# Local equivalence and odd refinements of the $s$-invariant

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# Slice Knots

**Definition**

Let $K \subset S^3$ be a knot and $i\colon S^1 \to K$ a smooth embedding. Then $K$ is called *slice*, if $i$ extends to a smooth embedding $j\colon D^2 \to D^4$.

For example, if $K$ is a knot and $\overline{K}$ its mirror, then $K \# \overline{K}$ is slice.

**Definition**

The *smooth concordance group* $\mathcal{C}$ is the set of concordance classes of knots, where $K_1, K_2$ are concordant if and only if $K_1 \# \overline{K_2}$ is slice.

Connected sum turns it into an abelian group.

Dunfield–Gong have a project to classify all prime slice knots with up to 19 crossings. Of the roughly 352 million such knots only about 17 thousand have open slice status (Dec '23). They found about 1.6 million slice knots.

Owens–Swenson studied alternating knots with up to 21 crossings. Of 1.2 billion such knots only a bit over 3 thousand have unresolved slice status.

# Slice Obstructions

Well-known computable topological slice obstructions are

- Fox–Milnor criterion for the Alexander polynomial.
- In particular, $\det(K)$ is square for $K$ slice.
- The signature, $\operatorname{sgn}(K) = 0$ for $K$ slice.
- Herald–Kirk–Livingston criterion for twisted Alexander polynomials.

Knot homologies tend to give smooth slice obstructions, for example,

- Heegaard Floer invariants $\tau$, $\nu$, and $\varepsilon$.
- Rasmussen's $s$-invariants coming from Khovanov homology.
- Lipshitz–Sarkar refinements of the $s$-invariant over $\mathbb{F}_2$.

# The Lipshitz–Sarkar refinements

Lipshitz and Sarkar defined a stable homotopy type for Khovanov homology and as an application they got refinements $s^\alpha, r^\alpha$ of the Rasmussen invariant $s_{\mathbb{F}}$ for any stable cohomology operation $\alpha\colon \tilde{H}^*(\cdot;\mathbb{F}) \to \tilde{H}^{*+n}(\cdot;\mathbb{F})$ with $n > 0$ and $\mathbb{F}$ a field.

These are functions $s^\alpha, r^\alpha\colon \mathcal{C} \to 2\mathbb{Z}$ satisfying

- $s_{\mathbb{F}}(K) \leq r^\alpha(K) \leq s^\alpha(K) \leq s_{\mathbb{F}}(K) + 2$ for any knot $K$.
- $s^\alpha(U) = r^\alpha(U) = 0$ for the unknot $U$.

For example,

- $\alpha = \mathrm{Sq}^2$: Good refinement. Slow to compute.
- $\alpha = \mathrm{Sq}^1$: Poor refinement. Fast to compute.
- $\alpha = \mathrm{Sq}^3$: Poor refinement. Slow to compute.

Sarkar–Scaduto–Stoffregen generalized this to odd Khovanov homology, but did not give any calculations.

- $\mathrm{Sq}^1_{\mathrm{odd}}$: Good refinement. Fast to compute.

# Frobenius systems and Link homology

**Definition**

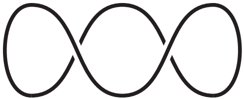
A *Frobenius system* is a tuple $(R, A, \iota, \varepsilon, \Delta)$, where $\iota\colon R \to A$ is an inclusion of commutative rings, $\Delta\colon A \to A \otimes_R A$ a co-associative, co-commutative $A$-bimodule map, $\varepsilon\colon A \to R$ a $R$-module map, and $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$.

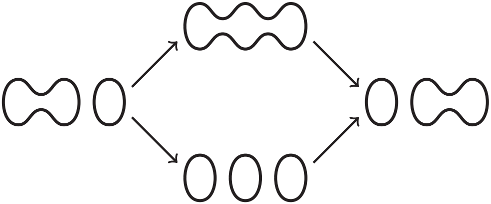
Khovanov showed that rank 2 Frobenius systems give rise to link homology theories. Popular examples include

- (even Khovanov homology) $A = \mathbb{K}[X]/(X^2)$,
$$\Delta(1) = X \otimes 1 + 1 \otimes X, \qquad \Delta(X) = X \otimes X.$$
- (graded Lee homology) $R = \mathbb{K}[t]$, $A = \mathbb{K}[X, t]/(X^2 - t)$,
$$\Delta(1) = X \otimes 1 + 1 \otimes X, \qquad \Delta(X) = X \otimes X + t \otimes 1.$$
- (Bar-Natan homology) $R = \mathbb{K}[h]$, $A = \mathbb{K}[X, h]/(X^2 - Xh)$,
$$\Delta(1) = X \otimes 1 + 1 \otimes X - h \otimes 1, \qquad \Delta(X) = X \otimes X.$$

# Frobenius systems and Link homology

Given a link diagram $D$ and a Frobenius system, the link complex is built from the hypercube of smoothings of $D$.





$$\begin{tikzcd}
 & A \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[ur, "m"] \arrow[dr, "\Delta \otimes \mathrm{id}"'] & & A \otimes A \\
 & A \otimes A \otimes A \arrow[ur, "-\mathrm{id} \otimes m"'] &
\end{tikzcd}$$

# Odd Khovanov homology

Even Khovanov homology can also be defined using a symmetric algebra $S_{\mathbb{K}}[\mathcal{S}]$ over every smoothing $\mathcal{S}$, generated by the circles $C$ in $\mathcal{S}$ subject to $C^2 = 0$. The role of $\Delta$ is then played by multiplication with $C + D$.

Ozsvath–Rasmussen–Szabo defined *odd* Khovanov homology by replacing the symmetric algebra with an exterior algebra, and the role of $\Delta$ is played by multiplication with $C - D$.

Typical properties are

- Over $\mathbb{F}_2$ it agrees with even Khovanov homology.
- Has structure of $\mathbb{K}[X]/(X^2)$-module using a basepoint on $L$.
- $\mathrm{Kh}_o^{*,q}(L;\mathbb{K}) \cong \widetilde{\mathrm{Kh}}_o^{*,q-1}(L;\mathbb{K}) \oplus \widetilde{\mathrm{Kh}}_o^{*,q+1}(L;\mathbb{K})$ for any $\mathbb{K}$.
- There does not seem to be an analogue for an odd Lee or Bar-Natan complex.

# LEO-triples

Let $\mathcal{R} = \mathbb{Z}[X,h]/(X^2 - Xh)$.

## Definition

A *local even-odd (LEO) triple* $(C, D, f)$ consists of a finitely generated, free, bigraded cochain complex $C$ over $\mathbb{Z}[X]/(X^2)$, a finitely generated, free, bigraded cochain complex $D$ over $\mathcal{R}$, and a bigraded chain homotopy equivalence $f\colon C \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to D \otimes_{\mathcal{R}} \mathcal{R}/(2,h)$ such that

- the map $f$ is a homomorphism of cochain complexes over $\mathbb{F}_2[X]/(X^2)$.
- the localization $h^{-1}D = D \otimes_{\mathcal{R}} h^{-1}\mathcal{R}$ is homotopy equivalent to a free graded module of rank 1 over $h^{-1}\mathcal{R}$ supported in homological degree 0 and odd quantum gradings.

There is a reduced version with $C$ over $\mathbb{Z}$ and $D$ over $\mathbb{Z}[h]$ and localization $h^{-1}\mathbb{Z}[h] = \mathbb{Z}[h, h^{-1}]$.

# Local equivalence

**Definition**

Given LEO-triples $(C, D, f)$ and $(C', D', f')$, a *local map* from $(C, D, f)$ to $(C', D', f')$ consists of bigrading-preserving chain maps $\alpha\colon C \to C'$ and $\beta\colon D \to D'$ so that

- the induced map $\beta\colon h^{-1}D \to h^{-1}D'$ is a homotopy equivalence.
- the following diagram commutes up to homotopy:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\alpha"] & C \otimes \mathbb{Z}/(2) \arrow[r, "f"] \arrow[d, "\alpha"] & D \otimes \mathcal{R}/(2,h) \arrow[d, "\beta"] & D \arrow[l] \arrow[d, "\beta"] \\
C' \arrow[r] & C' \otimes \mathbb{Z}/(2) \arrow[r, "f'"] & D' \otimes \mathcal{R}/(2,h) & D' \arrow[l]
\end{tikzcd}$$

We say $(C, D, f)$ and $(C', D', f')$ are *locally equivalent*, if local maps exist in both directions.

# LEO-triples of knots

For $K$ a knot we have the LEO-triples

$$\mathsf{LEO}(K) = (\mathsf{CKh}_o(K), \mathsf{CKh}_{BN}(K), \mathrm{id}),$$

and

$$\mathsf{LEE}(K) = (\mathsf{CKh}(K), \mathsf{CKh}_{BN}(K), \mathrm{id}).$$

In the special case of the unknot $U$ we have

$$\mathsf{LEO}(U) = \mathsf{LEE}(U) = (\mathbb{Z}[X]/(X^2)\{1\}, \mathcal{R}\{1\}, \mathrm{id}).$$

## Proposition

*If the knots $K, K'$ are concordant, then $\mathsf{LEO}(K)$ and $\mathsf{LEO}(K')$ are locally equivalent. The same holds for $\mathsf{LEE}$. In particular, if $K$ is slice then $\mathsf{LEO}(K)$ and $\mathsf{LEE}(K)$ are locally equivalent to $\mathsf{LEO}(U)$.*

# The group of local equivalence classes

Denote the set of local equivalence classes of LEO-triples by $\mathcal{C}_{\mathsf{LEO}}$. The reduced analogue is denoted by $\widetilde{\mathcal{C}}_{\mathsf{LEO}}$.

**Theorem (Dunfield–Lipshitz–S)**

*Both $\mathcal{C}_{\mathsf{LEO}}$ and $\widetilde{\mathcal{C}}_{\mathsf{LEO}}$ have the structure of an abelian group and there is an epimorphism $\pi\colon \mathcal{C}_{\mathsf{LEO}} \to \widetilde{\mathcal{C}}_{\mathsf{LEO}}$. Furthermore, the assignment $K \mapsto \mathsf{LEO}(K)$ defines a homomorphism $\mathcal{C} \to \mathcal{C}_{\mathsf{LEO}}$.*

Similar constructions have been done in involutive Heegaard Floer homology (Hendricks–Manolescu–Zemke, Dai–Hom–Stoffregen–Truong). Also, Lewark recently constructed a group which can be shown to be a direct summand of $\widetilde{\mathcal{C}}_{\mathsf{LEO}}$.

Given a LEO-triple $(C, D, f)$, one can form the LEO-triple $(D \otimes_{\mathcal{R}} Z[X]/(X^2), D, \mathrm{id})$, and such triples generate a subgroup $\mathcal{C}_{\mathsf{LEE}}$ which is a direct summand of $\mathcal{C}_{\mathsf{LEE}}$. The reduced analogue is Lewark's group.

# The $s$-invariant

Let $(C, D, f)$ be a LEO-triple. For $j$ an odd number we have the change of coefficients map $i \colon H^{0,j}(D) \to H^{0,j}(h^{-1}D) \cong \mathbb{Z} \oplus \mathbb{Z}$.

If $\mathbb{F}$ is a field, we can change coefficients and define

$$s_{\mathbb{F}}^{+}(C, D, f) = \max\{j \in 2\mathbb{Z}+1 \mid i_{\mathbb{F}} \text{ is non-zero}\} - 1,$$

and

$$s_{\mathbb{F}}^{-}(C, D, f) = \max\{j \in 2\mathbb{Z}+1 \mid i_{\mathbb{F}} \text{ is surjective}\} + 1.$$

If $(C, D, f)$ is a reduced LEO-triple, $H^{0,j}(h^{-1}D) \cong \mathbb{Z}$ and we define

$$s_{\mathbb{F}}(C, D, f) = \max\{j \in 2\mathbb{Z} \mid i_{\mathbb{F}} \text{ is surjective}\}.$$

Then $s_{\mathbb{F}} \colon \widetilde{\mathcal{C}}_{\mathsf{LEO}} \to 2\mathbb{Z}$ is a homomorphism, but $s_{\mathbb{F}}^{+}$ and $s_{\mathbb{F}}^{-}$ are not. However, $s_{\mathbb{F}}^{+}(\mathsf{LEO}(K)) = s_{\mathbb{F}}^{-}(\mathsf{LEO}(K)) = s_{\mathbb{F}}(\mathsf{LEO}(K))$.

# The $s$-invariant

The $s$-invariant only depends on the characteristic, and if $(C, D, f)$ is a reduced LEO-triple, then $s_{\mathbb{Q}}(C, D, f) = s_{\mathbb{F}_p}(C, D, f)$ except for finitely many primes.

**Proposition**

*The tuple*

$$(s_{\mathbb{Q}}(C, D, f), s_{\mathbb{Q}}(C, D, f) - s_{\mathbb{F}_2}(C, D, f), s_{\mathbb{Q}}(C, D, f) - s_{\mathbb{F}_3}(C, D, f), \cdots)$$

*induces a surjective homomorphism*

$$\widetilde{\mathcal{C}}_{\mathrm{LEO}} \to \bigoplus_{n \in \mathbb{N}} \mathbb{Z}.$$

This follows by considering ‘staircase complexes’, compare Iltgen–Lewark–Marino.

# Invariants of LEO-triples – Bockstein refinements

Given a reduced LEO-triple $(C, D, f)$ and $n$ a positive integer, we get a Bockstein homomorphism

$$\beta_n \colon H^{k,q}(C; \mathbb{Z}/(2^n)) \to H^{k+1,q}(C; \mathbb{Z}/(2)).$$

### Definition

We say the integer $q$ is *$\beta_n$-reduced full*, if there exists $\check{a} \in H^{-1,q}(C; \mathbb{Z}/(2^n))$ and $a \in H^{0,q}(D; \mathbb{F}_2)$ with $f \circ \beta_n(\check{a}) = p(a)$ and $0 \neq i(a) \in H^{0,q}(h^{-1}D; \mathbb{F}_2)$.

$$\begin{tikzcd} & H^{0,q}(D; \mathbb{F}_2) \arrow[r, "i"] \arrow[d, "p"] & H^{0,q}(h^{-1}D; \mathbb{F}_2) \\ H^{-1,q}(C; \mathbb{Z}/(2^n)) \arrow[r, "f \circ \beta_n"] & H^{0,q}(D_{h=0}; \mathbb{F}_2) & \end{tikzcd}$$

# Invariants of LEO-triples – Bockstein refinements

Observe that if $q$ is $\beta_n$-reduced full, then

- $q \leq s_{\mathbb{F}_2}(C, D, f)$.
- $q - 2$ is also $\beta_n$-reduced full: use $ha \in H^{0,q-2}(D; \mathbb{F}_2)$ and $0 \in H^{-1,q-2}(C; \mathbb{Z}/(2^n))$.

### Definition

Given a reduced LEO-triple $(C, D, f)$, define

$$\tilde{s}^{\beta_n}(C, D, f) = \max\{q \in 2\mathbb{Z} \mid q \text{ is } \beta_n\text{-reduced-full}\} + 2.$$

The definition is so that

$$\tilde{s}^{\beta_n}(\mathbb{Z}, \mathbb{Z}[h], \mathrm{id}) = 0.$$

Also, for $n < m$ we have

$$s_{\mathbb{F}_2}(C, D, f) \leq \tilde{s}^{\beta_n}(C, D, f) \leq \tilde{s}^{\beta_m}(C, D, f) \leq s_{\mathbb{F}_2}(C, D, f) + 2$$

# Invariants of LEO-triples – Bockstein refinements

The case $n = 1$ leads to the first Steenrod square. In particular,

$$r^{\mathrm{Sq}^1_o}(K) \leq \tilde{s}^{\beta_1}(\mathrm{LEO}(K)) \leq s^{\mathrm{Sq}^1_o}(K),$$

and

$$r^{\mathrm{Sq}^1}(K) \leq \tilde{s}^{\beta_1}(\mathrm{LEE}(K)) \leq s^{\mathrm{Sq}^1}(K).$$

Also, note we have two operators

$$\mathrm{Sq}^1, \mathrm{Sq}^1_o \colon \widetilde{\mathrm{Kh}}^{-1,q}(K; \mathbb{F}_2) \to \widetilde{\mathrm{Kh}}^{0,q}(K; \mathbb{F}_2).$$

We get another invariant $\tilde{s}^{\beta}(C, D, f)$ corresponding to their sum by looking at

$$\beta = \beta_1 + f^{-1} \circ \beta_1 \circ f \colon H^{-1,q}(C; \mathbb{F}_2) \to H^{0,q}(C; \mathbb{F}_2).$$

# Invariants of LEO-triples – Comprehensive refinements

Consider

$$\begin{tikzcd}
 & H^{0,q}(D) \arrow[r, "i"] \arrow[d, "p"] & H^{0,q}(h^{-1}D) \\
H^{0,q}(C) \arrow[r, "f \circ j"] & H^{0,q}(D_{h=0}; \mathbb{F}_2) &
\end{tikzcd}$$

**Definition**

Let $(C, D, f)$ be a reduced LEO-triple. An integer $q$ is called *oddly reduced full*, if there exist $\check{a} \in H^{0,q}(C)$ and $a \in H^{0,q}(D; \mathbb{F}_2)$ such that $f \circ j(\check{a}) = p(a)$ and $0 \neq i(a) \in H^{0,q}(h^{-1}D; \mathbb{F}_2)$.
Also, $q$ is called *completely reduced full*, if there exist $\check{a} \in H^{0,q}(C)$ and $a \in H^{0,q}(D)$ such that $f \circ j(\check{a}) = p(a)$ and $i(a)$ generates $H^{0,q}(h^{-1}D) \cong \mathbb{Z}$.

# Invariants of LEO-triples – Comprehensive refinements

### Definition

For a reduced LEO-triple $(C, D, f)$ define

$$\tilde{s}_o(C, D, f) = \max\{q \in 2\mathbb{Z} \mid q \text{ is oddly reduced full}\},$$
$$\tilde{s}_c(C, D, f) = \max\{q \in 2\mathbb{Z} \mid q \text{ is completely reduced full}\}.$$

Then

$$s_{\mathbb{F}_2}(C, D, f) - 2 \leq \tilde{s}_o(C, D, f) \leq s_{\mathbb{F}_2}(C, D, f),$$
$$s_{\mathbb{Z}}(C, D, f) - 2 \leq \tilde{s}_c(C, D, f) \leq s_{\mathbb{Z}}(C, D, f).$$

where

$$s_{\mathbb{Z}}(C, D, f) = \max\{q \in 2\mathbb{Z} \mid i \text{ is surjective}\}.$$

Note $s_{\mathbb{Z}}(C, D, f) \leq s_{\mathbb{F}}(C, D, f)$ for any field $\mathbb{F}$.

# Example – $9_{42}$

For the mirror of $9_{42}$ the odd Khovanov homology is given by

| $q \backslash h$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| $6$ | | | | | | | $\mathbb{Z}$ |
| $4$ | | | | | | $\mathbb{Z}$ | |
| $2$ | | | | | $\mathbb{Z}$ | | |
| $0$ | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | | |
| $-2$ | | | $\mathbb{Z}$ | | | | |
| $-4$ | | $\mathbb{Z}$ | | | | | |
| $-6$ | $\mathbb{Z}$ | | | | | | |

It is known that $s_{\mathbb{F}}(\overline{9_{42}}) = 0$ for any $\mathbb{F}$, and we get

$$\tilde{s}^{\beta_1}(\overline{9_{42}}) = 2.$$

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|---|---|---|---|---|---|---|---|
| $6$ | | | | | | | $\mathbb{F}_2$ |
| $4$ | | | | | | $\mathbb{F}_2$ | |
| $2$ | | | | | $\mathbb{F}_2$ | | |
| $0$ | | | | $(\mathbb{F}_2)^2$ | $\mathbb{F}_2$ | | |
| $-2$ | | | $\mathbb{F}_2$ | | | | |
| $-4$ | | $\mathbb{F}_2$ | | | | | |
| $-6$ | $\mathbb{F}_2$ | | | | | | |

It is known that $s_{\mathbb{F}}(\overline{9_{42}}) = 0$ for any $\mathbb{F}$, and we get

$$\tilde{s}^{\beta_1}(\overline{9_{42}}) = 2.$$

# Example – $12^n_{475}$

For $12^n_{475}$ odd Khovanov homology is given by

| $q \backslash h$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 12 | | | | | | | | $\mathbb{Z}$ |
| 10 | | | | | | | $\mathbb{Z}$ | |
| 8 | | | | | | $\mathbb{Z}$ | | |
| 6 | | | | | $\mathbb{Z}^2$ | | | |
| 4 | | | | $\mathbb{Z}$ | | | | |
| 2 | | | $\mathbb{Z} \oplus \mathbb{Z}_3$ | | | | | |
| 0 | | $\mathbb{Z}_8$ | | | | | | |
| $-2$ | $\mathbb{Z}_3$ | | | | | | | |

Again, $s_{\mathbb{F}}(12^n_{475}) = 0$ for any $\mathbb{F}$. Here

$$\tilde{s}_o(12^n_{475}) = -2, \;\; \tilde{s}^{\beta_n}(12^n_{475}) = 0, \;\; \tilde{s}^{\beta_3}(\overline{12^n_{475}}) = 2$$

# Example – $12^n_{475}$

For $12^n_{475}$ odd Khovanov homology is given by

| $q \backslash h$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 12 | | | | | | | | $\mathbb{F}_2$ |
| 10 | | | | | | | $\mathbb{F}_2$ | |
| 8 | | | | | | $\mathbb{F}_2$ | | |
| 6 | | | | | $(\mathbb{F}_2)^2$ | | | |
| 4 | | | | $\mathbb{F}_2$ | | | | |
| 2 | | | $\mathbb{F}_2$ | | | | | |
| 0 | $\mathbb{F}_2$ | $\mathbb{F}_2$ | | | | | | |
| $-2$ | | | | | | | | |

Again, $s_{\mathbb{F}}(12^n_{475}) = 0$ for any $\mathbb{F}$. Here

$$\tilde{s}_o(12^n_{475}) = -2, \;\; \tilde{s}^{\beta_n}(12^n_{475}) = 0, \;\; \tilde{s}^{\beta_3}(\overline{12^n_{475}}) = 2$$

# Properties of the refinements

**Proposition**

*All Bockstein and comprehensive refinements only depend on the local equivalence class. In particular, they all vanish on slice knots.*

**Theorem A (Dunfield–Lipshitz–S)**

*A reduced triple* $(C, D, f)$ *represents* $0$ *in* $\widetilde{C}_{\mathrm{LEO}}$ *if and only if*

$$\tilde{s}_c(C, D, f) = 0 = \tilde{s}_c(C^*, D^*, f^*).$$

Basic idea: If $a, \check{a}$ witness $\tilde{s}_c(C, D, f) = 0$, one can define a local map $(\mathbb{Z}, \mathbb{Z}[h], \mathrm{id})$ to $(C, D, f)$. To get the local map in the other direction, use the dual and $0 = \tilde{s}_c(C^*, D^*, f^*)$.

# Properties of the refinements

A similar result holds for $\tilde{s}_o$ with a group $\widetilde{C}^o_{\mathrm{LEO}}$ that involves triples $(C, D, f)$ where $D$ is defined over $\mathbb{F}_2[h]$.
This can be used to show:

### Proposition

*Let $(C, D, f)$ be a reduced LEO-triple with dual $(C^*, D^*, f^*)$. If $\tilde{s}^{\beta_n}(C, D, f) \neq s_{\mathbb{F}_2}(C, D, f)$, then $\tilde{s}^{\beta_n}(C^*, D^*, f^*) = s_{\mathbb{F}_2}(C^*, D^*, f^*)$*

The point is that $\tilde{s}^{\beta_n}(C, D, f) \neq s_{\mathbb{F}_2}(C, D, f)$ implies $\tilde{s}_o(C, D, f) = s_{\mathbb{F}_2}(C, D, f)$. If the dual also differs from the $s$-invariant, we get $(C, D, f)$ trivial in $\widetilde{C}^o_{\mathrm{LEO}}$ (up to grading shift).

A similar behaviour works for $\tilde{s}_o$, but the proof is more difficult.

### Proposition

*Let $(C, D, f)$ be a reduced LEO-triple with dual $(C^*, D^*, f^*)$. If $\tilde{s}_o(C, D, f) \neq s_{\mathbb{F}_2}(C, D, f)$, then $\tilde{s}_o(C^*, D^*, f^*) = s_{\mathbb{F}_2}(C^*, D^*, f^*)$*

# Properties of the refinements

We can define a relation on $\widetilde{C}^o_{\mathrm{LEO}}$ by $[(C, D, f)] \geq [(C', D', f')]$, if there is a local map from $(C', D', f')$ to $(C, D, f)$. Because of the previous Proposition, this gives a total order

**Theorem B**

*There is a translation invariant total order on $\widetilde{C}^o_{\mathrm{LEO}}$, characterised by $[(C, D, f)] \geq 0$ if and only if $\tilde{s}_o(C, D, f) \geq 0$.*

We do not know if this is true for $\widetilde{C}_{\mathrm{LEO}}$, but the previous Proposition does not hold for $\tilde{s}_c$. Finally, a possibly disappointing result:

**Proposition**

*Let $K$ be an alternating knot. Then $\mathrm{LEO}(K)$ and $\mathrm{LEE}(K)$ are locally equivalent to $(\mathbb{Z}\{s\}, \mathbb{Z}[h]\{s\}, \mathrm{id})$, where $s$ is the signature of $K$.*

# Computations

The following table shows the number of prime knots for which the refinement differs from the $s$-invariant.

| crossings | $\tilde{s}^{\mathrm{Sq}^1}(K)$ | $\tilde{s}^{\mathrm{Sq}^1_o}(K)$ | $\tilde{s}^{\beta}(K)$ | $\tilde{s}^{\beta_{15}}(K)$ | $\tilde{s}_c(K)$ |
|---|---|---|---|---|---|
| 9 | 0 | 1 | 1 | 1 | 1 |
| 10 | 0 | 2 | 2 | 2 | 2 |
| 11 | 0 | 10 | 10 | 10 | 10 |
| 12 | 0 | 49 | 49 | 50 | 50 |
| 13 | 0 | 286 | 285 | 297 | 297 |
| 14 | 2 | 1,718 | 1,717 | 1,797 | 1,797 |
| 15 | 41 | 11,244 | 11,239 | 11,808 | 11,819 |
| 16 | 162 | 73,814 | 73,787 | 77,873 | 77,929 |

# Computations

In April 2023 Dunfield–Gong were left with 17,991 knots among prime knots with up to 19 crossing for which the slice status was unknown. Of those knots 826 have non-zero $\tilde{s}^{\beta_{15}}$, and 64 have non-zero $\tilde{s}^{\beta}$ (no overlap between these knots). Also, $\tilde{s}_c$ detects exactly these 890 knots.

Manolescu–Piccirillo ('21) listed five topologically slice knots such that if any of them is smoothly slice, then an exotic $S^4$ exists. Nakamura ('22) showed that all of these knots are not slice.

For all five of these knots we also show that $\tilde{s}^{\beta}$ is non-zero.