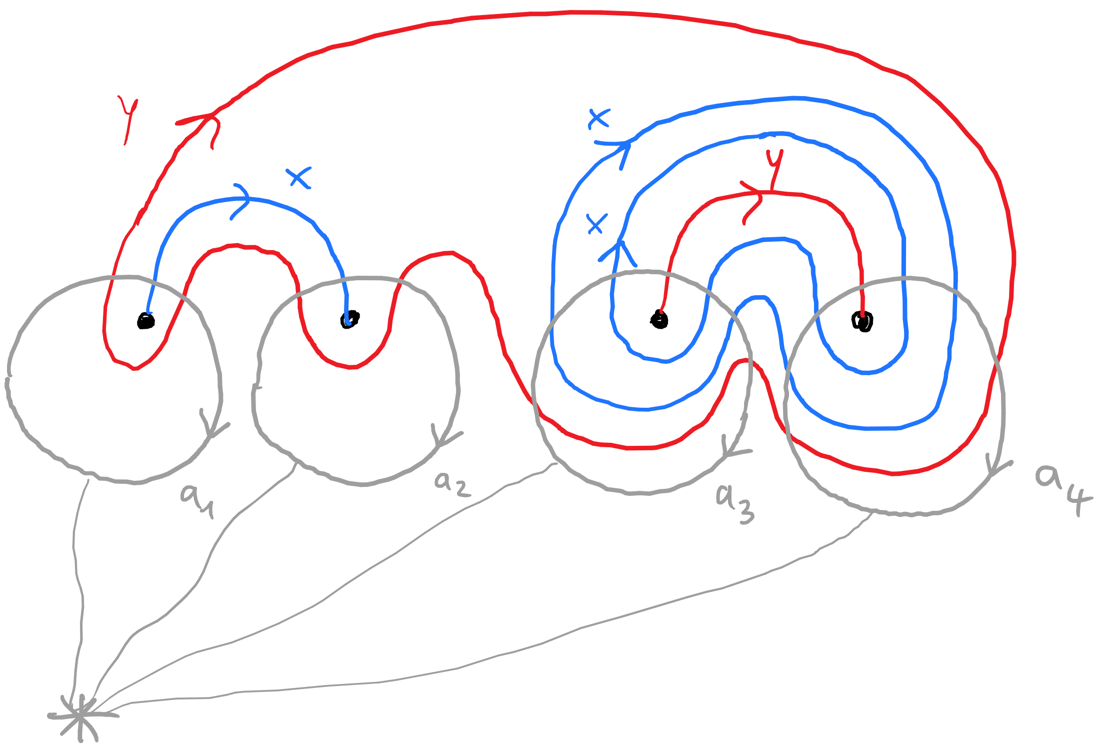


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



$$\pi_1(S^2 - \{4 \text{ bridge points}\}) \longrightarrow \pi_1(D^3 - \text{tangle})$$

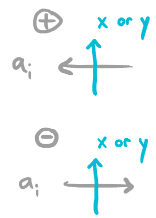
$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

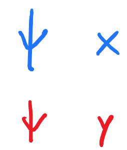
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

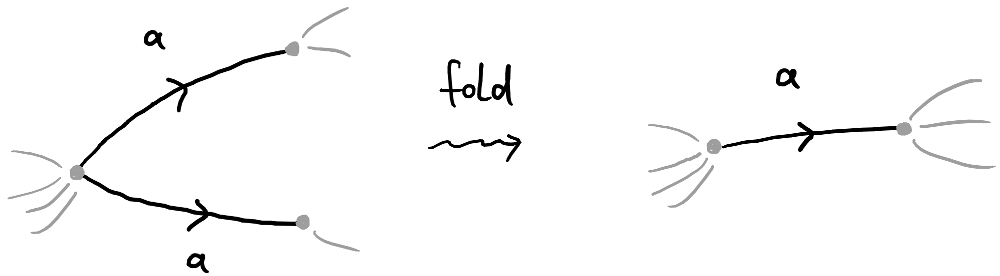
Signs:



Colour coding:

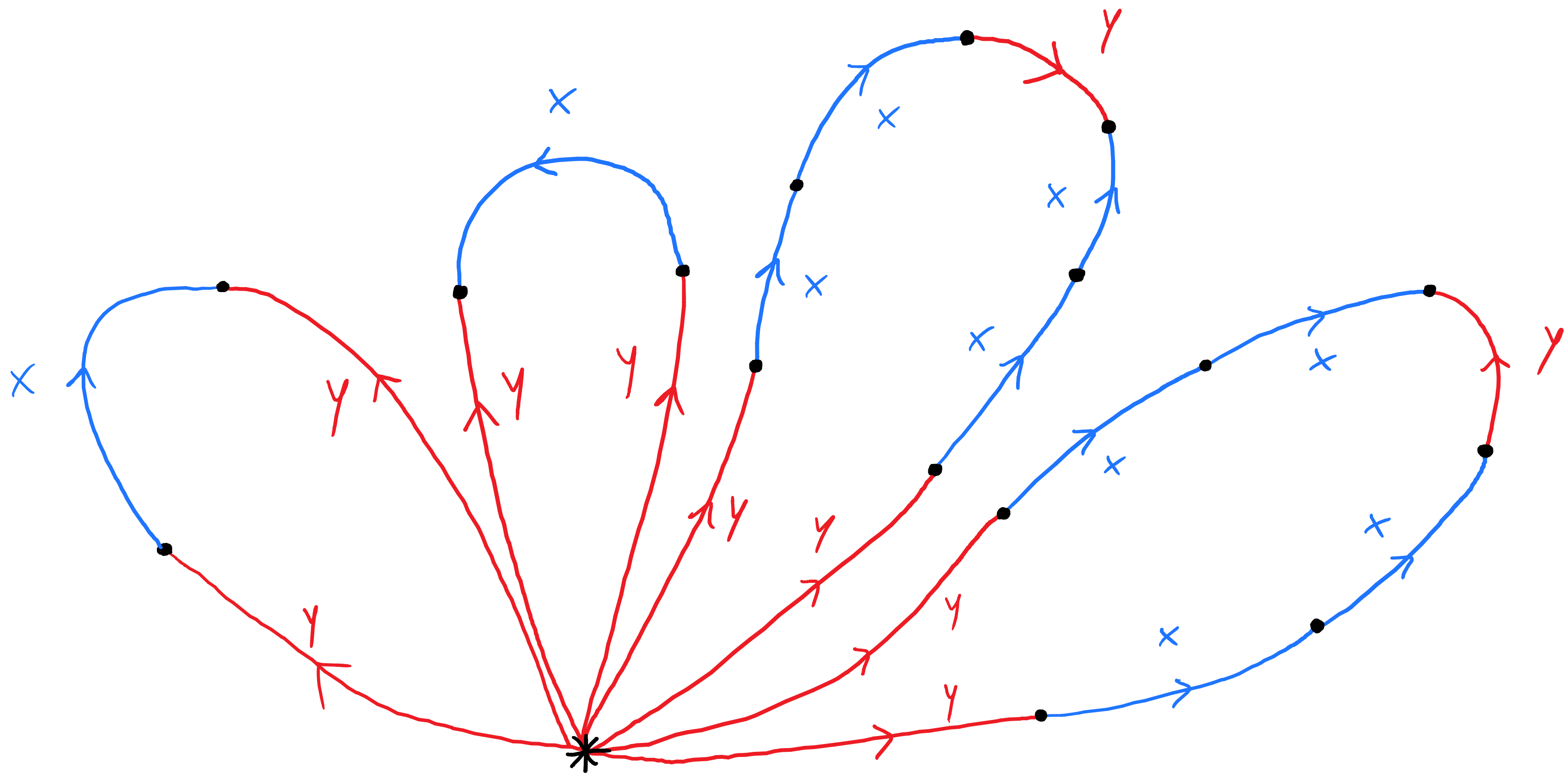


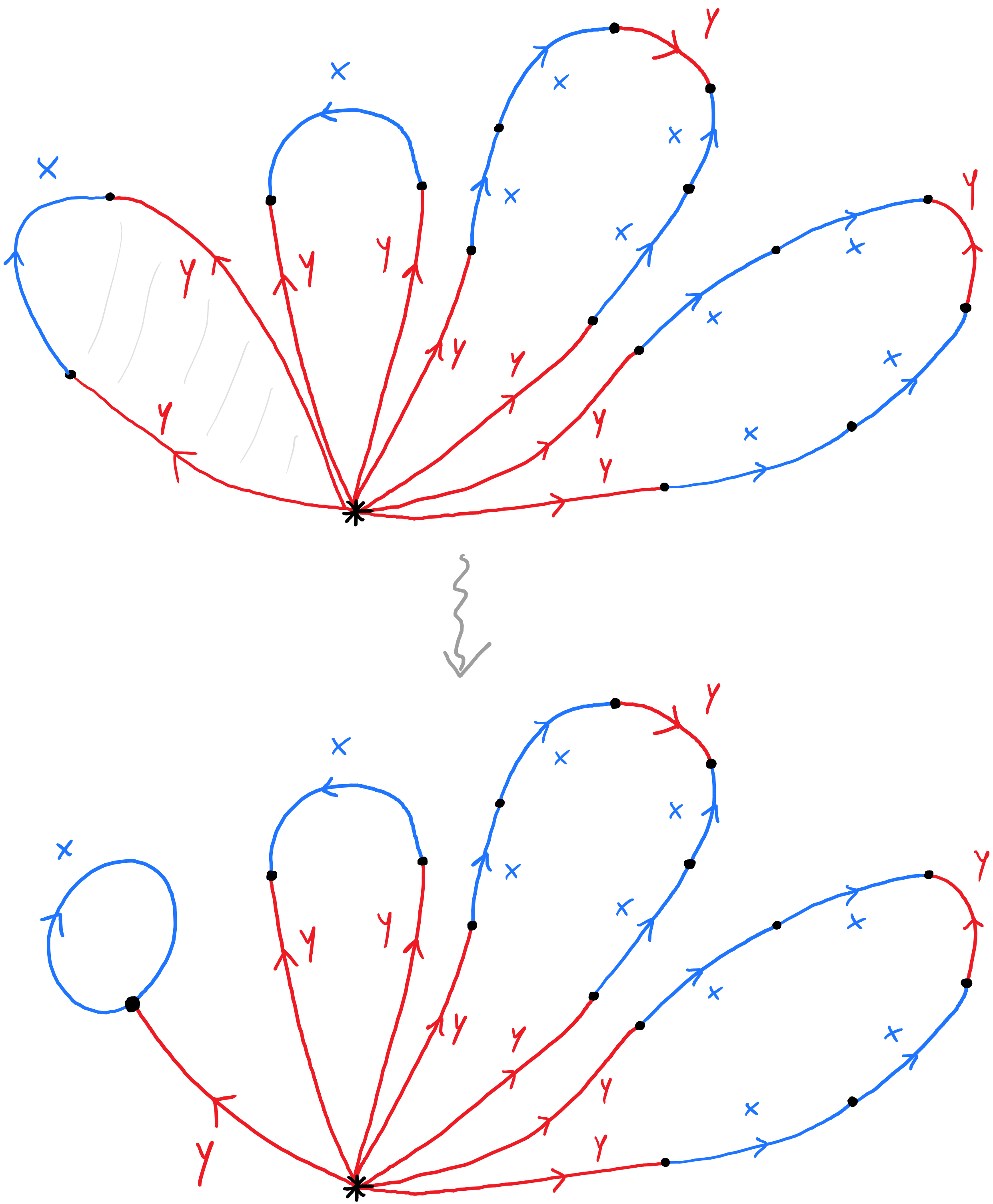
If there are closed circle components, we use band sums guided by <u>Stallings folding</u>

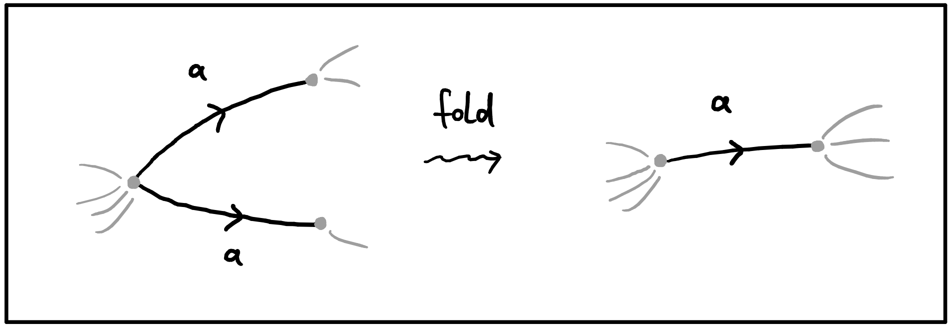


We would like to check whether $\langle yxy^{-1}, yx^{-1}y^{-1}, yxxyx^{-1}x^{-1}y^{-1}, yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$

generates the free group $\langle x, y \rangle$





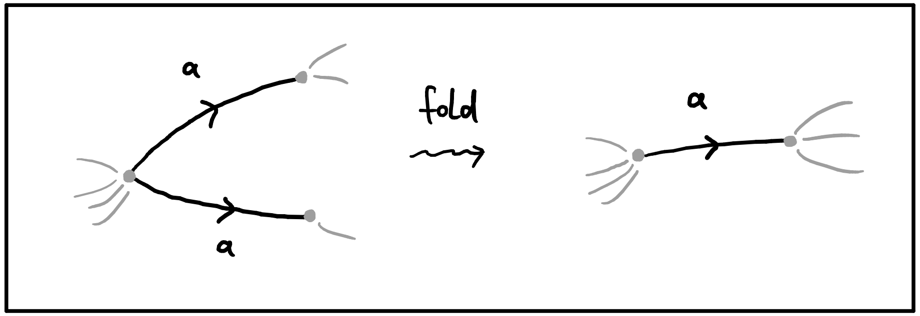


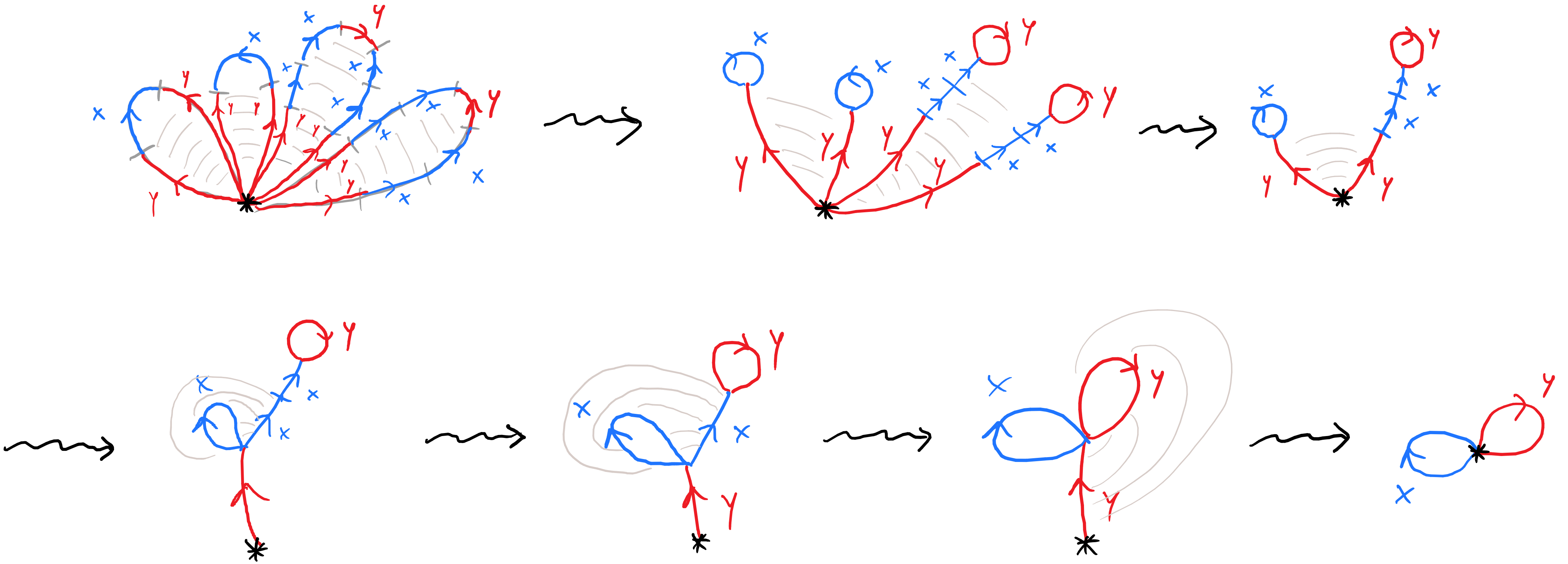
[Stallings]

Sequence of folds which show that

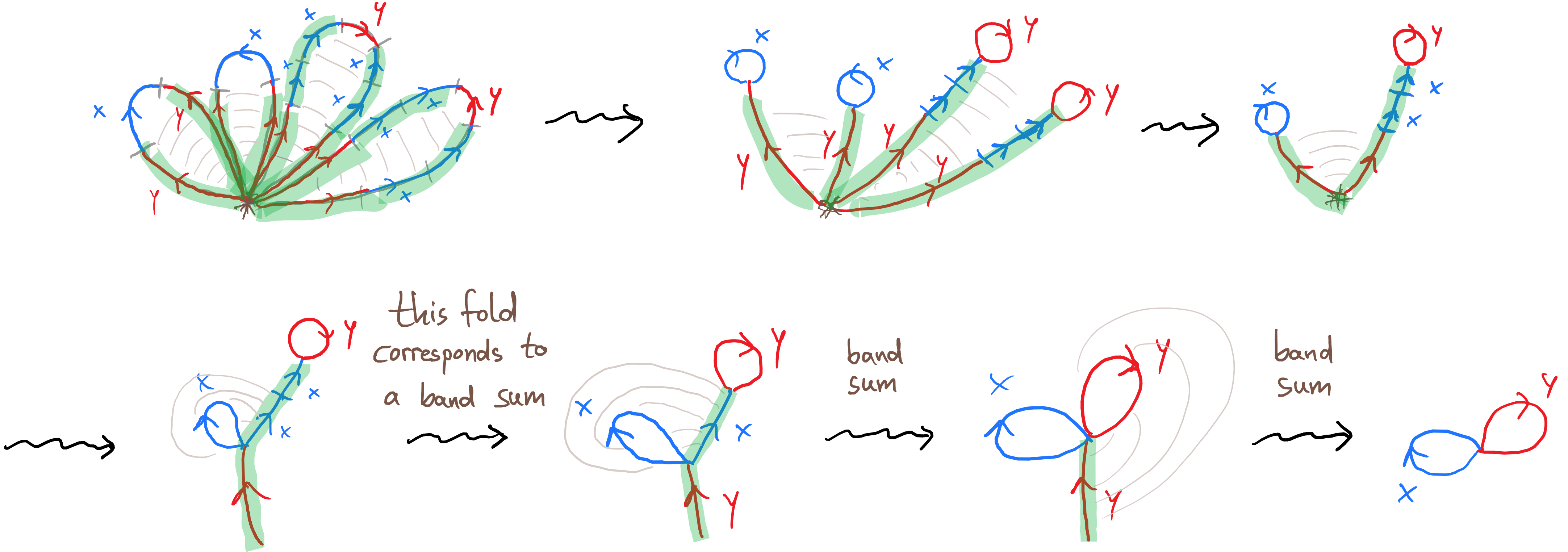
$$\langle yxy^{-1},\ yx^{-1}y^{-1},\ yxxyx^{-1}x^{-1}y^{-1},\ yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$$

generates the free group $\langle x, y \rangle$

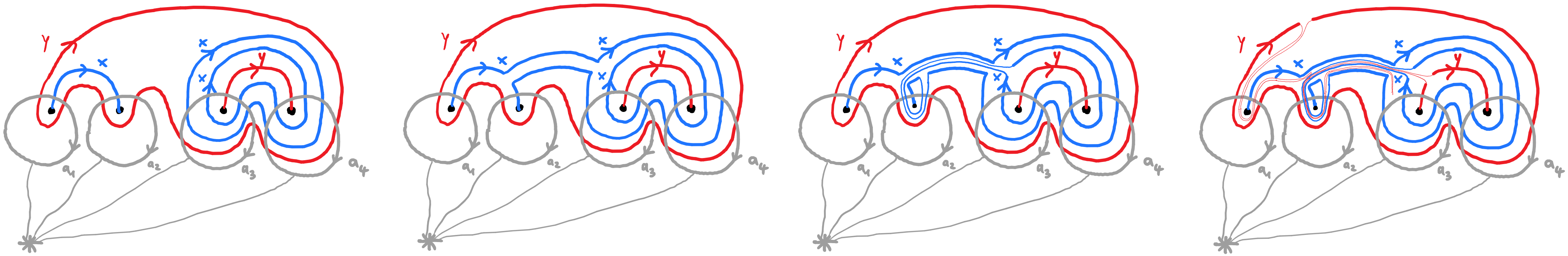




[Stallings]



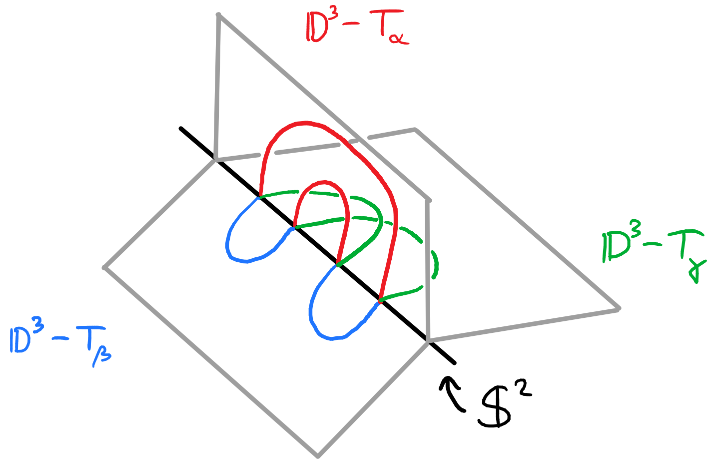
this fold
corresponds to
a band sum
band
sum
band
sum



y
x
a1
a2
a3
a4

(based, parameterized)

{ bridge trisections of a smoothly knotted surface $K^2 \subset S^4$



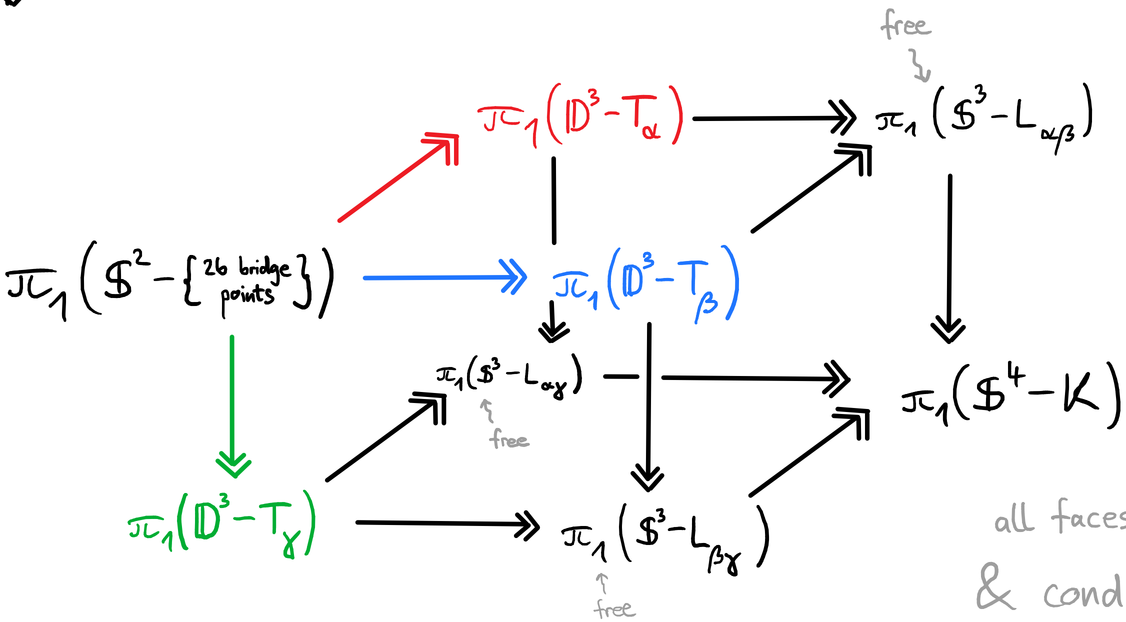
}

take π_1 of pieces ↓

1:1

↑ [Blackwell-Kirby-Klug-Longo-R, 2021]

{ trisected knotted surface group $\pi_1(S^4 - K)$



}

all faces are push-outs
& conditions apply

We take inspiration from:

- [Stallings: How not to prove the Poincaré conjecture (1965)]
- [Jaco: Heegaard splittings and splitting homomorphisms (1968)]

  [Jaco: Stable equivalence of splitting homomorphisms (1970)]
- [Abrams, Gay, Kirby: Group trisections and smooth 4-manifolds (2018)]

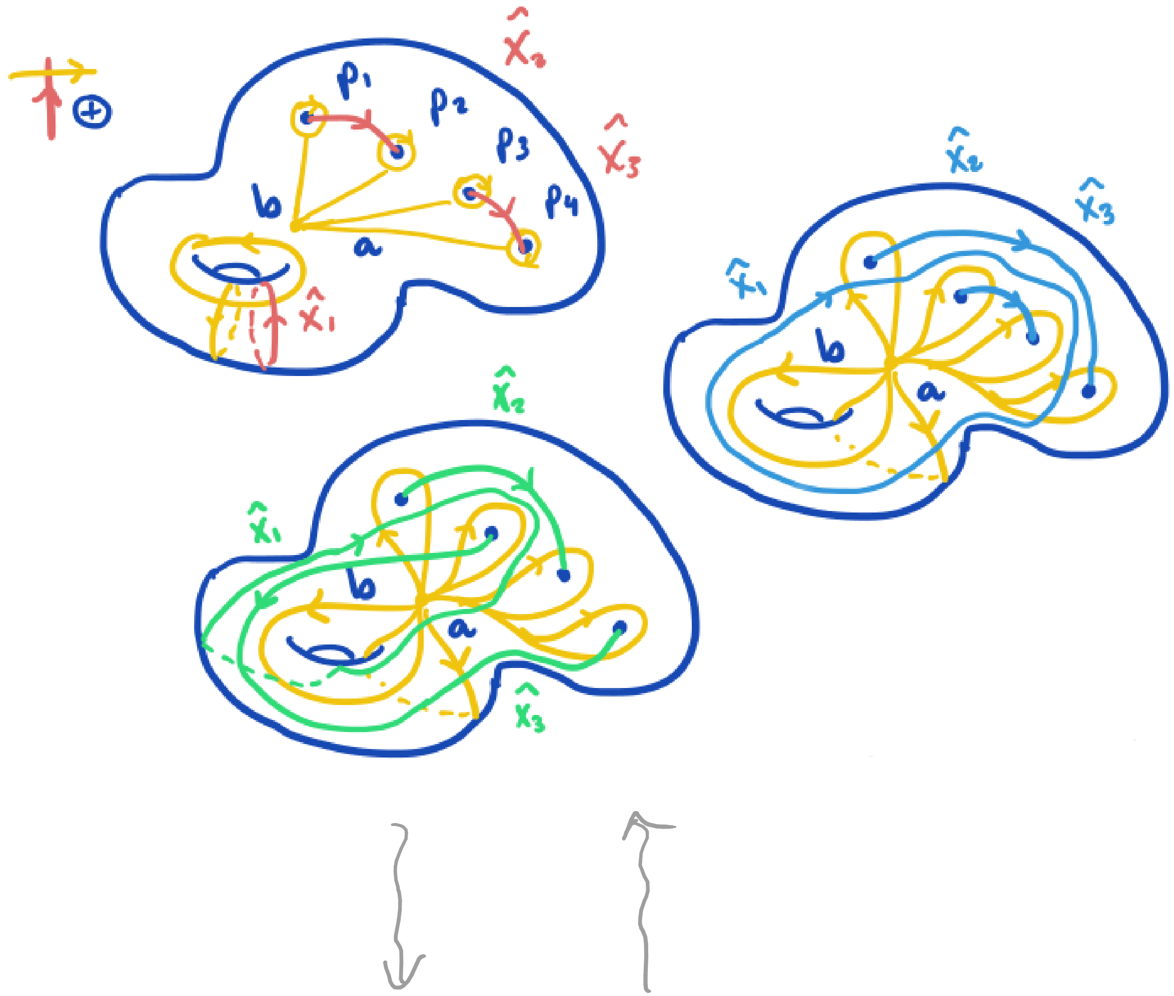
Thanks!

Example of a bridge trisected surface in a trisected 4-manifold:

bridge position of real $\mathbb{RP}^2$

$\cap$

genus 1 trisection of $\mathbb{CP}^2$



corresponding group trisection

$a \mapsto 1$	$a \mapsto x_1$	$a \mapsto \bar{x}_1 x_3$
$b \mapsto x_1$	$b \mapsto 1$	$b \mapsto x_1$
$p_1 \mapsto x_2$	$p_1 \mapsto \bar{x}_1 x_2 x_1$	$p_1 \mapsto x_3 \bar{x}_1 x_2 x_1 \bar{x}_3$
$p_2 \mapsto \bar{x}_2$	$p_2 \mapsto x_3$	$p_2 \mapsto x_3$
$p_3 \mapsto x_3$	$p_3 \mapsto \bar{x}_3$	$p_3 \mapsto \bar{x}_1 \bar{x}_2 x_1$
$p_4 \mapsto \bar{x}_3$	$p_4 \mapsto \bar{x}_1 \bar{x}_2 x_1$	$p_4 \mapsto \bar{x}_1 \bar{x}_3 x_1$