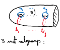
* Invariants of $\mathbb{Q}$-HS and the mod p Torelli group.
(joint work with W. Pitsch)

1) Preliminaries
2) $\mathbb{Z}/p$-HS and the mod p Torelli group.
3) Invariants of $\mathbb{Q}$-HS from 2-cocycles.
4) Application: Invariants from the abelianization

1]

$$\Sigma_{g,1} \hookrightarrow S^3 = \mathcal{H}_g \cup_{\iota_g} - \mathcal{H}_g \quad \text{Heegaard splitting.}$$

$H = H_1(\Sigma_{g,1})$
$= \langle a_i, b_i \rangle_{1 \le i \le g}$



$$\mathcal{M}_{g,1} = MCG(\Sigma_{g,1})$$

3 subgroups:

$A_{g,1}$ = elems. $\mathcal{M}_{g,1}$ that ext. to $\mathcal{H}_g$
$B_{g,1}$ = " " " " $-\mathcal{H}_g$
$AB_{g,1}$ = " " " " S^3
= $A_{g,1} \cap B_{g,1}$ Waldh.

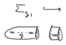
* Symplectic images

$$\mathcal{M}_{g,1} \xrightarrow{\Psi} Sp_{2g}(\mathbb{Z})$$
$$A_{g,1} \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$
$$B_{g,1} \longrightarrow \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \quad \simeq GL_g(\mathbb{Z}) \ltimes Sym_g(\mathbb{Z})$$
$$AB_{g,1} \longrightarrow \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \simeq GL_g(\mathbb{Z}).$$

* Stabilization.

the stab. map. $\mathcal{M}_{g,1} \hookrightarrow \mathcal{M}_{g+1,1}$.

$$\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$$



Th. (Singer 1933)

$$\lim_{g \to \infty} A_{g,1} \backslash \mathcal{M}_{g,1} / B_{g,1} \xrightarrow{\sim} \mathcal{V} = \text{diff class of 3-manifolds.}$$
$$\phi \longrightarrow \mathcal{H}_g \cup_{\iota_g \phi} - \mathcal{H}_g$$

$\mathbb{Z}$-HS. Torelli group $\mathcal{T}_{g,1}$

$$1 \to \mathcal{T}_{g,1} \to \mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z}) \to 1$$

Th. Morita (1989)

$$\lim_{g \to \infty} A_{g,1} \backslash \mathcal{T}_{g,1} / B_{g,1} \xrightarrow{\sim} \mathbb{Z}\text{-HS}$$

$$\mathcal{T}_{g,1} \longrightarrow \lim_{g \to \infty} A_{g,1} \backslash \mathcal{M}_{g,1} / B_{g,1}.$$

Question: mod p version of Morita Th?

$S^3(p) = \mathbb{Z}/p$-HS mod p Torelli group $\mathcal{M}_{g,1}[p]$

$$1 \to \mathcal{M}_{g,1}[p] \to \mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z}/p).$$

Q:
$$\lim_{g \to \infty} A_{g,1} \backslash \mathcal{M}_{g,1}[p] / B_{g,1} \to S^3[p] \overset{?}{=} S^3(p)$$

A: No in general. $S^3[p] = S^3(p)$ iff $p = 2, 3$.

$$\mathbb{Q}\text{-HS} = \bigcup_{p \text{ prime}} S^3[p]$$

If use $A_{g,1}[p] = A_{g,1} \cap \mathcal{M}_{g,1}[p]$, $B_{g,1}[p]$ ...

Prop.
$$\lim_{g \to \infty} \left(A_{g,1}[p] \backslash \mathcal{M}_{g,1}[p] / B_{g,1}[p] \right)_{AB_{g,1}} \xrightarrow{\sim} S^3[p].$$

3]

$F : S^3[p] \longrightarrow \mathbb{Z}/p$. normalized inv.
$F(S^3) = 0$.

$$\mathcal{M}_{g,1}[p] \longrightarrow \lim_{g \to \infty} \mathcal{M}_{g,1}[p]/\sim \longrightarrow S^3[p].$$

A family of fncl. $\{F_g : \mathcal{M}_{g,1}[p] \to \mathbb{Z}/p\}_{g \ge 4}$ that satisfy:

0) $F_g(Id) = 0$
i) comp. w/ stab. map.
ii) inv. under conj of $AB_{g,1}$
iii) zero on $A_{g,1}[p]$, $B_{g,1}[p]$

$$\begin{tikzcd} \mathcal{M}_{g,1}[p] \arrow[r, "i_g"] \arrow[dr, "F_g"'] & \mathcal{M}_{g+1,1}[p] \arrow[d, "F_{g+1}"] \\ & \mathbb{Z}/p \end{tikzcd}$$

Defect of F_g to be a hom.

$$C_g : \mathcal{M}_{g,1}[p] \times \mathcal{M}_{g,1}[p] \longrightarrow \mathbb{Z}/p$$
$$x, y \qquad F_g(x) + F_g(y) - F_g(xy).$$

1) comp. w/ stab.
2) inv. und $AB_{g,1}$ conj.
3) $C_g(x,y) = 0$ for $x \in A_{g,1}[p]$, $B_{g,1}[p]$.

Question: If consider $\{C_g\}_{g \ge 4}$ satisfying (1)-(3).

Answer: No, we need more:

4) C_g trivial.
5) $\exists$ $AB_{g,1}$-inv. triv.

Th (P-R) $\{C_g\}$ with (1)-(5) then we get an invariant.
(F_g (0)-(iii)).

F_g F_g' (0)-(iii) $\to$ C_g

$\underbrace{(F_g - F_g')}$ (0)-(iii) $\longrightarrow$ 0
is a hom.

Hom$(\mathcal{M}_{g,1}[p], \mathbb{Z}/p)$ (0)-(iii)

Th There is only one hom. (up to mult. cst.)

$$\varphi : \mathcal{M}_{g,1}[p] \to Sp_{2g}(\mathbb{Z}, p) \to sp_{2g}(\mathbb{Z}/p) \longrightarrow gl_g(\mathbb{Z}/p) \longrightarrow \mathbb{Z}/p$$
$$x \qquad Id + pX \mapsto X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to D \longrightarrow tr(D).$$

φ is an invariant! distinguish lens spaces $L(p,q)$.

4] 2-cocicle from $H_1(\mathcal{M}_{g,1}[p])$

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{T}_{g,1} \arrow[r] \arrow[d] & \mathcal{M}_{g,1}[p] \arrow[r, "\Psi"] \arrow[d, "\tau_1^{\mathfrak{s}}"] \arrow[dl, dashed, "\tau_1^2"'] \arrow[dr, dashed] & Sp_{2g}(\mathbb{Z}, p) \arrow[r] \arrow[d, "\alpha"] & 1 \\ 0 \arrow[r] & \Lambda^3 H_p \arrow[r] & H_1(\mathcal{M}_{g,1}[p]) \arrow[r] & sp_{2g}(\mathbb{Z}/p) \arrow[r] & 0 \end{tikzcd}$$

split as $\mathcal{M}_{g,1}$-modules.

Necessary cond. $C_g : H_1(\mathcal{M}_{g,1}[p]) \times H_1(\mathcal{M}_{g,1}[p]) \longrightarrow \mathbb{Z}/p$

(1') comp. stabl.
(2') $GL(\mathbb{Z})$-inv.
(3') $C_g(x,y) = 0$ $x \in \tau_1^{\mathfrak{s}}(A_{g,1}[p])$ or $y \in \tau_1^{\mathfrak{s}}(B_{g,1}[p])$.

$\Lambda^3 H_p$

$$\Lambda^3 H_p = \underset{\substack{a's \\ a_1 \wedge a_2 \wedge a_3}}{W_A} \oplus \underset{\substack{a's\ b's \\ a_1 \wedge b_2 \wedge a_3 \\ b_1 \wedge a_2 \wedge b_3}}{W_{AB}} \oplus \underset{\substack{b's \\ b_1 \wedge b_2 \wedge b_3}}{W_B} \quad \text{dec. as } GL\text{-modules.}$$

$$\tau_1^2(A_{g,1}[p]) = W_A \oplus W_{AB} \qquad \tau_1^2(B_{g,1}[p]) = W_B \oplus W_{AB}.$$

$${}^t J_g : W_B \otimes W_A \longrightarrow \mathbb{Z}/p \qquad {}^t J_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Id & 0 & 0 \end{pmatrix}$$
$$b_i \wedge b_j \wedge b_k,\ a_i \wedge a_j \wedge a_k \longrightarrow 1$$

Prop ${}^t J_g$ is the unique candidate.
(1)-(3')

Thm (P-R) $(\tau_1^2)^*({}^t J_g)$ is not trivial.

$sp_{2g}(\mathbb{Z}/p)$

$$sp_{2g}(\mathbb{Z}/p) = sl_g(\mathbb{Z}/p) \oplus Sym_g^A(\mathbb{Z}/p) \oplus Sym_g^B(\mathbb{Z}/p) \qquad GL_g(\mathbb{Z})\text{-modules.}$$
$$\begin{pmatrix} \alpha & \gamma \\ \beta & -\alpha^t \end{pmatrix} = \alpha \qquad \beta \qquad \gamma$$

$$(\alpha \circ \Psi)(A_{g,1}[p]) = sl_g(\mathbb{Z}/p) \oplus Sym_g^A(\mathbb{Z}/p)$$
$$(\alpha \circ \Psi)(B_{g,1}[p]) = sl_g(\mathbb{Z}/p) \oplus Sym_g^B(\mathbb{Z}/p).$$

(I) ${}^t K_g : Sym_g^B(\mathbb{Z}/p) \otimes Sym_g^A(\mathbb{Z}/p) \longrightarrow \mathbb{Z}/p.$
$\qquad \gamma \qquad \beta \longmapsto tr(\gamma \cdot \beta).$

(II) Pull-back of 2-cocycles on $\mathbb{Z}/p$ to $sp(\mathbb{Z}/p)$ by $(tr \circ \pi_{gl})$

$\pi_{gl} : sp_g(\mathbb{Z}/p) \to sl_g(\mathbb{Z}/p)$

Prop (I) + (II) generate all candidates.

* Pull-back to $\mathcal{M}_{g,1}[p]$ by $(\alpha \circ \Psi)^*$

(II) only the triv. 2-cocycle gives a triv. pull-back.

$$H^2(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p = \langle d \rangle$$

$$(tr \circ \pi_{gl})^* : H^2(\mathbb{Z}/p, \mathbb{Z}/p) \hookrightarrow H^2(sp(\mathbb{Z}/p), \mathbb{Z}/p)$$

Triv. 2-coc on $\mathbb{Z}/p \longrightarrow x_0 + x_1 \varphi + x_2 \varphi^2 + \cdots + x_{p-1} \varphi^{p-1}$.

$\langle d \rangle \longrightarrow \varphi^*(d)$. non triv.

(I) $(\alpha \circ \Psi)^*({}^t K_g)$ is not triv.

(I) + (II) $(\alpha \circ \Psi)^*({}^t K_g) - \varphi^*(d)$ this is triv. and has an $AB_{g,1}$-triv.

Th. it induces an invariant.