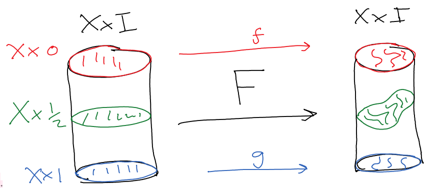
Pseudo-isotopies of 4-Manifolds

I. Pseudo-isotopies, high dimensional background



Def^n A (smooth) *Pseudo-isotopy* of a Manifold X is a diffeomorphism

$$F: X\times[0,1] \longrightarrow X\times[0,1]$$

Such that $F(X\times\{i\}) = X\times\{i\}$, for $i=0,1$ and $F|_{\partial X\times I} = \mathbb{1}_{\partial X\times I}$.

Denoting $f = F|_{X\times\{0\}}: X\longrightarrow X$, $g = F|_{X\times\{1\}}: X\longrightarrow X$ we say f and g are (smoothly) *Pseudo-isotopic*.

Remark
- A pseudo-isotopy that is level preserving, i.e. $F(X\times\{i\}) = X\times\{i\}$ For $i\in[0,1]$ is an isotopy!
- Isotopy implies Pseudo-isotopy!

Motivation Given self-diffeomorphisms $f,g: X\to X$ which are homotopic, one has a continuous map $F: X\times[0,1]\to X\times[0,1]$ With $F|_{X\times\{0\}} = f$, $F|_{X\times\{1\}} = g$ in certain favourable situations, surgery theory allows us to turn this map into a homeomorphisms/diffeomorphism.

Question 1. Given $f,g: X\longrightarrow X$ pseudo-isotopic maps, when are they isotopic?

Equivalently,

Question 1'. Given $f: X\longrightarrow X$ pseudo-isotopic to $\mathbb{1}_X$, when is f isotopic to $\mathbb{1}_X$?

Denote the set of diffeomorphisms pseudo-isotopic to the identity by $\mathrm{Diff}_{PI}(X,\partial X)$

Question 1''. Is $\pi_0\,\mathrm{Diff}_{PI}(X,\partial X) = 0$?

Question 2. Given a pseudo-isotopy $F: X\times I \longrightarrow X\times I$ with $F|_{X\times\{0\}} = \mathbb{1}_X$ is F isotopic to an isotopy?
i.e. does there exist $H_t: X\times I \longrightarrow X\times I$ for $t\in[0,1]$ with $H_0 = F$, H_1 an isotopy.

$$\text{W.L.O.G.} \bullet H_t|_{X\times\{0\}} = \mathbb{1}_X$$

Note: W.L.O.G.
- $H_t|_{X\times\{0\}} = \mathbb{1}_X$ $\forall t\in[0,1]$. Since if not we can take
- $H_1 = \mathbb{1}_{X\times 1}$ since any level preserving map $F: X\times[0,1]\to X\times$

Proof: $H_t(x,s) = F(x,ts)$ is an isotopy with

(with C^∞ topology)

Def^n Let $\mathcal{P}(X) = \{\text{pseudo-isotopies } F: X\times[0,1] \xrightarrow{\text{diffeo}} X\times[0,1]\ ;\ F|_{X\times\{0\}} = \mathbb{1}_X\}$

Note this is a group under composition!

Question 2'. What is $\pi_0\mathcal{P}(X)$?

Relationship Between $\mathcal{P}(X)$ & $\mathrm{Diff}_{PI}(X,\partial X)$

$$F: X\times I \to X\times I \qquad F|_{X\times 0} = id$$

$$0 \longrightarrow \pi_0\mathcal{I}(X) \longrightarrow \pi_0\mathcal{P}(X) \longrightarrow \pi_0\mathrm{Diff}_{PI}(X,\partial X) \longrightarrow 0$$

$$F \longmapsto F|_{X\times 1}$$

Where $\mathcal{I}(X) = \{\text{pseudo-isotopies from } \mathbb{1}_X \text{ to } \mathbb{1}_X\} \subset \mathcal{P}(X)$

When X is simply connected

Dimension ≥ 5

Theorem (Cerf) If X is a simply-connected manifold of dimension ≥ 5, every smooth pseudo-isotopy is smoothly isotopic to an isotopy.

Corollary: $\pi_0\mathcal{P}(X) = \pi_0\mathrm{Diff}_{PI}(X,\partial X) = 0$

Dimension 4

$$X\times I \longrightarrow X\times I$$

Theorem (Quinn) For X a simply connected manifold of dimension 4 every topological pseudo-isotopy is isotopic to a topological isotopy.

$$H'_t = \left(H_t|_{X\times\{0\}}^{-1} \times \mathbb{1}_{[0,1]}\right) \times H_t$$

$\times[0,1]$ fixing $X\times\{0\}$ is isotopic to $\mathbb{1}_{X\times[0,1]}$

$$H_1 = H, \quad H_0 = \mathbb{1}_{X\times[0,1]}$$

)

Corollary (Quinn, after a 5-dimensional surgery argument)

$$\frac{\mathrm{Homeo}(X)}{\mathrm{Isotopy}} \xrightarrow{\cong} \mathrm{Aut}(H_2X, \lambda_X)$$

Intersection form

Dimension 4 Examples of $\pi_0 \mathrm{Diff}_{PI}(X,\partial X) \neq 0$ from gauge theory. (Ruberman, Baraglia-Konno, Kronheimer-Mrowka, and Lin)

When X is non-simply connected

High Dimensions

Theorem (Hatcher + Wagoner + Igusa) (Igusa) For X a smooth manifold of dimension $n \geq 6$ there is an exact sequence

$$K_3\mathbb{Z}[\pi_1 X] \xrightarrow{\chi} Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) \rightleftarrows \pi_0 P(X) \xrightarrow{\Sigma} Wh_2(\pi_1 X) \to 0$$

For $n \geq 4$ there are maps

The two invariants of this talk

$$\begin{cases} \Sigma : \pi_0 P(X) \longrightarrow Wh_2(\pi_1 X) & \text{first pi inv.} \\ \Theta : \ker \Sigma \longrightarrow Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) / \chi(K_3\mathbb{Z}[\pi_1 X]) \end{cases}$$

- When $n \geq 5$, Σ and Θ are surjective.
- When $k_1 X = 0$ $n \geq 7$ $\Sigma + \Theta : \pi_0 P(X) \longrightarrow Wh_2(\pi_1 X) \oplus Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)$ is an isomorphism (Hatcher & Wagoner)

first postnikov invariant

$Wh_1(\pi_1 X; \mathbb{Z}$

The groups Wh_1

Whitehead Torsion group

Dimension 4

Previous examples from Budney-Gabai and Watanabe of $\pi_0 \mathrm{Diff}_{PI}(X, \partial X) \neq 0$.

$\underbrace{}_{S^3 \times S^1,\ B^3 \times S^1}$ $\underbrace{}_{\Sigma(2,3,5) \times S^1}$

Recent examples (S.) and concurrently Igusa of $\mathrm{Diff}_{PI}(X, \partial X) \neq 0$ using the invariant Θ.

$\underbrace{}_{S^1 \times S^1 \times S^2,\ (M_1 \# M_2) \times S^1}$ $\underbrace{}_{((S^1 \times S^2) \# M) \times S^1}$

M_i^3 closed orientable aspherical

M^3 non-simply connected

Further examples (S.) of $\pi_0 P(X) \neq 0$.

$P(X) \neq 0$ when certain subgroup of $Wh_1(\pi_1 X, \mathbb{Z}_2 \times \pi_2 X) / \chi(K_3\mathbb{Z}[\pi_1 X])$ is $\neq 0$

$P(X \# S^2 \times S^2) \neq 0$ when $Wh_1(\pi_1 X, \mathbb{Z}_2 \times \pi_2 X) / \chi(K_3 \mathbb{Z}[\pi_1 X]) \neq 0$

$P(X \#^n S^2 \times S^2) \neq 0$ when $Wh_2(\pi_1 X) \neq 0$.

for n sufficiently large

II. Hatcher and Wagoners Invariants

Pseudo-isotopies as paths of functions

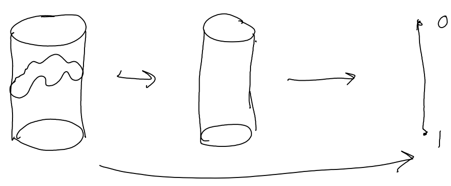
Def^n.
$$\mathcal{M}(X) = \{ f : X \times [0,1] \xrightarrow{\text{Smooth}} [0,1] \ ;\ f(X \times \{i\}) = i \text{ for } i = 0,1 \}$$
$$\mathcal{M}_0(X) = \{ f \in \mathcal{M}(X) \ ;\ f \text{ has no critical points} \}$$

Let p denote the standard projection $p : X \times [0,1] \longrightarrow [0,1]$

We can construct a map $\pi : P(X) \longrightarrow \mathcal{M}(X)$

$$F \longmapsto p \circ F$$
$$X \times I \to X \times I \to I$$

Since F is a diffeomorphism, and p has no critical points $\pi(P(X)) \subseteq \mathcal{M}_0(X)$



Lemma $\pi_* : \pi_0 P(X) \longrightarrow \pi_0 \mathcal{M}_0(X)$ Is an isomorphism!

So since $\mathcal{M}(X) \simeq *$ we have

$$\pi_0 P \underset{\pi_*}{\cong} \pi_1(\mathcal{M}(X), \mathcal{M}_0(X)).$$

Proof injective: ✓
surjective: $f \in \mathcal{M}_0(X)$, let $F = \phi_{s,t}(x, 0)$ □

flow of f

We will define Σ and Θ on $\pi_1(\mathcal{M}(X), \mathcal{M}_0(X))$

Paths in $\mathcal{M}(X)$

Any path $f_t \in \mathcal{M}(X)$ can be deformed to a 1 parameter family of morse functions (with some standard extra singular points

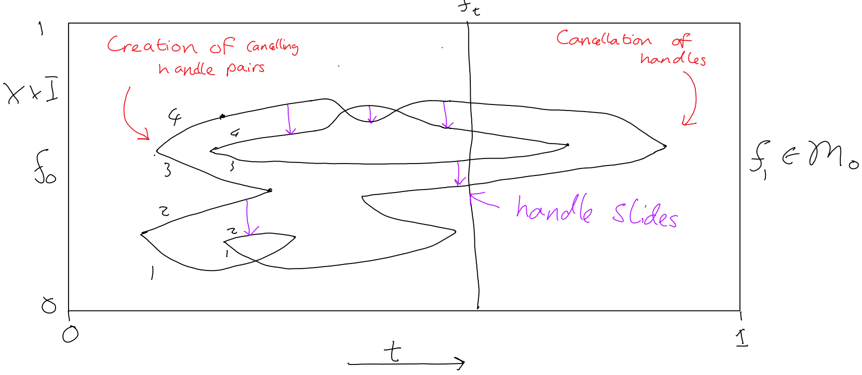
$$\mathbb{Z}_2 \times \pi_2 X) = \frac{(\mathbb{Z}_2 \times \pi_2 X)[\pi_1 X]}{\langle \alpha\sigma - \alpha^{\tau}\tau\sigma\tau^{-1},\ \beta \cdot 1 \mid \alpha, \beta \in \mathbb{Z}_2 \times \pi_1 X,\ \tau, \sigma \in \pi_1 X \rangle}$$

, Wh_2 are defined using algebraic K-Theory. In the non-relative case $Wh_1(G)$ is the

$Wh_1(G) = \frac{GL(\mathbb{Z}[G])}{E}$ arising in the s-cobordism theorem.

Paths in $\mathcal{M}(X)$

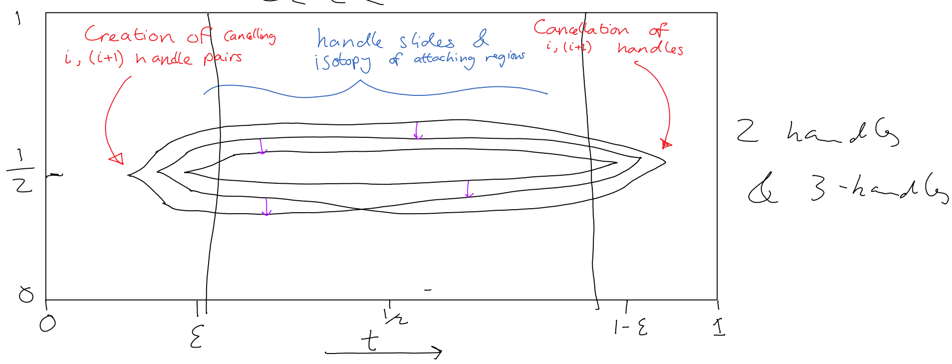
Any path $f_t \in \mathcal{M}(X)$ can be deformed to a 1-paramater family of morse functions (with some standard extra singular points corresponding to creation of cancelling handle pairs). We can also make the Cerf diagram "nice":



(Hatcher & Wagoner)

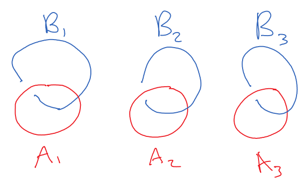
Thm Let X be an n-manifold. For any $2 \leq i \leq n-2$ and any path $f_t \in \mathcal{M}(X)$ with $f_0, f_1 \in \mathcal{M}_0(X)$ we can deform f_t fixing the ends so it has the following Cerf graphic

$2 \leq 2 \leq 2 = 4-2$

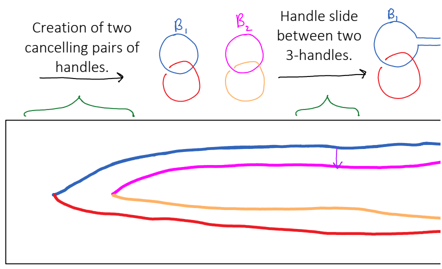


4-dim picture

- Looking in the "middle 4-manifold" $V_t = f_t^{-1}\left(\frac{1}{2}\right)$ we see first a copy of X for $t < \varepsilon$.
- Between $t=$ and $t=\varepsilon$ we see the creation of m cancelling pairs - the middle 4-manifold becomes $X \#^m S^2\times S^2$
  - We see two sets of embedded spheres $\underbrace{A_1, A_2, \dots, A_n}_{\text{2-handle Belt Spheres}}$ ($S^2\times\{0\}$) and $\underbrace{B_1, B_2, \dots, B_n}_{\text{3-handle attaching spheres}}$ ($\{0\}\times S^2$) with $A_i \cap B_j = \delta_{ij}$ points

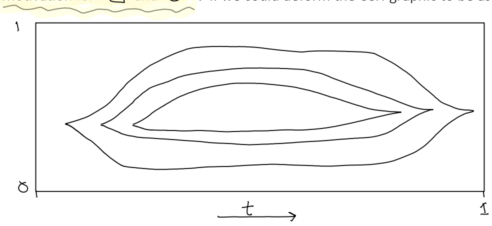


- Between $t=\varepsilon$ and $t=1-\varepsilon$ some weird stuff can happen!
  - Handles can slide over each other (resulting in tubing an A-sphere to a parallel copy of another A-sphere, or tubing a B-sphere to a parallel copy of another B-sphere
  - The spheres can move about. The A-spheres must be mutually disjoint, as must the B-spheres, but the A's and B's can intersect! By general position, we can perturb f_t so we see a collection of finger moves and Whitney moves



- At $t = 1-\varepsilon$ each A-sphere has a B-sphere that it intersects in a single point. The 2 and 3 handles then cancel in pairs!

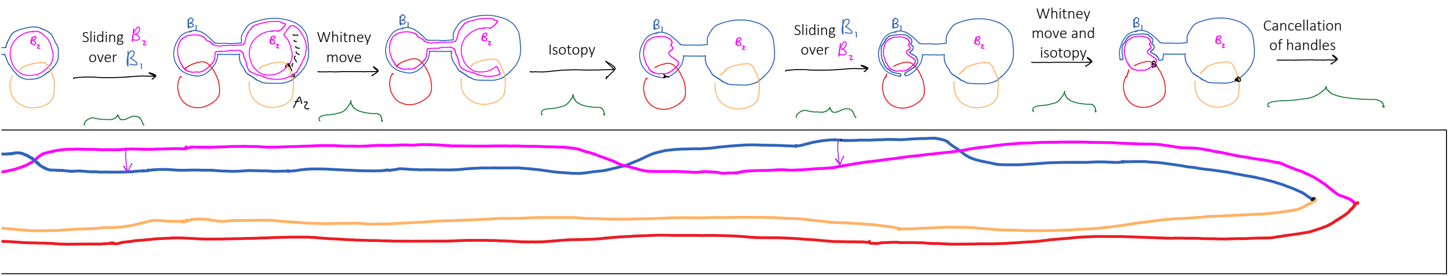
Motivation for Σ and Θ: If we could deform the Cerf graphic to be as below



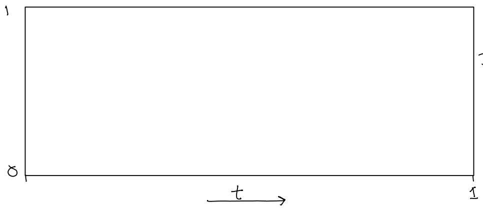
no handle slides!
no finger Moves!
no Whitney Moves!

then we could "cancel the eyes"

Θ : Ker Σ



$(M(X), M_\gamma(X))$



$f_t \in \mathcal{M}_0 \ \forall t$

So $[f_t] = 0 \in \pi_1(\mathcal{M}(X), \mathcal{M}_0(X)) = \pi_0 \mathcal{P}(X)$

Def^n of Σ

Intuition: Σ counts handle slides.

$$GL(\mathbb{Z}[\pi_1 X]) = \lim_{n\to\infty} GL_n(\mathbb{Z}[\pi_1 X])$$

$$\langle e_{ij}^{\lambda} \rangle = E(\mathbb{Z}[\pi_1 X]) \subset GL(\mathbb{Z}[\pi_1 X])$$

$$e_{ij}^{\lambda} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & \cdots & \lambda & \\ 0 & 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & & 1 & \\ & & & & & \ddots \end{pmatrix} \begin{array}{c} \\ \\ j \\ \\ \\ \\ \end{array}$$

easy relations:

- $e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$
- $[e_{ij}^{\lambda}, e_{k\ell}^{\mu}] = 0$ for $i \neq k$, $j \neq \ell$
- $[e_{ij}^{\lambda}, e_{jk}^{\mu}] = e_{ik}^{\lambda\mu}$ i,j,k distinct

Def^n The Steinberg group $St(\mathbb{Z}[\pi_1 X])$ is the group freely generated by the symbols x_{ij}^{λ} subject to the relations

- $x_{ij}^{\lambda} x_{ij}^{\mu} = {}^{\lambda+\mu}_{ij}$
- $[x_{ij}^{\lambda}, x_{k\ell}^{\mu}] = 0$ for $i \neq k$, $j \neq \ell$
- $[x_{ij}^{\lambda}, x_{jk}^{\mu}] = e_{ik}^{\lambda\mu}$ i,j,k distinct

There is a map $\pi: St(\mathbb{Z}[\pi_1 X]) \longrightarrow E(\mathbb{Z}[\pi_1 X])$

$$x_{ij}^{\lambda} \longmapsto e_{ij}^{\lambda}$$

$$K_2(\mathbb{Z}[\pi_1 X]) := \operatorname{Ker}(\pi) \leqslant St(\mathbb{Z}[\pi_1 X])$$

$$Wh_2(\pi_1 X) := K_2(\mathbb{Z}[\pi_1 X]) \bmod \langle x_{ij}^{\pm g} x_{ji}^{\mp g^{-1}} x_{ij}^{\pm g} \rangle \cap K_2(\mathbb{Z}[\pi_1 X])$$

Label the i-handles and the $(i+1)$-handles, and pick some arcs to a basepoint. Then you can consider the morse differential at time t

$$\partial_{i+1}^{t} \in GL_m(\mathbb{Z}[\pi_1 X]) \subset GL(\mathbb{Z}[\pi_1 X])$$

at $t = \varepsilon$, $\partial_{i+1}^{\varepsilon} = I$

When a handle slide occurs, ∂_{i+1}^{t} changes by left or right multiplication by e_{ij}^{λ}

at $t > \varepsilon$, $\partial_{i+1}^{t} = e_{i_1 j_1}^{\lambda_1} \cdots e_{i_k j_k}^{\lambda_k}$

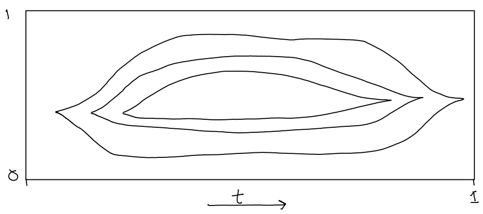
Right before all of the handles are cancelled we must have $A_i \cap B_j = \delta_{i\sigma(j)}$ for σ some permutation

at $t = 1-\varepsilon$ $\partial_{i+1}^{\varepsilon} = e_{i_1 j_1}^{\lambda_1} \cdots e_{i_k j_k}^{\lambda_k} = P \cdot D$

Fact: $\exists\ w \in \langle x_{ij}^{\pm g} x_{ji}^{\mp g^{-1}} x_{ij}^{\pm g} \rangle$ with $\pi(w) = P \cdot D$

Def^n: $\Sigma(f_t) = x_{i_1 j_1}^{\lambda_1} \cdots x_{i_k j_k}^{\lambda_k} w^{-1}$

Thm (Hatcher-Wagoner) When $\Sigma(f_t) = 0$ and $n \geqslant 4$, f_t can be deformed to have the following Cerf Graphic



no handle slides!

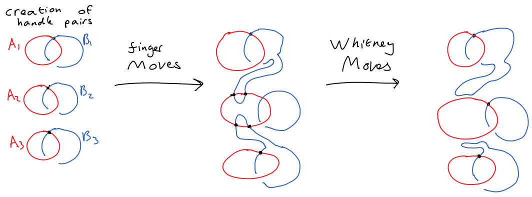
Some finger/whitney moves

In the middle level we now see

creation of handle pairs

$\xrightarrow{t}$ $\quad$ i

In the middle level we now see



$X \times I \times I$

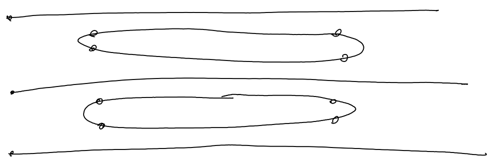
$\uparrow$

A_i, B_i $\quad S^2 \times I$

$A_i \times I, B_j \times I$ $\quad S^2 \times I$

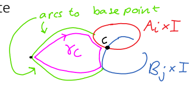
Consider the trace of intersections $A_i \times I \cap B_j \times I$: these are lines and circles

$\Theta : \ker \Sigma \longrightarrow Wh_1$



Intuition for Θ: counts circles of intersection

To each circle of intersection C, associate



$\gamma_c \in \pi_1 X$

$\sigma_c \in \pi_2 X$

$C \subset A_i \times I$ so bounds a disk $D_a \subset A_i \times I$

$C \subset B_j \times I$ so bounds a disk $D_b \subset B_j \times I$

$S^2 \subset X \times I \times I$

$S_c \in \mathbb{Z}_2$ Framings!

Matrix of intersections: $M_{ij} = \sum\limits_{\substack{\text{circles of intersection} \\ C \subset A_i \times I \cap B_j \times I}} (S_c, \sigma_c)\gamma_c \in (\mathbb{Z}_2 \times \pi_2 X)[\pi_1 X]$

$$\Theta(f_t) = tr(I + M_{ij}) \in \frac{(\mathbb{Z}_2 \times \pi_2 X)[\pi_1 X]}{\sim} = \frac{Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)}{\chi(k_3 \mathbb{Z}[\pi_1 X])}$$

III Realisation of Hatcher and Wagoner's invariants in dimension 4

Recall When $n \geqslant 5$, Σ and Θ are surjective.

Thm (S.) (Stable theorems) Σ and Θ are (in some sense) surjective after taking connect sums with $S^2 \times S^2$

Thm (S.) For X a 4-manifold Hatcher/Wagoner's invariant Θ surjects onto the following subgroup

$$\langle (s, \sigma)\gamma \mid w_2^X(\sigma) \neq 0 \text{ or } s = 0,\ \sigma \in \pi_2 X,\ s \in \mathbb{Z}_2,\ \gamma \in \pi_1 X \rangle$$

$$\subset \frac{(\mathbb{Z}_2 \times \pi_2 X)[\pi_1 X]}{\sim} = \frac{Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)}{\chi(k_3 \mathbb{Z}[\pi_1 X])}$$

Give examples of $P(X) = 0$

$X_n = X \#$

$Wh_2(\pi_1$

$\frac{Wh_1(\pi_1 X; \mathbb{Z}_2 \times}{\chi(k_3 \mathbb{Z}[}$

Given $a \in Wh$

Given $a \in \frac{Wh_1(}{}$

Examples of $Diff_{PI}(X, \partial X) \neq 0$ via:

$$\begin{tikzcd}
0 \arrow[r] & \pi_0 \mathcal{I}(X) \arrow[r] \arrow[d, "\Sigma"] & \pi_0 P(X) \arrow[r] \arrow[d, "\Sigma"] & \pi_0 Diff_{PI}(X, \partial X) \arrow[r] \arrow[d, "\hat{\Sigma}"] & 0 \\
0 \arrow[r] & \Sigma(\mathcal{I}(X)) \arrow[r] & Wh_2(\pi_1 X) \arrow[r] & \frac{Wh_2(\pi_1 X)}{\Sigma(\mathcal{I}(X))} \arrow[r] & 0
\end{tikzcd}$$

$$0 \longrightarrow \pi_0 \mathcal{I}(X) \cap \ker \Sigma \longrightarrow \ker \Sigma \longrightarrow \ker \hat{\Sigma} \longrightarrow 0$$

$^{n} S^2 \times S^2$

$X) = Wh_2(\pi_1 X_n)$ $\quad \Sigma$

$$\frac{\times \pi_2 X)}{\pi_1 X])} \leqslant \frac{Wh_1(\pi_1 X_n; \mathbb{Z}_2 \times \pi_2 X_n)}{\chi(K_3 \mathbb{Z}[\pi_1 X_n])} \quad \Theta$$

$\underline{{}_2(\pi_1 X)}$ $\underline{\exists\, n}$ and $F \in \mathcal{P}(X_n)$ with $\Sigma(F) = a$.

$\frac{\pi_1 X;\, \mathbb{Z}_2 \times \pi_2 X)}{\chi(K_3 \mathbb{Z}[\pi_1 X]}$ there exists $F \in \mathcal{P}(X_1)$ with $\Sigma(F) = 0$, $\Theta(F) = a$.

$$\begin{tikzcd}
0 \arrow[r] & \pi_0 \mathcal{I}(X) \cap \ker \Sigma \arrow[r] \arrow[d, "\Theta"] & \ker \Sigma \arrow[r] \arrow[d, "\Theta"] & \ker \hat{\Sigma} \arrow[r] \arrow[d, "\hat{\Theta}"] & 0 \\
0 \arrow[r] & \Theta(\mathcal{I}(X) \cap \ker \Sigma) \arrow[r] & Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) \arrow[r] & \dfrac{Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)}{\Theta(\mathcal{I}(X) \cap \ker \Sigma)} \arrow[r] & 0
\end{tikzcd}$$

Define $\Sigma(f) = \Sigma(F) \in \dfrac{Wh_2(\pi_1 X)}{\Sigma(\mathcal{I}(X))}$

$\Theta(f) = \Theta(F) \in \dfrac{Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) / \chi(K_3 \mathbb{Z}[\pi_1 X])}{\Theta(\mathcal{I}(X) \cap \ker \Sigma)}$

$\Sigma(\mathcal{I}(X))$ and $\Theta(\mathcal{I}(X) \cap \ker \Sigma)$ are in general very complicated.

When $X = M^3 \times I$, $\Theta(\mathcal{I}(X) \cap \ker \Sigma)$ and $\Sigma(\mathcal{I}(X))$ can be understood algebraically!

$K_3 \mathbb{Z}[\pi_1 M^3]$ can be calculated using the Farell-Jones conjecture for 3-manifold groups! (Bartels, Farell, Lück)

$\overset{\shortparallel}{H_3}(B(\pi_1 M^3); K(\mathbb{Z}))$

K theory spectrum of $\mathbb{Z}$

M closed

Lemma (Igusa) $\pi_0 \mathrm{Diff}_{PI}(M \times I, \partial(M \times I)) \hookrightarrow \pi_0 \mathrm{Diff}_{PI}(M \times S^1)$

Thm (S.) Let X be the 4-manifold $S^2 \times S^1 \times I$, $S^2 \times S^1 \times S^1$, $(M_1 \# M_2) \times I$, or $(M_1 \# M_2) \times S^1$ for M_1, M_2 closed, orientable, aspherical 3-manifolds. Then there is a subgroup $K \subset \pi_0 \mathrm{Diff}_{PI}(X, \partial X)$ and a surjective map

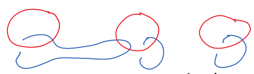
$$\Theta' : K \twoheadrightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$$

and hence infinitely many distinct pseudo-isotopy classes of diffeomorphism of X.

Stable Σ realisation:

recipe to build path f_t with $\Sigma(f_t) = \sigma$

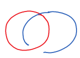
- Have some element $\sigma \in Wh_2(\pi_1 X)$
- Write it in terms of generators
  $\sigma = x_{ij}^{\pm} \cdots\cdots x_{ij}^{\pm}$
- Create N cancelling pairs of handles
- Do handle slides of 3-handles according to generators
  At the end:



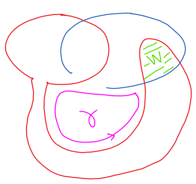
- A_i and B_j intersect algebraically as $\pi(\sigma) = I$, but intersect weirdly geometrically!
- In high dimensions can sort this out using the Whitney trick - in 4-dimensions use $\#^N S^2 \times S^2$
- Once $A_i \cap B_j = \delta_{ij}$ cancel the handles!

Θ realisations

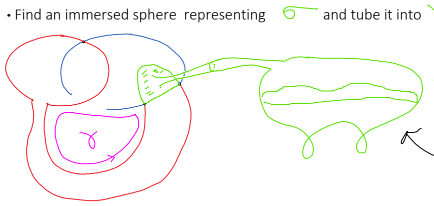
- Have $(s, \sigma)\gamma \in \dfrac{(\mathbb{Z}_2 \times \pi_2 X)[\pi_1 X]}{\sim} = Wh_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)$
- Create a cancelling pair of handles



- Do a finger move around γ:



- Find an immersed sphere representing σ and tube it into W



rep^n some π_2 elt

- Do boundary twists to make W' framed, and to arrange for the correct framing coefficient S

- Worry about intersections.
- Successfully resolve intersections, except when $w_2^X(\sigma) = 0$ and $s = 1$ - in this case you need an $S^2 \times S^2$ to help.