p-dg structures in link homology

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I) p-dg algebras

- For a prime p, and a 3-man. M^3, the WRT-invariant $Z(M^3) \in \mathcal{O}_p$ where
$$\mathcal{O}_p = \frac{\mathbb{Z}[q,q^{-1}]}{(\Phi_p(q^2))} \quad \text{where } \Phi_p(q) = q^{p-1} + q^{p-2} + \cdots + 1.$$

In order to categorify this inv., we want to categorify $\mathcal{O}_p$.
Want a category $\mathcal{V}_p$ s.t. $K_0(\mathcal{V}_p) \cong \mathcal{O}_p$.

- Let $\Bbbk$ be a field of char. p.
$$H_p = \frac{\Bbbk[\partial]}{(\partial^p)} \qquad \deg \partial = 2$$

H_p has a unique simple module up to $\cong$ and grade shift.

Let L be the simple module concentrated deg 0.

Then $K_0(H_p\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$

$$[L\langle r\rangle] \mapsto q^r.$$

H_p has a filtration whose subquotients are $L, L\langle 2\rangle, \ldots, L\langle 2(p-1)\rangle$

Thus $[H_p] \longmapsto 1 + q^2 + \cdots + q^{2(p-1)} = \Phi_p(q^2)$

Want a cat. where $H_p \cong 0$.

Let $\underline{H_p\text{-gmod}}$ = stable cat of H_p-gmod

objects = same as H_p-gmod.

Morphisms = $\mathrm{Hom}_{\underline{H_p\text{-gmod}}}(M,N) = \dfrac{\mathrm{Hom}_{H_p\text{-gmod}}(M,N)}{I(M,N)}$

$$I(M,N) = M \longrightarrow \text{proj} \longrightarrow N$$

$H_p \cong 0$ in stable category.

Prop (Khovanov): $K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathcal{O}_p$

The indecomposable objects in this cat. are

$$\bullet \qquad \bullet \to \bullet \qquad \bullet \to \bullet \to \bullet \; \cdots\cdots \qquad \cancel{\bullet \to \bullet \cdots \to \bullet}$$

p-complexes

• WRT inv. could be defined using $u_q(\mathfrak{sl}_2)$

an alg over $\mathcal{O}_p$.

Want a procedure for categorifying $\mathcal{O}_p$-modules:

Let A be a $\mathbb{Z}$-graded alg. / $\Bbbk$.
equipped with a der. $\partial : A \to A$ of
degree 2 s.t. $\partial^p = 0$.
(This is called a p-dg alg)

Let $N \in A$-pdgmod. Let $M \in H_p$-gmod.

Then $M \otimes N \in A$-pdgmod where

$a.(m \otimes n) = m \otimes an$ and

$\partial(m \otimes n) = \partial m \otimes n + m \otimes \partial n$.

So there's a funct

$H_p\text{-gmod} \times A\text{-pdgmod} \longrightarrow A\text{-pdgmod}$.

So $K_0(A\text{-pdgmod})$ is a module over

$K_0(H_p\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$

- Let $f : N_1 \longrightarrow N_2$ in A-pdgmod

f is said to be null-homotopic if

$$f = \sum_{i=0}^{p-1} \partial^i H \partial^{p-1-i} \quad \text{where } H : N_1 \to N_2$$

Let $A\text{-}\underline{\text{pdgmod}}$ = homotopy cat.

(quotient by null-homotopic maps)

$H_p\text{-}\underline{gmod} \times A\text{-}p\underline{dgmod} \longrightarrow A\text{-}p\underline{dgmod}$

So $K_0(A\text{-}p\underline{dgmod})$ is a module over
$K_0(H_p\text{-}\underline{gmod}) \cong \mathcal{O}_p$

- Let $f: N_1 \to N_2$ in $A\text{-}p\underline{dgmod}$
  f is said to be a qis if
  $\mathrm{Res}(f)$ in $H_p\text{-}\underline{gmod}$ is an $\cong$.
  $\mathcal{D}(A,\partial) \simeq$ derived cat.

- Want to cat. Jones poly at a root of unity

1) Khovanov homology
   Problem: Alg. used here are too small for non-trivial p-dg structures

2) Webster homology
   a) [Khovanov-Qi-S] cat. $\underbrace{V_r \otimes V_1^{\otimes n}}_{u_q(sl_2)}$
   b) [Qi-S] cat. Burau rep of braid gp

at a root of unity

3) Construction of $sl(-2)$ homology:

Cautis

Robert-Wagner

Queffelec-Rose-Sartori

II) Homflypt homology

- Foundational structure: Khovanov's Homflypt cat.

Let $R = \mathbb{K}[x_1, \dots, x_n]$

$R^i = R^{S_1 \times S_1 \times \dots \times S_2 \times S_1 \times \dots S_1}$

Rouquier cat. a braid gp action:

Let $B_i = R \otimes_{R^i} R$ (Soergel bimodule)

There are bimodule homom.

$br: B_i \to R$ $\quad 1 \otimes 1 \mapsto 1$

$rb: R \to B_i$ $\quad 1 \mapsto x_{i+1} \otimes 1 - 1 \otimes x_i$

Let $T_i := B_i \to R$ $\qquad$ $T_i^{-1} := R \to B_i$

$\sigma_i = | \cdots \times \cdots |$ $\qquad$ $\sigma_i^{-1} = | \cdots \times \cdots |$

Any braid β could be written as $\sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_n}^{\varepsilon_n}$.

To β, we have a complex of bimodules:

$T_\beta := T_{i_1}^{\varepsilon_1} \cdots T_{i_n}^{\varepsilon_n}$

Thm (Rouquier): The functors T_i, T_i^{-1} satisfy braid gp rel. in $K((R,R)\text{-gmod})$

Sketch of (R2): $T_i T_i^{-1}$

$$= (B_i \to R) \otimes (R \to B_i)$$

$$= B_i \longrightarrow \begin{matrix} R \\ B_i \otimes B_i \end{matrix} \longrightarrow B_i$$

$$\cong \cancel{B_i} \longrightarrow \begin{matrix} R \\ \cancel{B_i} \\ \cancel{B_i} \end{matrix} \longrightarrow \cancel{B_i}$$

$$\cong R \quad \text{(identity functor)}$$

· Khovanov extended this to a link inv. (based on earlier work with Rozansky)

$T_{\mathcal{B}} = \cdots \to C_i \to C_{i-1} \to \cdots$

Apply $HH_\bullet$ to each term to get a complex of v.s's:

$HH_\bullet(T_{\mathcal{B}}) = \cdots \to HH_\bullet(C_i) \to HH_\bullet(C_{i-1}) \to \cdots$

where $HH_i(M) = H_i(M \otimes_{(R,R)} R)$

Complex above has 3 gradings

1) Homological grading (t-grading)
2) grading from R (q-grading)
3) Hoch. grading (a-grading)

Thm (Khovanov, Khovanov-Rozansky, Rouquier):
Let L be closure of braids $\mathcal{B}_1$ and $\mathcal{B}_2$.
Then $HHH(\mathcal{B}_1) \cong HHH(\mathcal{B}_2)$
so $HHH(L)$ is a link inv.

$\chi_{q,a}$ HHH(L) = Homfly-pt poly of L.

III) p-dg Jones homology

- Two constructions we'll use:

1) Khovanov-Rozansky constructs an action of (half of) Witt alg. on HHH(L)
2) Cautis constructed a diff ∂_C on HHH(L) which gives rise to a bigraded theory cat. Jones poly at generic q. (Robert-Wagner, Queffelec-Rose-Sartori)

- We use part of Witt action on HHH(L) to define a der on R

$\partial(x_i) = x_i^2$ extended by lin. and Leibniz rule.

This make R a p-dg alg.

Extend this p-dg structure to $B_i = R \otimes_{R^i} R$

$\partial(m \otimes n) = \partial m \otimes n + m \otimes \partial n$

The complex $T_i = B_i \to R$ respects ∂.

$T_i^{-1} = R \to B_i$ doesn't.

Put a new p-dg structure in B_i (call it $B_i^{-e_1}$)

$\partial(1 \otimes 1) = -e_1 (1 \otimes 1)$ $\quad e_1 = x_i + x_{i+1}$

Let $T_i^{-1} = R \to B_i^{-e_1}$ respects ∂.

- Prop: The functors T_i, T_i^{-1} satisfy braid relations in $K_{rel}((R,R)\text{-gmod})$.

$C \cong D$ in rel. homotopy cat if

$\varphi : C \to D$ (p-dg morphism of bimodules)

s.t. φ is an $\cong$ in homotopy cat.

Sketch of pf:

(R2) like before.

(R3) $T_i T_{i+1} T_i$ is a complex with 8 objects

$$\cong \begin{tikzcd} & B_{i+1} \circ B_i \arrow[r] \arrow[dr] & B_i \arrow[dr] & \\ B_{i,i+1} \arrow[ur] \arrow[dr] & & & R \\ & B_i \circ B_{i+1} \arrow[r] \arrow[ur] & B_{i+1} \arrow[ur] & \end{tikzcd}$$

sym. in $i, i+1$

$\cong T_{i+1} T_i T_{i+1}$

- There's a p-dg version of Hochschild homology

To a braid β we have a complex of p-dg bimodules

$$pHH_\cdot(T_\beta) = \cdots \to pHH_\cdot(C_i) \to pHH_\cdot(C_{i-1}) \to \cdots$$

- Notice $HH^1(R)$ has a special der

$$\partial_c = \sum_{i=1}^{n} x_i^2 \frac{\partial}{\partial x_i}$$

This acts on $pHH_\cdot(M)$ and commutes with ∂

So on complex $pHH_\cdot(T_\beta)$ there are 3 commuting diffs:

1) One from complex
2) ∂
3) ∂_C

Let ∂_T = sum of diffs

Let $pH_\cdot(\beta)$ = homology of $pHH_\cdot(T_\beta)$ w.r.t. ∂_T.

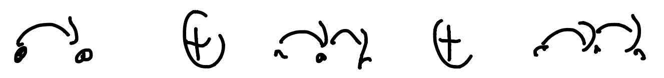
Thm (Qi-S.): Let L be the closure of braids β_1 and β_2. Then $pH_\cdot(\beta_1) \cong pH_\cdot(\beta_2)$

So $pH_\cdot(L)$ is a link inv.

$\chi_q \, pH_\cdot(L)$ = Jones poly at a root of unity

Ex: Let $T_{2,n}$ be a $(2,n)$ torus link.

If $n=2$ Hopf link



If $n=3$ trefoil.

$\overset{\frown}{\cdot \ \cdot} \oplus \overset{\frown}{\cdot \ \cdot} \oplus \overset{\frown}{\cdot \ \cdot}$ if $p \neq 3$

$\overset{\frown}{\cdot \ \cdot} \oplus \cdot \ \cdot \oplus \overset{\frown}{\cdot \ \cdot}$ if $p = 3$

$\mathbb{H}(\Sigma X)$ has a koszul res!

$$\left(\mathbb{H}(\Sigma X)^{\times} \otimes \mathbb{H}(\Sigma X)^{\times} \xrightarrow{x \otimes 1 - 1 \otimes y} \mathbb{H}(\Sigma X) \otimes \mathbb{H}(\Sigma X) \right) \dashrightarrow \mathbb{H}(\Sigma X)$$