A slicing obstruction from the 10/8 + 4 theorem

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K-OS Knot Online Seminar

Slicing Obstructions

Def. A knot $K \subseteq S^3 \cong \partial B^4$ is (smoothly) slice if it bounds a smoothly, properly embedded disk in B^4.

ex. $K \# -K$ slice

ex. ribbon knots are slice

Obstructions to sliceness:

- Fox-Milnor condition on $\Delta_K(t)$
  if K slice $\Rightarrow \Delta_K(t) = f(t) f(t^{-1})$
- Knot Floer homology invariants
  - $\tau(K)$ (Ozsváth-Szabó)
  - $\varepsilon(K)$ (Hom)
  - $\Upsilon_t(K)$ (Ozsváth-Stipsicz-Szabó)
  - $\varphi_i(K)$ (Dai-Hom-Stoffregen-T.)
  - involutive $\underline{V}_0(K)$, $\overline{V}_0(K)$ (Hendricks-Manolescu)
  - more...
- Khovanov homology invariants
  - s-invariants (Rasmussen), more...
- 2-handlebody theory ← 4-mfd obtained by attaching 4-d 2-handles to B^4

combined w/ 10/9 thms.

Theorem (Dowdd-Vafaee)
Let $K \subseteq S^3$ be smoothly slice knot and X spin 2-handlebody with $\partial X \cong S^3_0(K)$.
Then $b_2(X) = 1$ or
$$b_2(X) \geq \frac{10}{8}|\sigma(X)| + 3$$

- This bound relies on Furuta's 10/8 theorem

Theorem (T.)
Let $K \subseteq S^3$ be smoothly slice knot and X spin 2-handlebody with $\partial X \cong S^3_0(K)$.
If $(b_2(X), \sigma(X)) \neq (1,0), (3,0)$, or $(23, \pm 16)$
then $$b_2(X) \geq \frac{10}{8}|\sigma(X)| + 5.$$

- This bound relies on Hopkins-Lin-Shi-Xu 10/8 + 4 theorem.

Rem. These slice obstructions can obstruct a knot K from being slice even if $K \# K$ is slice.

e.g. 4_1

The 11/8 Conjecture (Matsumoto 1982)

Conj. (version 1) The form $2pE_8 \oplus q\begin{pmatrix}0&1\\1&0\end{pmatrix}$ ($p \geq 0$, $q \geq 0$) is spin realizable if and only if $q \geq 3p$.

[i.e. can be realized as the intersection form of a closed smooth spin 4-mfd M.

"only if" $\Updownarrow$
$$b_2(M) = 2q + 16p$$
$$\sigma(M) = 16p$$

Conj (version 2) Any closed, smooth spin 4-mfd M w. indefinite intersection form must satisfy $b_2(M) \geq \frac{11}{8}|\sigma(M)|$.

The 10/8 Theorem (Furuta)

For $p \geq 1$, the bilinear form $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is spin realizable only if $q \geq 2p+1$.

$\iff$ Any closed smooth spin 4-mfd M w. indef int. form has

$$b_2(M) \geq \frac{10}{8}|\sigma(M)| + 2.$$

The 10/8 + 4 Theorem (Hopkins, Lin, Shi, Xu)

For $p \geq 2$, the bilinear form $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is spin realizable only if $q \geq 2p+2$.

In fact,
$$q \geq \begin{cases} 2p+2 & p \equiv 1,2,5,6 \mod 8 \\ 2p+3 & p \equiv 3,4,7 \mod 8 \\ 2p+4 & p \equiv 0 \mod 8 \end{cases}$$

$\iff$ Any closed smooth spin 4-mfd M with $(b_2(M), \sigma(M)) \neq (0,0), (2,0), (22, \pm 16)$ must satisfy $b_2(M) \geq \frac{10}{8}|\sigma(M)| + 4$.

Theorem. (Donald-Vafaee)

Let $K \subset S^3$ be a smoothly slice knot and X be a spin 2-handlebody with $\partial X \cong S^3_0(K)$. Then either $b_2(X) = 1$ or

$$b_2(X) \geq \frac{10}{8} |\sigma(X)| + 3.$$

- This bound relies on Furuta's 10/8 theorem.

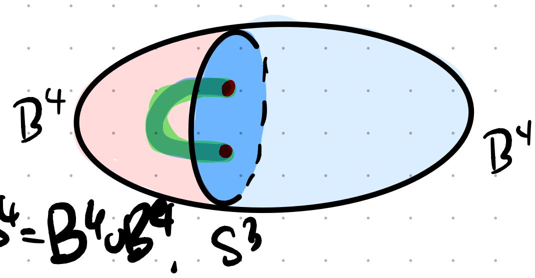
Theorem (T.)

Let $K \subset S^3$ be a smoothly slice knot and X be a spin 2-handlebody with $\partial X \cong S^3_0(K)$. If $(b_2(X), \sigma(X)) \neq (1,0), (3,0)$, or $(23, \pm 16)$, then

$$b_2(X) \geq \frac{10}{8} |\sigma(X)| + 5.$$

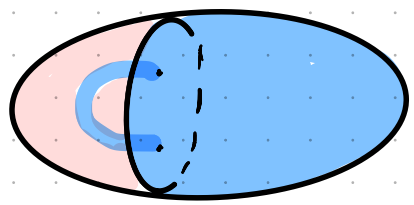
- This bound relies on HLSX 10/8+4 theorem.

Sketch of Pf.



- If K is smoothly slice, then $S^3_0(K)$ embeds smoothly in $S^4 = B^4 \cup B^4$.

- The embedding splits S^4 into 4-mfds U and V



$$S^4 = U \cup_{S^3_0(K)} -V$$

$H_*(U;\mathbb{Z}) \cong H_*(S^2 \times D^2;\mathbb{Z})$ ⇐ Mayer-Vietoris
$H_*(V;\mathbb{Z}) \cong H_*(S^1 \times D^3;\mathbb{Z})$

- Let X be a spin 2-handlebody w. bdry $\partial X \cong \partial V$.

Then $W = X \cup_{S^3_0(K)} (-V)$ closed, spin

- $\sigma(W) = \sigma(X) + \sigma(-V) = \sigma(X)$
- $\chi(W) = \chi(X) + \chi(-V) = 1 + b_2(X)$

LES (W,X) ⇒
- $b_1(W) = b_3(W) = 0 \Rightarrow \chi(W) = 2 + b_2(W)$
- $b_2(W) = b_2(X) - 1$

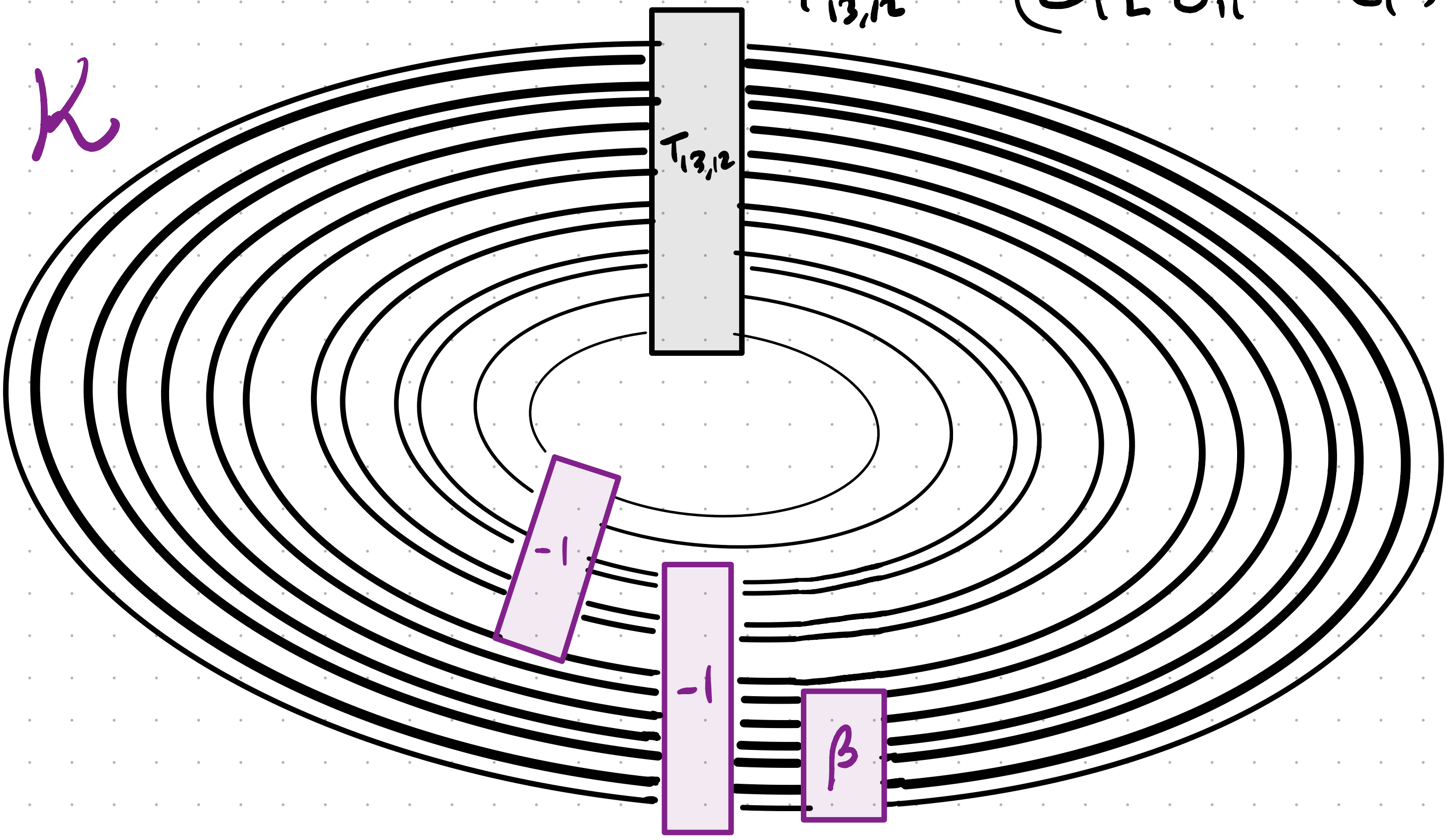
[HLSX] ⇒ $b_2(W) \geq \frac{10}{8}|\sigma(W)| + 4$ (rule out exceptions)
$10/8 + 4$

$\Rightarrow b_2(X) \geq \frac{10}{8}|\sigma(W)| + 5.$ □

Example. Let K be a generalized twisted torus knot.

$$T_{13,12} = (\sigma_{12}\sigma_{11}\cdots\sigma_1)^{12}$$

K



$$\beta = (\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_1)^{-2}(\sigma_3\sigma_4)^{-1}\sigma_5^{-2}$$

$S^3_0(K)$ bounds a spin 2-handlebody X with $b_2(X) = 13$ and sig $\sigma(X) = 8$.

$\Longrightarrow$ The knot K is not smoothly slice.

$13 < \frac{10}{8}|8| + 5$

but

$13 \geq \frac{10}{8}\cdot 8 + 3.$

To construct a spin 2-handlebody X bounding $S^3_0(K) = \partial X$:

<u>Def.</u> A framed link $L \subseteq S^3$ $L = L_1 \cup \cdots \cup L_m$, if $i=j$ $lk(L_i, L_j) = $ framing of L_i.

A <u>characteristic sublink L' of L</u>:

$lk(L_i, L_i) \equiv lk(L_i, L') \mod 2 \quad \forall i = 1, \ldots, m$

Given X a 2-handlebody w. Kirby diagram given by a framed link $L \subseteq S^3$

spin str on ∂X $\overset{1-1}{\longleftrightarrow}$ characteristic sublinks of L

spin str extends over $X \iff$ empty characteristic sublink

① $D^4 \cup \{$ 0-framed 2-handle along $K\}$ is a 2-handlebody w. bdry $S^3_0(K)$.

The 0-framed knot K reps a nonempty characteristic sublink $\mapsto$ spin str. does not extend over the 4-mfd.

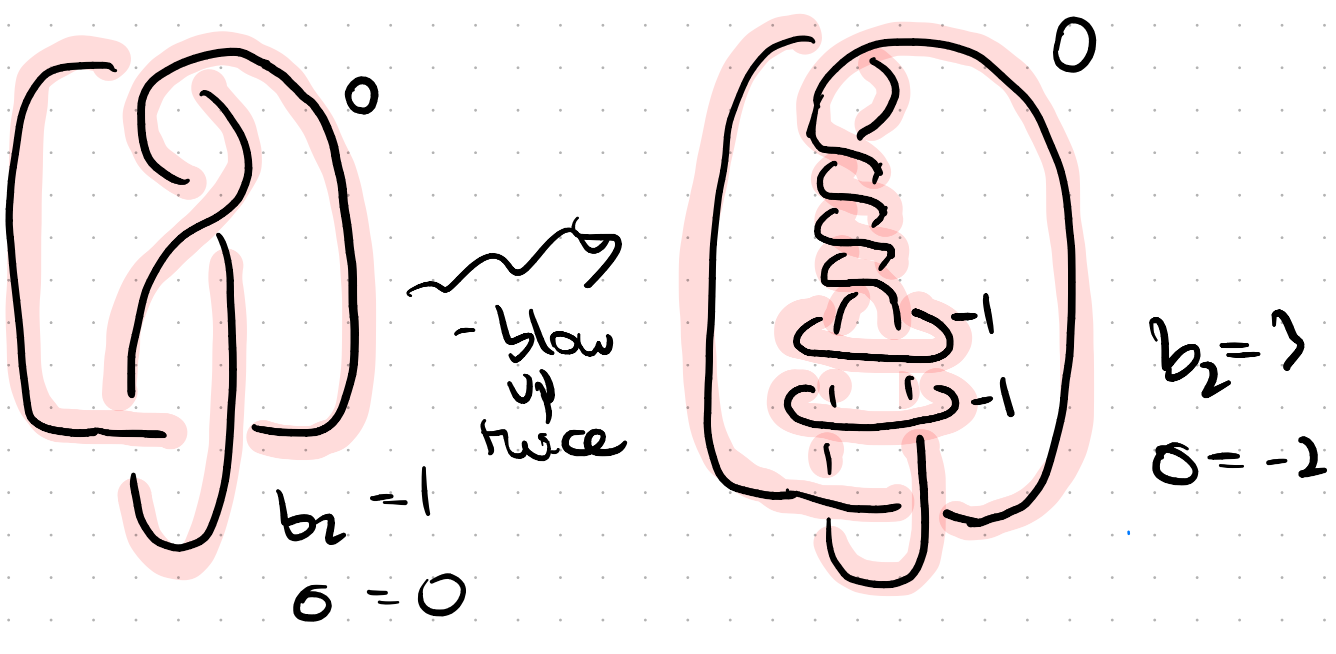
② Alter the 4-mfd w/out changing its bdry by a seq of blow ups/downs + handleslides until characteristic sublink is empty. $\Rightarrow$ spin 4-mfd X, $\partial X = S^3_0(K)$

③ Apply slicing obstruction to X.

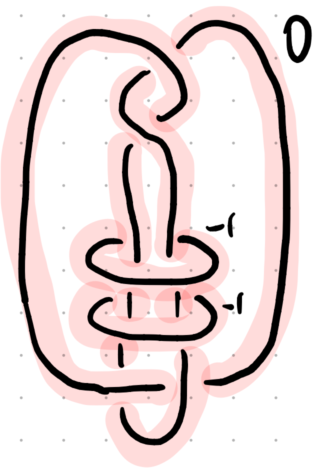
<u>Another example</u> (Donald-Vafaee)

The figure eight knot 4_1 is not slice.

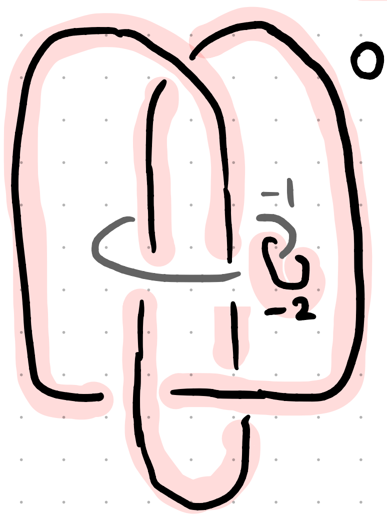
(A similar procedure works for the knot K previously drawn).



characteristic sublink



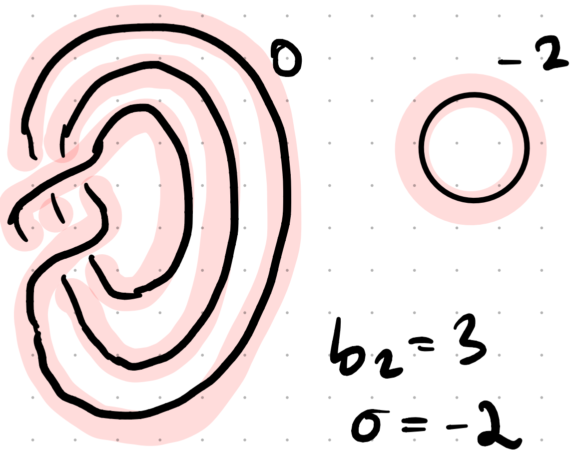
handleslide



$b_2 = 3$

$\sigma = -2$

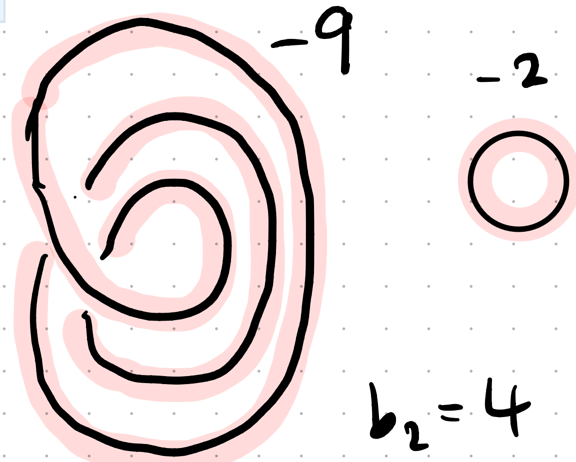
Below, only the characteristic link is drawn.



$b_2 = 3$

$\sigma = -2$

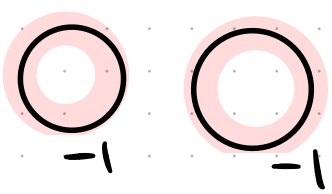
-blow up



$b_2 = 4$

$\sigma = -3$

nine +blowups



$b_2 = 13$

$\sigma = 6$

-blow down twice

$\emptyset$ char link

X spin 2-handlebody

$b_2 = 11$

$\sigma(X) = -8$

But $11 < \frac{10}{8} \cdot 8 + 3 \Rightarrow 4_1$ isnot slice.

Q: What other knots can be shown to be not slice using this obstruction?

Related work: H-sliceness obstruction

Def. X^4 smooth, closed conn. or.

If $K \subseteq S^3$ bounds a properly embedded null-homologous disk in $X \setminus \mathring{B}^4$, we say K is **H-slice in X**. **ex.** S^4, H-slice = slice.

Thm. (Marengon-Manolescu-Piccirillo)

Let $K \subseteq S^3$ H-slice in closed spin 4-mfd X and W a spin 2-handlebody w. $\partial W = S^3_0(K)$. If $b_2(W) \neq 1, 3, 23$ then

$$b_2(X) + b_2(W) \geq \frac{10}{8} |\sigma(X) - \sigma(W)| + 5.$$

$\Rightarrow$ This theorem include as a special case ($X = S^4$) the slicing obstruction bd from earlier.

Q: (MMP) $\exists$? knot top'l but not smoothly H-slice in every closed X^4?