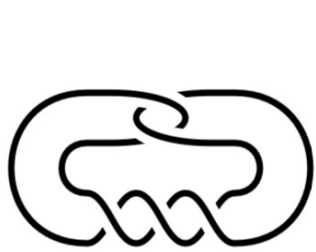


Floer homology and non-fibered knots

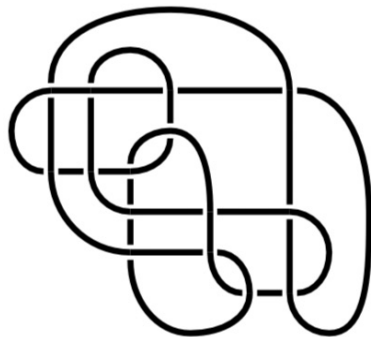
Steven Siuek (w/ John Baldwin)

Max Planck Institute for Mathematics /
Imperial College London

[K-OS] Knot Online Seminar, 15 Dec. 2022



5_2



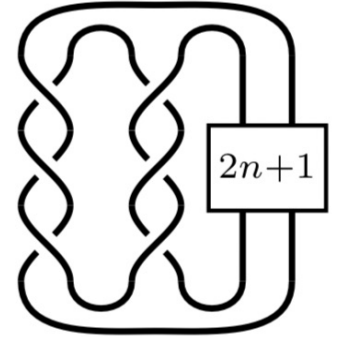
$15n_{43522}$



$Wh^-(T_{2,3}, 2)$



$Wh^+(T_{2,3}, 2)$



$P(-3, 3, 2n+1)$

Q: given a knot invariant, which knots can it detect?

Ex knot Floer homology $\widehat{HFK}(K) = \bigoplus_{m, a \in \mathbb{Z}} \widehat{HFK}_m(K, a)$


detects the Seifert genus $g(K)$, by


$$g(K) = \max \{ a \mid \widehat{HFK}(K, a) \neq 0 \} \quad (\text{Ozsváth-Szabó '03})$$

\rightsquigarrow \widehat{HFK} detects the unknot ;

and K is fibered iff $\dim \widehat{HFK}(K, g(K)) = 1$

(Ghiggini, Ni '06)

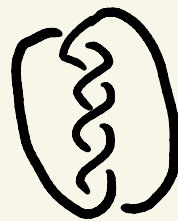
\rightsquigarrow \widehat{HFK} detects the trefoils 

and figure eight. 

(Ghiggini '06)

\widehat{HFK} is only known to detect one other knot (up to mirroring),

the **cinquefoil** $T(2,5)$.



(Farber-Reinoso-Wang '22)

Ex **Khovanov homology** $\overline{Kh}(K) = \bigoplus_{h,q} \overline{Kh}^{h,q}(K)$

admits spectral sequences

• $\overline{Kh}(K) \Rightarrow KHI(\bar{K})$

(Kronheimer-Mrowka '10)

• $\overline{Kh}(K) \Rightarrow \widehat{HFK}(\bar{K})$

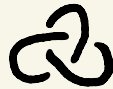
(Dowlin '18)

so it detects • the **unknot**



(Kronheimer-Mrowka '10)

• the **trefoils**



(Baldwin-S. '18)

• the **figure eight**



(Baldwin-Dowlin-Levine
- Libman-Sazdanović '20)

• the **cinquefoils**.



(Baldwin-Hu-S. '21)

Common feature: all of these rely on K being fibered, i.e. $\dim \widehat{\text{HFK}}(K, g(K)) = 1$.

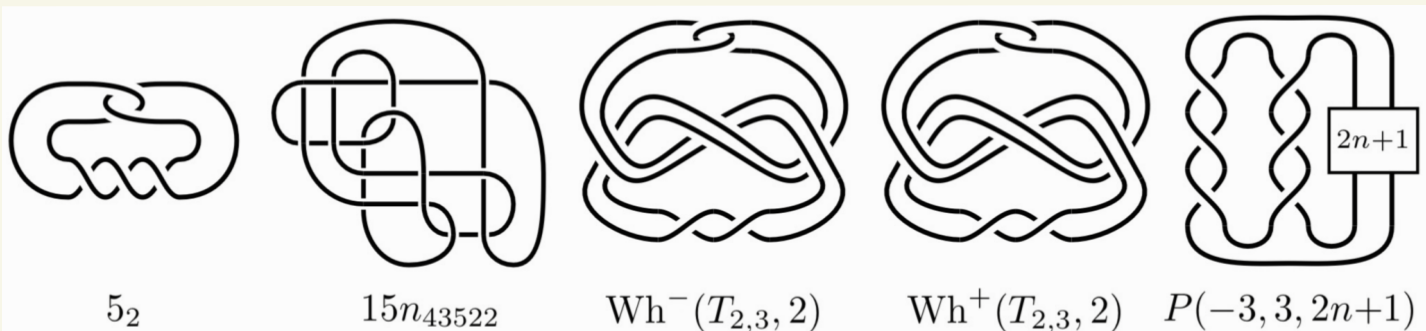
Goal: say something about non-fibered knots too!

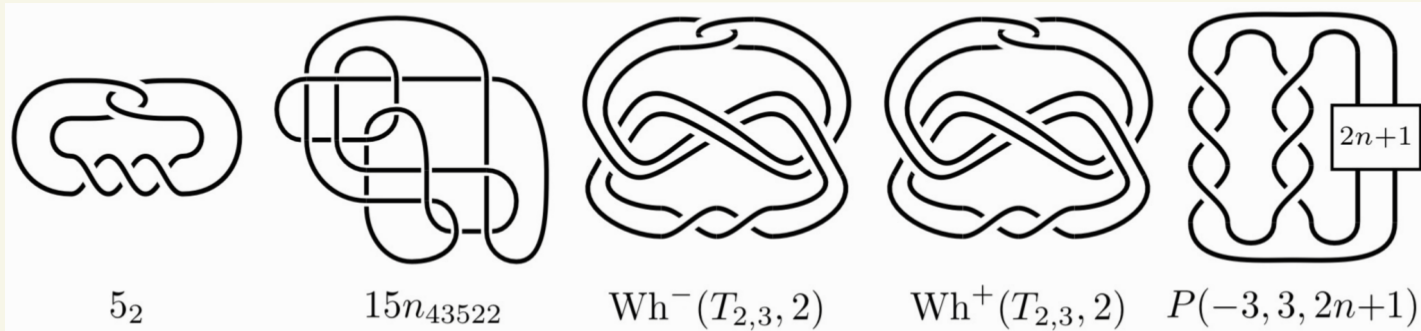
Def. A knot $K \subset S^3$ is nearly fibered if

$$\dim \widehat{\text{HFK}}(K, g(K)) = 2.$$

Main Thm (Baldwin-S '22)

A genus-1 knot is nearly fibered iff it is one of:



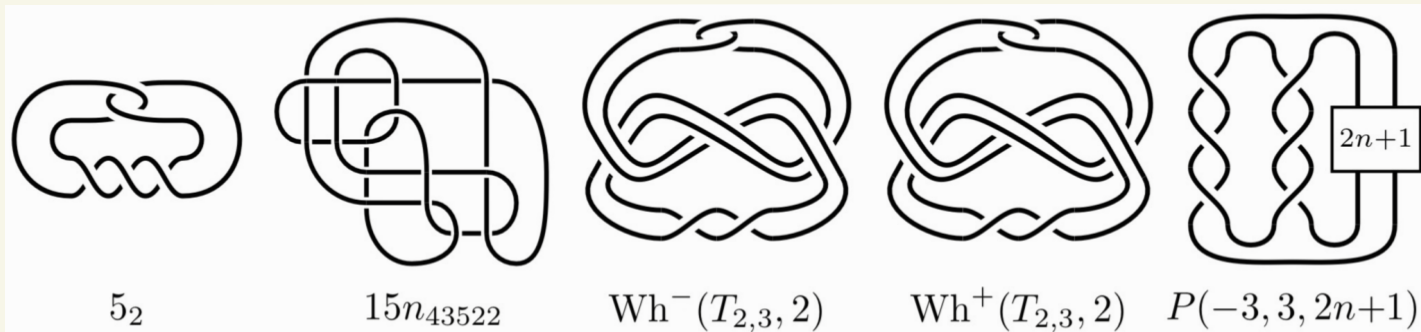


Cor. \widehat{HFK} detects 5_2 and $Wh^+(T_{2,3}, 2)$.

\widehat{HFK} detects membership in $\{15n_{43522}, Wh^-(T_{2,3}, 2)\}$

and in $\{P(-3, 3, 2n+1) \mid n \in \mathbb{Z}\}$.

K	$\widehat{HFK}(K, 1; \mathbb{Q})$	$\widehat{HFK}(K, 0; \mathbb{Q})$	$\widehat{HFK}(K, -1; \mathbb{Q})$
5_2	$\mathbb{Q}_{(2)}^2$	$\mathbb{Q}_{(1)}^3$	$\mathbb{Q}_{(0)}^2$
$15n_{43522}$	$\mathbb{Q}_{(0)}^2$	$\mathbb{Q}_{(-1)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-2)}^2$
$Wh^-(T_{2,3}, 2)$	$\mathbb{Q}_{(0)}^2$	$\mathbb{Q}_{(-1)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-2)}^2$
$P(-3, 3, 2n + 1)$	$\mathbb{Q}_{(1)}^2$	$\mathbb{Q}_{(0)}^5$	$\mathbb{Q}_{(-1)}^2$
$Wh^+(T_{2,3}, 2)$	$\mathbb{Q}_{(-1)}^2$	$\mathbb{Q}_{(-2)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-3)}^2$



Cor. $\bar{K}h$ detects S_2 .

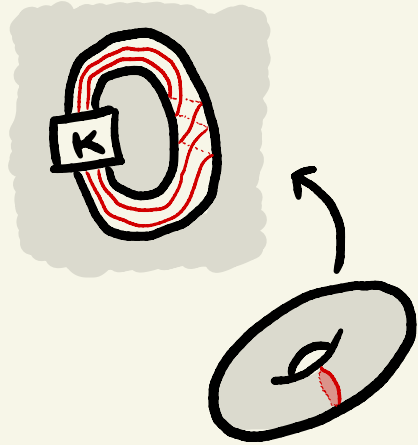
$\bar{K}h(K) + \Delta_K(t)$ detect each $P(-3, 3, 2n+1)$.

HomFLY homology detects each $P(-3, 3, 2n+1)$.

K	$\widehat{HFK}(K, 1; \mathbb{Q})$	$\widehat{HFK}(K, 0; \mathbb{Q})$	$\widehat{HFK}(K, -1; \mathbb{Q})$
5_2	$\mathbb{Q}_{(2)}^2$	$\mathbb{Q}_{(1)}^3$	$\mathbb{Q}_{(0)}^2$
$15n_{43522}$	$\mathbb{Q}_{(0)}^2$	$\mathbb{Q}_{(-1)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-2)}^2$
$Wh^-(T_{2,3}, 2)$	$\mathbb{Q}_{(0)}^2$	$\mathbb{Q}_{(-1)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-2)}^2$
$P(-3, 3, 2n+1)$	$\mathbb{Q}_{(1)}^2$	$\mathbb{Q}_{(0)}^5$	$\mathbb{Q}_{(-1)}^2$
$Wh^+(T_{2,3}, 2)$	$\mathbb{Q}_{(-1)}^2$	$\mathbb{Q}_{(-2)}^4 \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}_{(-3)}^2$

Applications to Dehn surgery

Def. A slope $r \in \mathbb{Q}$ is **characterizing** for $K \subset S^3$ if $S_r^3(J) \cong S_r^3(K)$ implies $J \cong K$.



Ex. All slopes are characterizing for

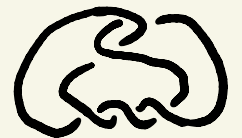
- the unknot (Kronheimer-Mrowka-Ozsváth-Szabó '03)
- the trefoils and figure eight. (Ozsváth-Szabó '06)

Ex. For many knots, slopes of large denominator are characterizing.
(Ni-Zhang; Lackenby, McCoy, Sorya, Wakelin)

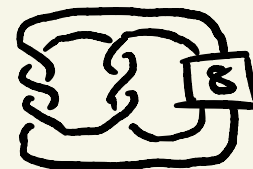
Thm (Baldwin-S. '22)

arXiv: 2209.09805

Every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is characterizing for S_2 .



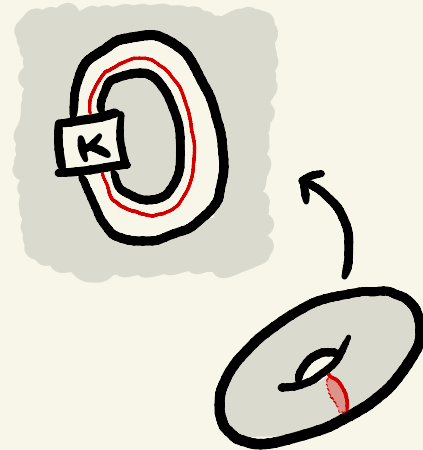
Note $S_1^3(S_2) \cong S_1^3(P(-3,3,8))$.



$\mathbb{Q} = \mathbb{Z}$

Applications to Dehn surgery

Def. A slope $r \in \mathbb{Q}$ is **characterizing** for $K \subset S^3$ if $S_r^3(J) \cong S_r^3(K)$ implies $J \cong K$.



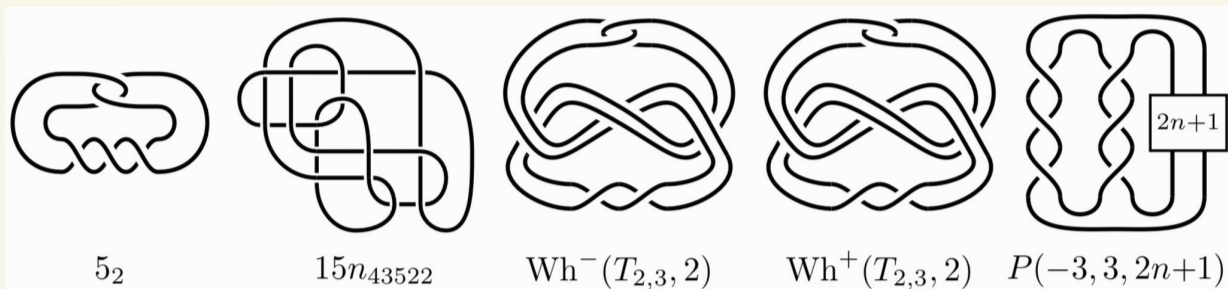
Thm 0-surgery characterizes the unknot, trefoils, figure eight.

(Gabai '87)

Thm (Baldwin-S. '22)

arXiv: 2211.04280

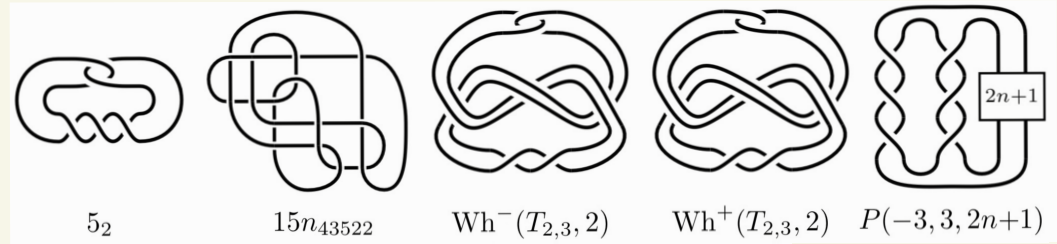
0-surgery characterizes each of the following:



Thm (Cheetham-West '22)

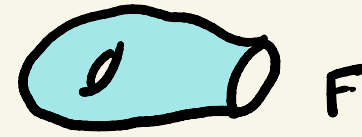
$\pi_1(S^3 \setminus K)$ is determined by its finite quotients for $K = 5_2, 15n_{43522}, P(-3, 3, 2n+1)$.

Proof outline, part 1



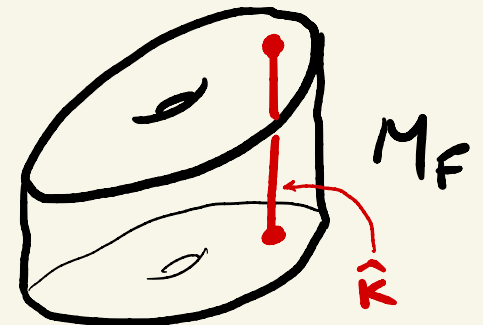
Let K have genus 1, $\dim \widehat{HFK}(K, 1) = 2$.

$F =$ genus-1 Seifert surface



\leadsto incompressible torus $\widehat{F} \subset S^3_0(K)$.

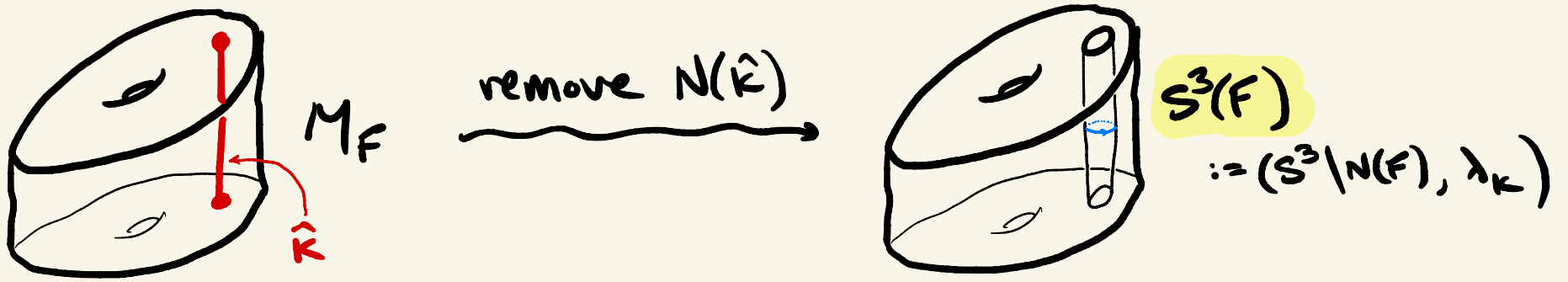
Cut open: $M_F := S^3_0(K) \setminus N(\widehat{F})$.



Thm (Cantwell-Goulon) if $K = S_2$ or $P(-3, 3, 2n+1)$

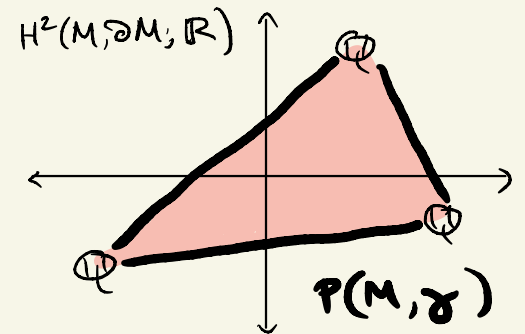
then M_F is the complement of a $(2, 4)$ torus link.

Idea: use $\dim \widehat{HFK}(K, 1) = 2$ to determine M_F .



Thm (Juhász) $SFH(S^3(F)) \cong \widehat{HFK}(K, 1) \cong \mathbb{Q}^2$.

Idea: SFH detects the sutured Thurston norm via the width of the sutured Flier polytope.

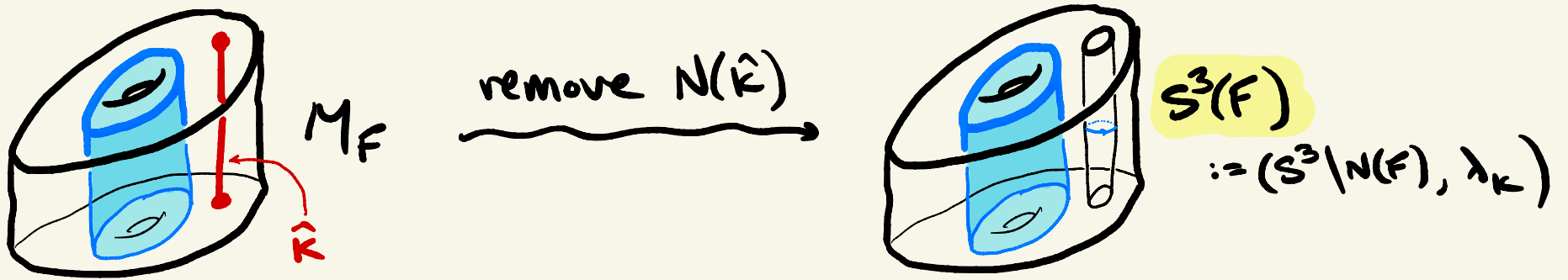


So width 0 in some direction \rightsquigarrow annuli in (M, γ) :

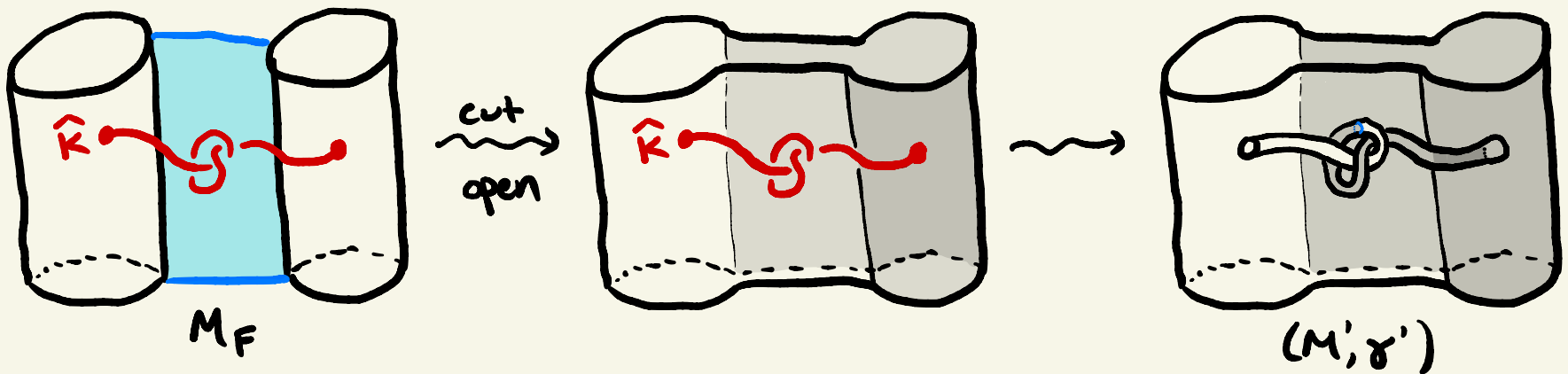
Thm (Juhász) $H_2(M) = 0$, (M, γ) taut, horizontally prime, reduced

$$\Rightarrow \underbrace{\dim SFH(M, \gamma)}_{=2 \text{ for } S^3(F)} > \underbrace{b_1(M)}_{=2 \text{ for } S^3(F)} = \frac{1}{2} b_1(\partial M).$$

Cor.: we find an essential annulus in $S^3(F)$, hence in M_F .



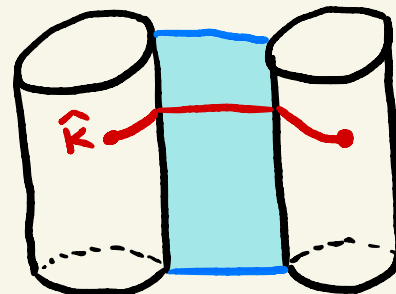
Note: the **annulus** identifies M_F as the complement of a cable of JCY .



Then $SFH(M', \gamma') \cong \mathbb{Q}^2 \Rightarrow$ essential annulus in (M', γ') !

One piece of complement has $\dim SFH = 1 \Rightarrow$ it's a product.

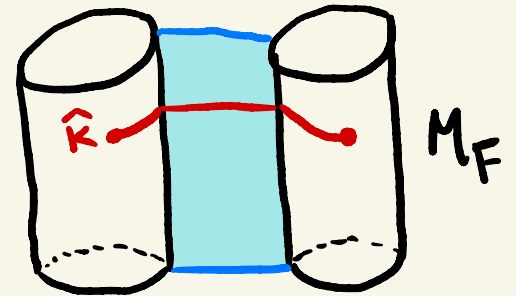
Upshot: \hat{K} can be isotoped into the **cabling annulus**.



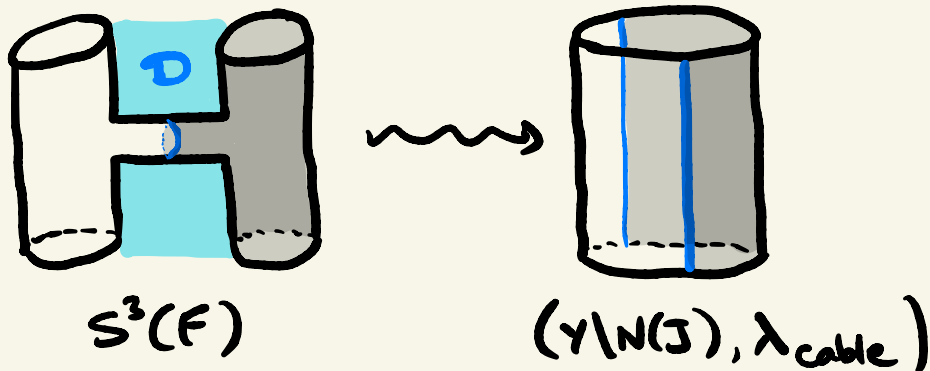
So far: $M_F := S^3_0(K) \setminus N(\hat{K})$

is the complement of a cable of $J \subset Y$,

and \hat{K} lies in the cabling annulus.



In $S^3(F)$, remove $D = \text{annulus} \setminus N(\hat{K})$



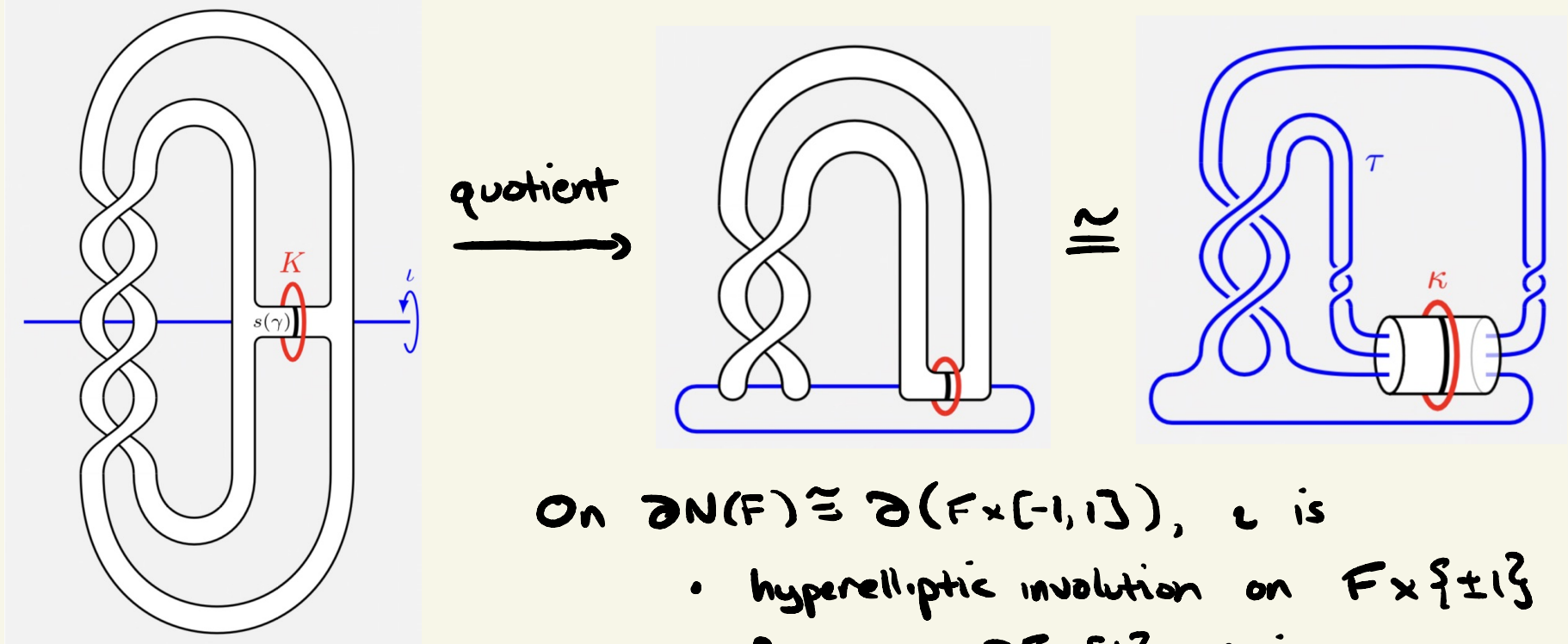
- so
- $Y \setminus N(J)$ embeds in S^3 ;
 - $\text{SFH}(Y \setminus N(J), \lambda_{\text{cable}}) \cong \text{SFH}(S^3(F)) \cong \mathbb{Q}^2$.

Thm If K is nearly fibered of genus 1, then M_F is the complement of the $(2,4)$ -cable of \bigcirc or $\left(\begin{array}{c} \text{?} \\ \text{?} \end{array}\right)$.

Thm If K is nearly fibered of genus 1, then M_F is the complement of the $(2,4)$ -cable of \bigcirc or $\left(\begin{smallmatrix} \bigcirc \\ \bigcirc \end{smallmatrix}\right)$.

Proof outline, part 2: classify all such K .

$S^3(F)$ admits an involution ι :



On $\partial N(F) \cong \partial(F \times [-1, 1])$, ι is

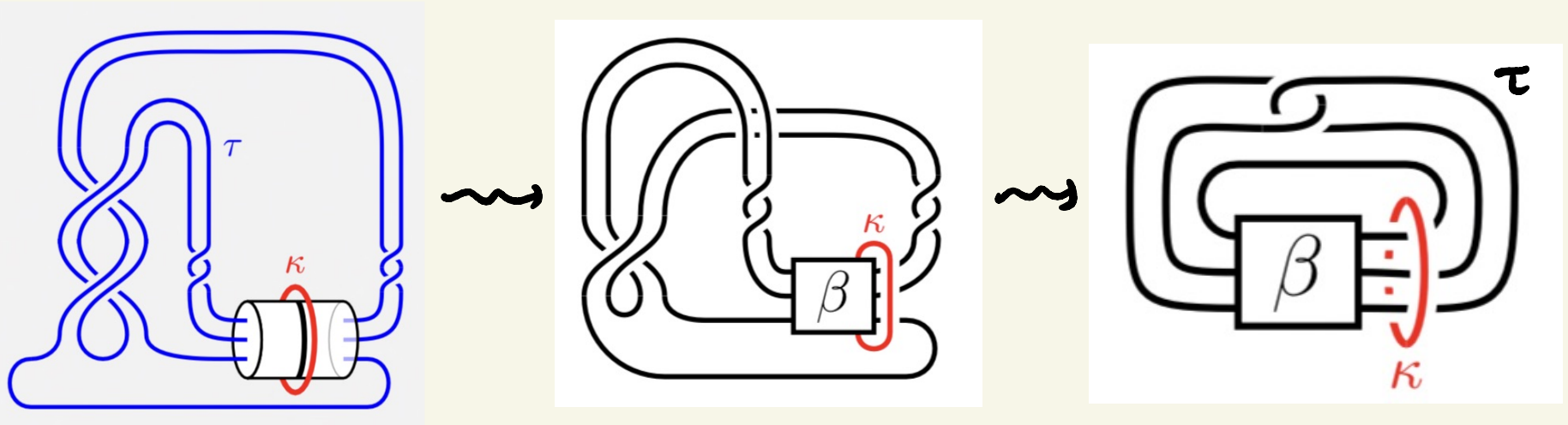
- hyperelliptic involution on $F \times \{\pm 1\}$
- fixes each $\partial F \times \{t\}$ setwise.

So ι extends over $F \times [-1, 1]$, with quotient $D^2 \times [-1, 1]$.

In other words: $S^3 \cong (S^3 \setminus N(F)) \cup N(F)$

is the branched double cover of $\tau \cup \beta$

for some 3-braid $\beta \subset D^2 \times I$.

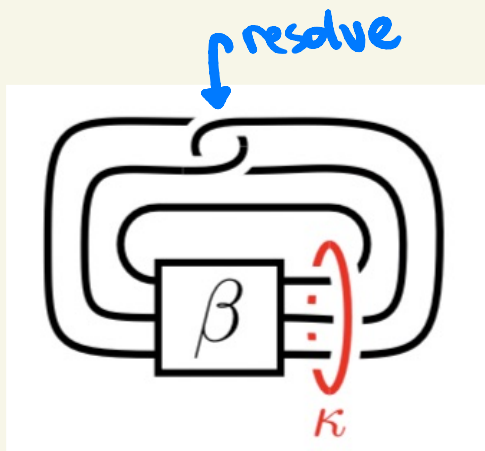


We need to

- find all β such that $\tau \cup \beta$ is unknotted;

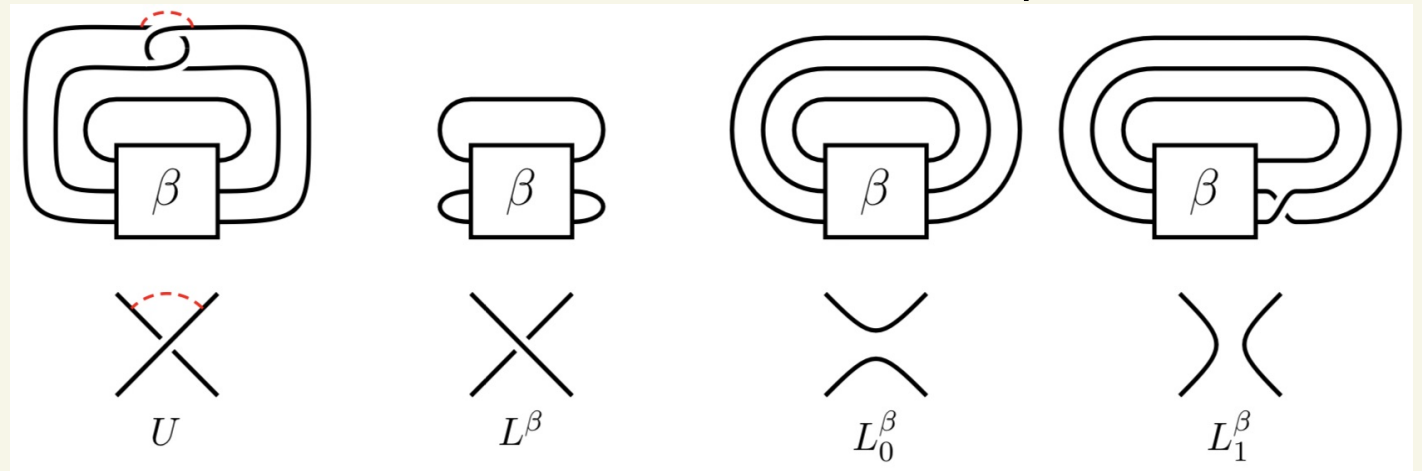
- lift K to $\Sigma_2(\tau \cup \beta) \cong S^3$

to find the corresponding knot K .



How to find all β such that $\tau \cup \beta = 0$?

Resolve a **crossing** in several ways:



Branched double covers:

$$S^3$$

$$S^3_{\frac{2n+1}{2}}(\gamma)$$

$$S^3_n(\gamma)$$

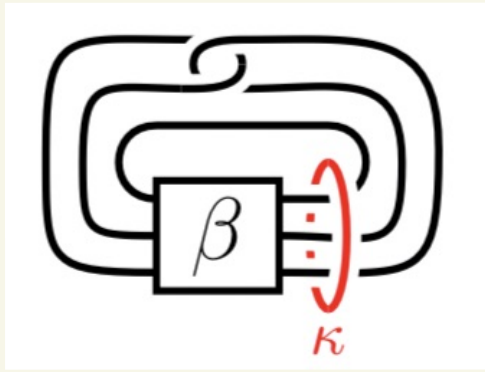
$$S^3_{n+1}(\gamma)$$

Then L^β is **2-bridge** $\rightsquigarrow S^3_{\frac{2n+1}{2}}(\gamma)$ is a **lens space**.

Cyclic surgery thm $\Rightarrow \gamma$ is a torus knot T_{ab} , $\frac{2n+1}{2} = \frac{2ab \pm 1}{2}$;
or unknot, $n \in \mathbb{Z}$.

Now $\Sigma(L^\beta) \cong S^3_n(\gamma)$ is a **lens space** or $L(a,b) \neq L(b,a)$, so

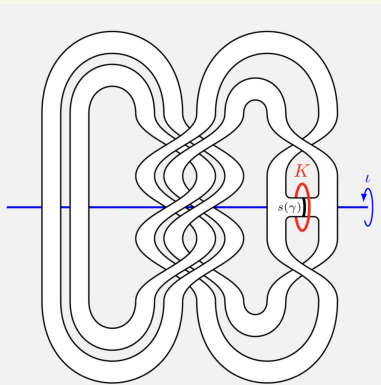
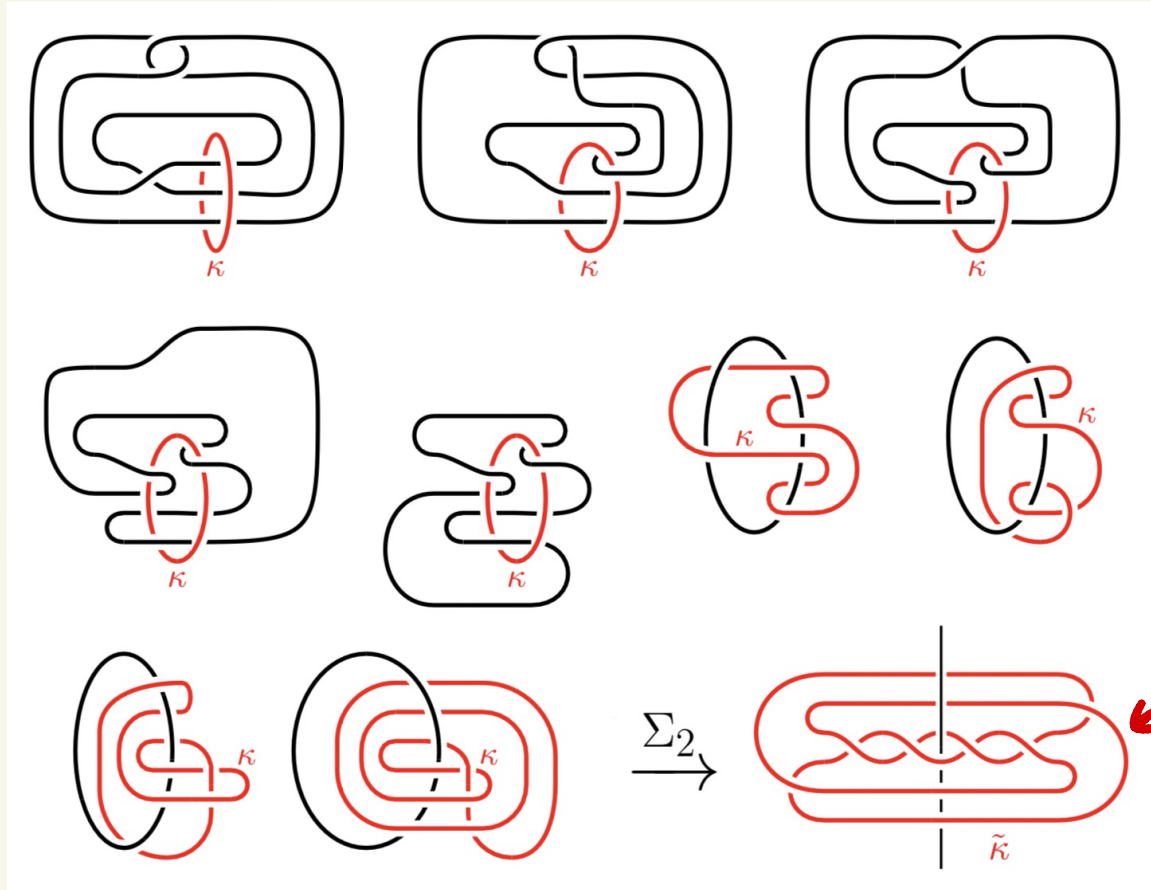
L_0^β is **2-bridge** (or sum of 2-bridge) with **braid index ≤ 3** . (\Rightarrow Murasugi)



This leads to a complete list of β .

To recover K , e.g. when $\beta = \underline{\underline{\infty}}$:

β	K
$\underline{\underline{\infty}}$	S_2
$\underline{\underline{\infty}} \text{ with a loop}$	$15N_{43522}$
$\underline{\underline{\infty}} \text{ with a square loop}$	$P(-3,3,2n+1)$



Similar analysis when $M_F \cong S^3 \setminus N(C_{2,4}(T_{2,3}))$ gives

$$K = \text{Wh}^+(T_{23}, 2) \text{ or } \text{Wh}^-(T_{23}, 2). \quad \blacksquare$$