The cosmetic surgery conjecture for pretzel knots

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Surgeries

Suppose that $Y = Y^3$ is a closed, oriented three-manifold.

- A framed knot $K \subset Y$ together with a surgery coefficient $r \in \mathbb{Q} \cup \{\infty\}$ defines a new three-manifold $Y_r(K) = (Y \setminus \nu(K)) \cup_\varphi S^1 \times D^2$ — this is *Dehn surgery*.
- The notion naturally extends to framed links.

Theorem (Lickorish, Wallace)

For any Y there is a link $(L, \Lambda) \subset S^3$ and $R = (r_1, \ldots, r_n)$ so that $S^3_R(L)$ is orientation preserving diffeomorphic to Y.

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Surgeries

- In S^3 we need links, knots are not sufficient (since $H_1(S^3_r(K);\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ if $r = \frac{p}{q}$). So for example T^3 is not surgery along a knot.
- The link is not unique — different choices can be connected by Kirby moves. E.g. 5-surgery along the RHT is the same as (-5)-surgery along the unknot (giving the lens space $L(5,1)$)
- Sometimes the knot and the coefficient is determined by the three-manifold (e.g. the Poincaré homology sphere can be only surgered along a single knot, the trefoil).

The (purely) cosmetic surgery conjecture

"If we fix the knot, then the result determines the surgery coefficient."

Conjecture (Gordon, 1990)

Suppose that $K \subset S^3$ is a non-trivial knot. Suppose that for $r, s \in \mathbb{Q}$ we have that $S^3_r(K)$ and $S^3_s(K)$ are orientation preserving diffeomorphic three-manifolds. Then $r = s$.

If we drop 'orientation preserving', the situation is very different: we always have that $S^3_r(K)$ and $S^3_{-r}(m(K))$ for the mirror $m(K)$ are (orientation-reversing) diffeomorphic. Hence if K is amphichiral (i.e. K and $m(K)$ are isotopic, like for the Figure-8 knot, or for any knot of the form $K\#m(K)$), r and $-r$ give the same three-manifold.

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Results

Theorem (Ni-Wu)

Suppose that for a nontrivial knot K we have that $S^3_r(K) \cong S^3_s(K)$. Then

- $r = -s$.
- *if* $r = \frac{p}{q}$ *with* $(p, q) = 1$, *then* $q^2 \cong -1 \pmod{p}$.
- $\tau(K) = 0$.

Theorem (Wang)

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Results

Conjecture holds for:

- torus knots
- nontrivial connected sums, and cable knots (R. Tao)
- 3-braid knots (Varvarezos)
- two-bridge knots and alternating fibered knots (Ichihara-Jong-Mattman-Saito)
- Conway and Kinoshita-Terasaka knot families (Bohnke-Gillis-Liu-Xue)
- knots up to 16 crossings (Hanselman)

Main technical result

Theorem (Hanselman)

Suppose that the nontrivial knot K admits $r \neq s$ with $S^3_r(K) \cong S^3_s(K)$. Then, either

- $g(K) = 2$ *and* $\{r, s\} = \{\pm 2\}$, *or*
- $\{r, s\} = \{\pm \frac{1}{q}\}$ *for some* $q \in \mathbb{N}$ *satisfying*

$$q \leq \frac{th(K) + 2g(K)}{2g(K)(g(K) - 1)},$$

where $th(K)$ is the knot Floer 'thickness' of K.

In particular, if $g(K) > 2$ and $th(K) \leq 5$, then K satisfies the purely cosmetic surgery conjecture (PCSC).

Idea: compute $\widehat{\mathrm{HF}}(S^3_{\pm r}(K))$ from knot Floer homology and compare.

Pretzel knots

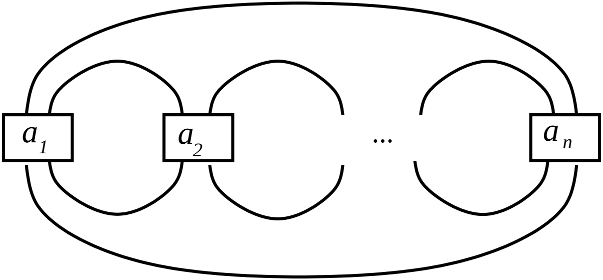

Figure: **The pretzel knot $P(a_1, \ldots, a_n)$.** The box with a_i in it means $|a_i|$ half twists (to the right if $a_i > 0$ and to the left if $a_i < 0$). We have a knot if a_1 is even and all others are odd, or all are odd and n is odd.

Pretzel knots

Simple observations:

- a_i's can be cyclically permuted
- if $a_i = \pm 1$ then it can be commuted with anything
- $a_i = 1$ and $a_{i+1} = -1$ cancel (by Reidemeister 2)

So assume that we do not have both 1 and -1. Also can assume that $a_1 \neq 0$ in case it is even (then P is just connected sum of torus knots).

Thickness of knots

Suppose that $V = \sum_{a \in \mathbb{R}} V_a$ is a finite dimensional graded vector space, V_a is the subspace of homogeneous elements of grading a.

Definition

The thickness $th(V)$ is defined as the largest possible difference of degrees, i.e.

$$th(V) = \max\{a \mid V_a \neq 0\} - \min\{a \mid V_a \neq 0\}.$$

Knot Floer homology: associates a bigraded vector space $\widehat{\mathrm{HFK}}(K) = \sum_{M,A} \widehat{\mathrm{HFK}}_M(K, A)$ (over the field $\mathbb{F} = \{0, 1\}$) to a knot, in such a way that the Poincaré polynomial $P_K(s, t) = \sum_{M,A} \dim \widehat{\mathrm{HFK}}_M(K, A) \cdot s^M t^A$ satsifies

- $P_K(-1, t) = \Delta_K(t)$, the Alexander polynomial of K
- For the polynomial $G_K(t) = P_K(1, t)$ the degree (highest power with nonzero coefficient) is equal to the genus $g(K)$
- leading coefficient is ± 1 if and only if K is fibered.
- If K is alternating, then $P_K(s, t)$ is determined by $\Delta_K(t)$ (and the signature $\sigma(K)$) of K.

Collapse the two gradings to $\delta = A - M$; the thickness of the resulting graded vector space $\widehat{\mathrm{HFK}}^{\delta}(K)$ is, by definition the *thickness* $th(K)$ of K.

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Thickness

Not hard: if K is alternating, then $th(K) = 0$ (so called *thin* knot). How to measure non-alternating?

Observation: consider an alternating diagram D; then any domain (connected component of the complement) has the property that any edge on the boundary connects an over- and an under-crossing

Definition

Suppose that D is a diagram of a knot $K \subset S^3$. A domain d is **good** *if every edge on its boundary connects an over- and an under-crossing; otherwise d is* **bad**. *Let $B(D)$ denote the number of bad domains.*

The knot invariant

$$\beta(K) = \min\{B(D) \mid D \text{ is a diagram of } K\}$$

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A bound on β

Suppose that K is non-alternating (that is, $\beta(K) > 0$). Then

Theorem (S-Szabó)

$th(K) \leq \frac{1}{2}\beta(K) - 1.$

This gives a convenient way to bound $th(K)$. For example:

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If K is a pretzel knot or a Montesinos knot, then $th(K) \leq 1$.

Indeed, in these cases it is easy to isotope the standard diagram of K so that the result has at most four bad domains, giving the claimed bound.

Combining with Zibrowius' theorem (stating that $\widehat{\mathrm{HFK}}^{\delta}(K)$ is mutation invariant) one can get bounds in other cases.

Can view the inequality as a bound on $\beta(K)$ provided by knot Floer homology (through the thickness $th(K)$).

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Back to pretzel knots

If $g(P) \neq 2$, then Hanselman's corollary shows that PCSC holds.
If a_1 is even, then there are a few families of knots with $g(K) = 2$, and they can be handled as follows.

Theorem (Boyer-Lines)

Suppose that the knot $K \subset S^3$ has Alexander-Conway polynomial $\nabla_K(z) = \sum_{i=0}^{d} a_{2i}(K) z^{2i}$ with $a_2(K) \neq 0$. Then K satisfies the PCSC.

Recall that ∇_K is defined by the skein relation $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$ and $\nabla_U = 1$; it satisfies the identity $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$. Indeed, $\Delta''_K(a) = 2a_2(K)$ and the proof of the above theorem follows from the fact that in case $\Delta''_K(1) \neq 0$ then the Casson-Walker invariants λ of $S^3_r(K)$ and of $S^3_{-r}(K)$ are different — which follows from a surgery formula for λ in terms of $\Delta''_K(1)$ and the surgey coefficient.

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If all a_i are odd in $P = P(a_1, \ldots, a_n)$ (hence n is odd), then by a result of Gabai the genus $g(P)$ is equal to $\frac{1}{2}(n-1)$, and hence we need to consider only $n = 5$.
In this case $a_2(P)$ can be determined: if $a_i = 2k_i + 1$ and s_i is the i^{th} symmetric polynomial of $\{k_i\}_{i=1}^5$, then

$$a_2(P) = s_2 + 2s_1 + 3$$

Here we need a further invariant: $\lambda_2(Y)$ of a rational homology sphere Y is a generalization of the Casson-Walker invariant $\lambda = \lambda_1$. It also admits a surgery formula, involving the knot invariants $a_2(K)$ and

$$w_3(K) = \frac{1}{72}V_K'''(1) + \frac{1}{24}V_K''(1),$$

where V_K is the (normalized) Jones polynomial of K (defined by the skein relation $q^{-1}V_{K_+}(t) - qV_{K_-}(t) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{K_0}(q)$).

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Now for five-strand pretzel knots we showed that

$$w_3(P) = \frac{1}{2}(5 + 3s_1 + s_1^2 + s_2 + \frac{1}{2}(s_3 + s_1 s_2))$$

A surprisingly simple argument shows that

Proposition

The quantities $a_2(P)$ and $w_3(P)$ cannot be zero at the same time.

Idea: If both hold then $s_2 = -2s_1 - 3$ and $s_3 = s_1 + 2$, the first is a degree-2, the second is a degree-3 equation, so we do not expect them to be satisfied at the same time.

Suppose that D is a non-alternating diagram; we want to show that $th(K) \leq \frac{1}{2}B(D) - 1$.

Recall that $th(K) = th(\widehat{\mathrm{HFK}}^{\delta}(K))$. By its definition, $\widehat{\mathrm{HFK}}^{\delta}(K)$ is the homology of a chain complex, a model of which can be given as follows.

Kauffman states

Put a marking on the diagram D, i.e. mark an edge by a point p (distinct from the crossings). Take the Kauffman states of (D, p), i.e. bijections κ between the set $Cr(D)$ of crossings and the set $Dom(D, p)$ of those domains which do not contain p on their boundary (by the Euler theorem the two sets have the same cardinality) which satisfy that the domain associated to a crossing is one of the four meeting at the crossing.

Kauffman states

Equip each Kauffman state by the gradings A, M (and $\delta = A - M$) as instructed by

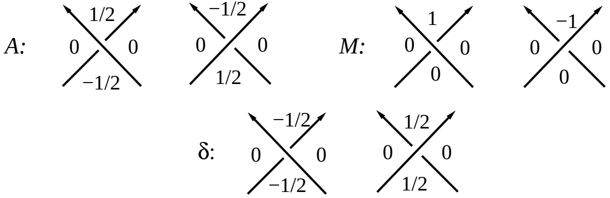

Figure: The local contributions to $M(\kappa), A(\kappa)$ and $\delta(\kappa)$.

Kauffman states

Consider now the vector space $C_{D,p}$ generated by the Kauffman states of (D,p), equipped by the bigrading (M,A).

Theorem (Ozsváth-Szabó)

There is an endomorphism $\partial\colon C_{D,p} \to C_{D,p}$ *of bidegree* $(-1,0)$ *with* $\partial^2 = 0$ *such that*

$$H(C_{D,p},\partial) = \widehat{\mathrm{HFK}}(K).$$

Since $th(H(C_{D,p}) \leq th(C_{D,p})$ (now viewed them with the δ-grading), we need to show that $th(C_{d,p}) \leq \frac{1}{2}B(D) - 1$.

The proof of the inequality

The δ-grading at a crossing is either 0 or $-\frac{1}{2}$ if the crossing is positive, and either 0 or $\frac{1}{2}$ if the crossing is negative. So we can express the δ-grading of a Kauffman state κ as the sum

$$-\frac{1}{4}\mathrm{wr}(D) + \sum_{c \in Cr} f(\kappa(c)),$$

where wr is the writhe of the diagram, and f is a function on the Kauffman corners, which is either $\frac{1}{4}$ or $-\frac{1}{4}$ (depending on the chosen quadrant at the crossing c).

For a good domain each corner in the domain gives the same f-value, hence for different Kauffman states the contributions from this particular domain are the same. For a bad domain the maximal difference for two Kauffman states on a bad domain is $\frac{1}{2}$.

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The proof of the inequality

By assumption D is not alternating, hence there is a bad domain, with an edge showing that it is bad. Place the marking p to this edge. Since this edge guarantees that the other domain it is on the boundary of, is also bad, while these two bad domains do not get Kauffman corners, we get that $th(C_{D,p})$ is bounded by $\frac{1}{2}(B(D)-2)=\frac{1}{2}B(D)-1$, concluding the proof.

Thank you!