Symmetries
of integrable
lattice models

Invariants of knots
and 3-manifolds

Vertex Operator
Algebras

$G \circlearrowright$ Symmetries of integrable lattice models

Invariants of knots and 3-manifolds

$G \circlearrowright$ Vertex Operator Algebras

$U_q(\mathfrak{g})$

Quantum groups

TQFT

Kazhdan-Lusztig

VOA / MTC

Theorem:

MTC $\longrightarrow$ 3d TQFT

Reshetikhin-Turaev construction

$$U_q(\mathfrak{g})$$

Quantum groups

Kazhdan-Lusztig

TQFT

VOA / MTC

$$Z(S^1 \times \Sigma_g) = \mathrm{sdim}\mathcal{H}(\Sigma_g)$$

$$= \sum_{\lambda} (S_{0\lambda})^{2-2g}$$

$G \circlearrowright U_q(\mathfrak{g})$

Quantum groups

Kazhdan-Lusztig

$G \circlearrowright$ TQFT

$G \circlearrowright$ VOA / MTC

$$
\begin{aligned}
Z(S^1 \times \Sigma_g) &= \mathrm{sdim}\mathcal{H}(\Sigma_g) \\
&= \sum_{\lambda} (S_{0\lambda})^{2-2g}
\end{aligned}
$$

$G \circlearrowright U_q(\mathfrak{g})$

Quantum groups

Non-semisimple

$G \circlearrowright$ TQFT

Kazhdan-Lusztig

Logarithmic

$G \circlearrowright$ VOA / MTC

:

C.Blanchet, F.Costantino, N.Geer, B.Patureau-Mirand

M.Barkeshli, P.Bonderson, M.Cheng, Z.Wang

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:

Non-semisimple TQFT's and BPS q-series

FRANCESCO COSTANTINO, SERGEI GUKOV, AND PAVEL PUTROV

Based on the spectacular success of the Khovanov homology, that categorifies the Jones polynomial,

$$J_K(q) = \sum_{i,j} (-1)^i q^j \, \dim Kh_{i,j}(K)$$

it is natural to ask whether Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds admit a similar categorification:

$$\mathrm{WRT}(M_3; {\color{red}k}) = \sum \, \ldots \, \dim H(M_3)$$

One immediate obstacle is that the WRT invariants, defined at roots of unity, do not come in the form of a polynomial / power series in $q = \exp(2\pi i/k)$ with integer coefficients, e.g.

$$\left(\frac{k}{2}\right)^{g-1} \sum_{j=1}^{k-1} \left(\sin \frac{\pi j}{k}\right)^{2-2g}$$

Possible ways around this challenge:

- Hopfological algebra — M.Khovanov, Y.Qi, A.Beliakova, ...
- Higher representation theory — R.Rouquier, A.Manion, ...
- Holomorphic q-series in $|q|<1$ — this talk

Surprise: multiple q-series

$$\widehat{Z}_b(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

S.G., P.Putrov, C.Vafa
S.G., M.Marino, P.Putrov

labeled by $b \in H_1(M_3;\mathbb{Z}) \;\cong\; \mathrm{Spin}^c(M_3)$

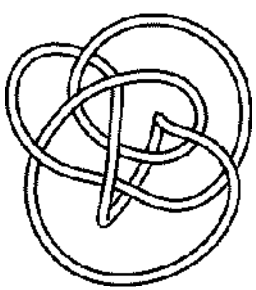
S.G., C.Manolescu
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

so that $\mathrm{WRT}(M_3,k) \;=\; \sum_b c_b^{\mathrm{WRT}} \; \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$

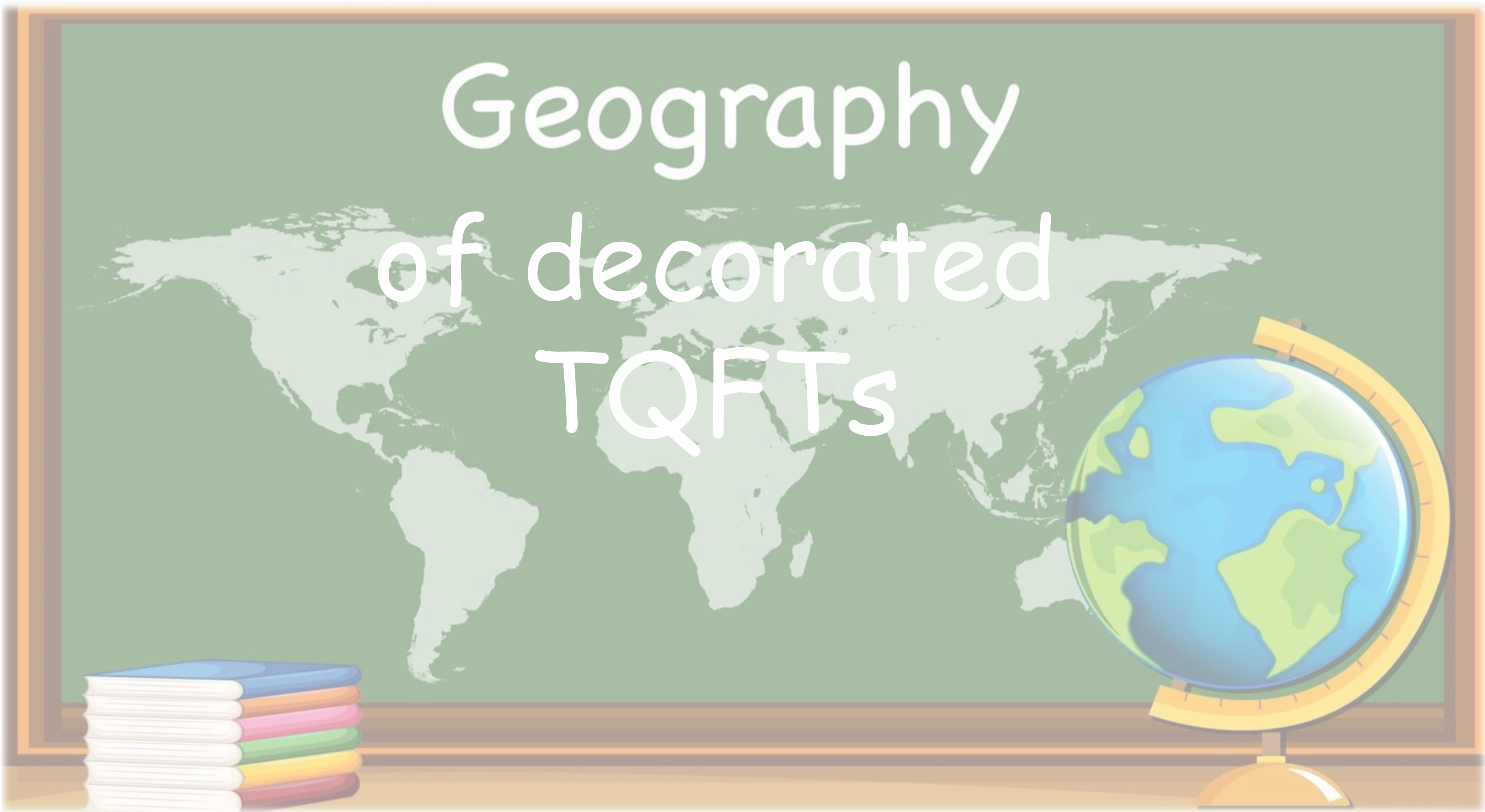
For knot and link complements, a very efficient diagrammatic approach based on the R-matrix for Verma modules and quantum groups at generic q was proposed by S. Park (2020, 2021).

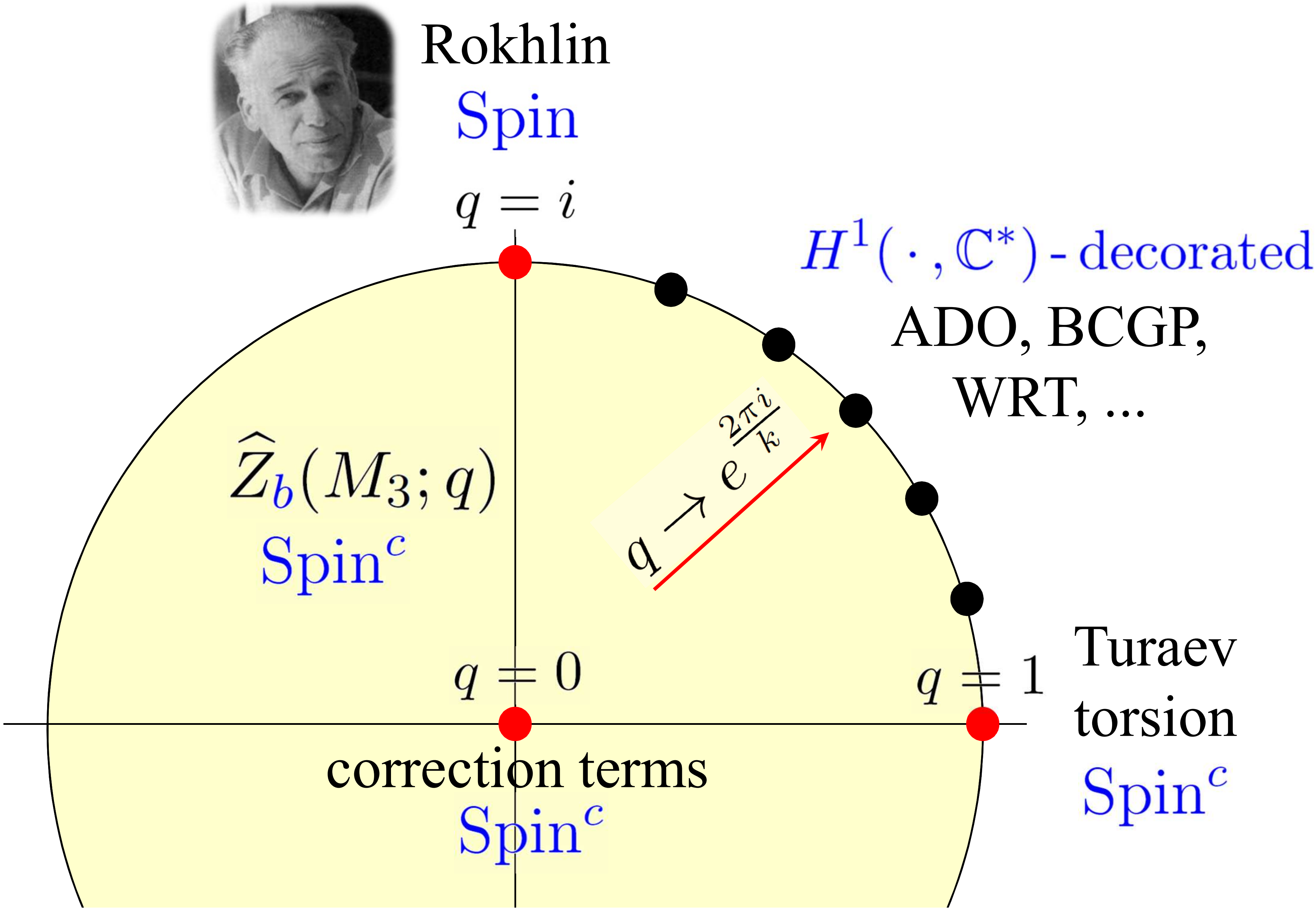
For example, using this approach and the GM surgery formula one finds:

$S^3_{+5}(\mathbf{10_{145}})$



$$\begin{aligned} b=2: &\quad q^{14/5}\left(-1+2q+2q^2+q^3+\ldots\right) \\ b=1: &\quad q^{11/5}\left(-1-2q^2-2q^3-4q^4+\ldots\right) \\ b=0: &\quad 2q^4+2q^7+2q^8+2q^9+4q^{10}+\ldots \\ b=-1: &\quad q^{11/5}\left(-1-2q^2-2q^3-4q^4+\ldots\right) \\ b=-2: &\quad q^{14/5}\left(-1+2q+2q^2+q^3+\ldots\right) \end{aligned}$$


Geography
of decorated
TQFTs



Rokhlin
Spin
$q = i$
$H^1(\cdot, \mathbb{C}^*)$ - decorated
ADO, BCGP,
WRT, ...
$q \to e^{\frac{2\pi i}{k}}$
$\widehat{Z}_b(M_3; q)$
Spin^c
$q = 0$
$q = 1$
Turaev
torsion
Spin^c
correction terms
Spin^c

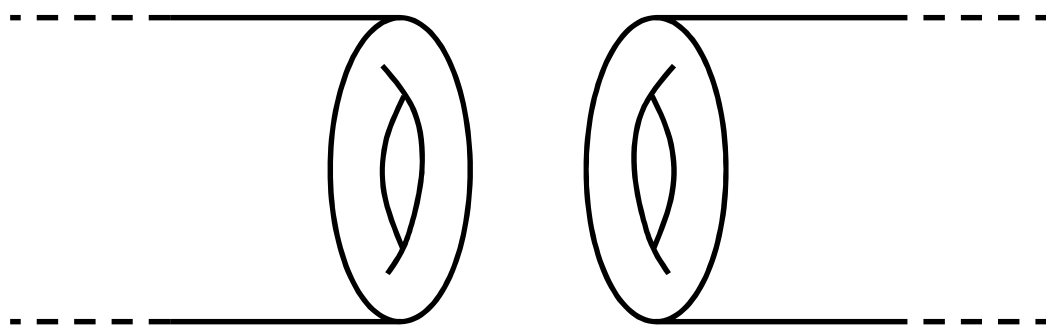
d-dimensional TQFT

Def: "n-form symmetry" G, abelian

(n+1)-form $A \in H^{n+1}(M_d; G)$

connection on a flat n-gerbe

$H^{n+1}(\cdot \, ; G)$–decorated TQFT



d=3:

$$\omega \in H^1(M_3;G) \cong \mathrm{Hom}(H_1(M_3;\mathbb{Z}),G)$$
$$\cong \mathrm{Hom}(H^2(M_3;\mathbb{Z}),G)$$

$M_3 = \Sigma \times S^1$:

$$\omega \in H^1(\Sigma \times S^1;G)$$
$$\cong \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}),G) \oplus \mathrm{Hom}(H_0(\Sigma;\mathbb{Z}),G)$$
$$\cong H^1(\Sigma;G) \oplus \mathrm{Hom}(H_0(\Sigma;\mathbb{Z}),G)$$

$\mathcal{H}(\Sigma)$	structure
graded by	$H^2(\Sigma;\widehat{G})$
decorated by	$H^1(\Sigma;G)$

$\widehat{G} = \mathrm{Hom}(G,U(1))$

Pontryagin dual

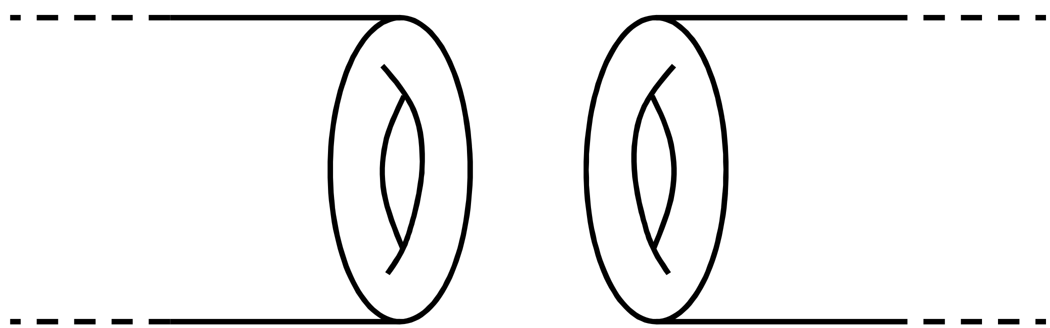
Example: $G = U(1)$ or $\mathbb{C}^*$ 0-form symmetry

$U(1)_k$ Chern-Simons TQFT

$$Z(M_3, \omega) = \int DA \exp\left(\frac{ik}{2\pi} \int_{M_3} AdA + 2\pi i \omega(c_1) \right)$$

$\omega \in H^1(M;G)$

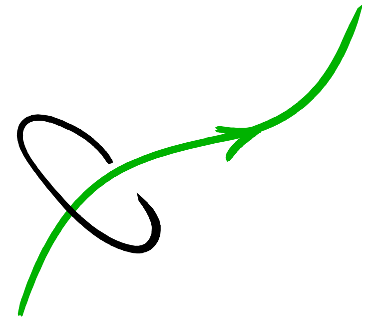
$\mathcal{H}(\Sigma)$	structure
graded by	$H^2(\Sigma;\widehat{G})$
decorated by	$H^1(\Sigma;G)$



Example: $G = U(1)$ or $\mathbb{C}^*$ 0-form symmetry

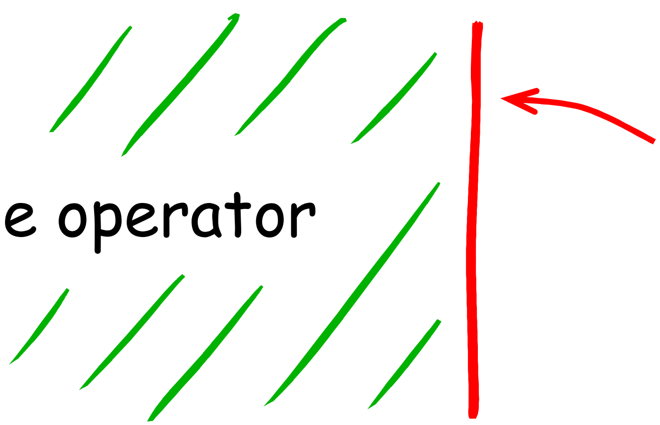
$U(1)_k$ Chern-Simons TQFT

$$\mathcal{H}_{\mathrm{TQFT}}(T^2) = \text{ line operators } = K^0(\mathcal{C})$$



$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

has k simple objects $\begin{cases} \mu \in \mathbb{C}/k\mathbb{Z} \\ \mu \equiv g \mod 1 \end{cases}$



surface operator

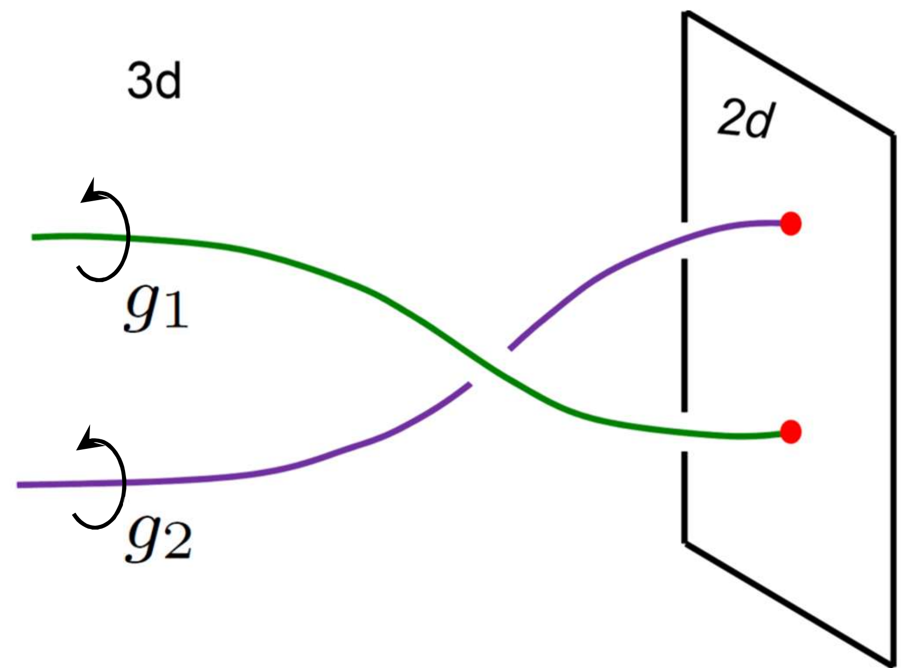
line operator

$$W_\mu(\gamma) = \exp\left(i\mu \int_\gamma A\right)$$

Quantum groups ⟷ Kazhdan-Lusztig ⟷ 2d VOA / CFT

$U_q(\mathfrak{g})$

$U(1)_k$ Chern-Simons TQFT “enriched” by $G = U(1)$ or $\mathbb{C}^*$ ⟷ $U(1)_k$ chiral algebra (lattice VOA)



$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

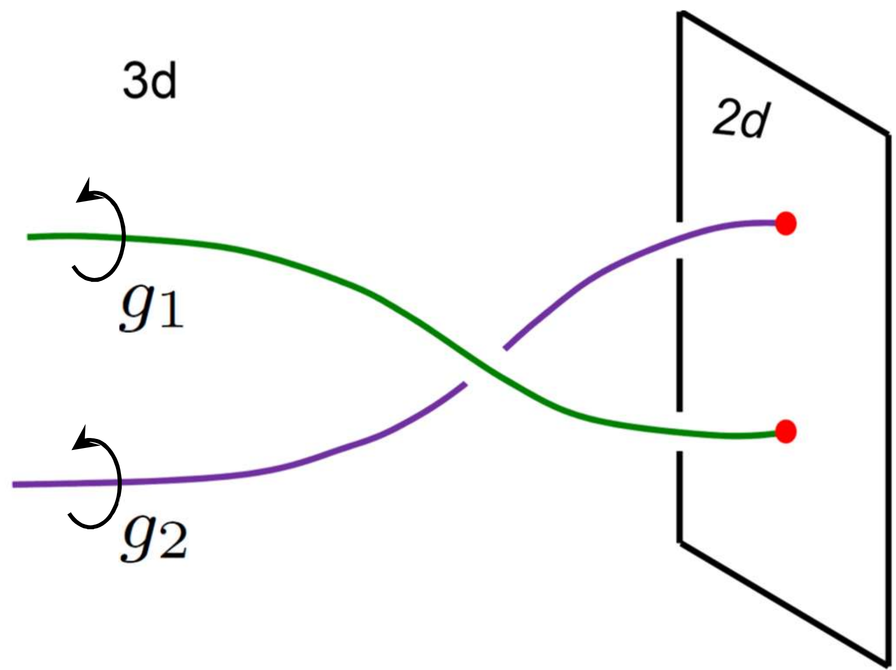
$$\mathcal{C}_{g_1} \boxtimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2}$$

F.Costantino, S.G., P.Putrov

Quantum groups ⟷ Kazhdan-Lusztig ⟷ 2d VOA / CFT

$U_q(\mathfrak{g})$

Decorated TQFTs,
G-crossed MTCs, ... ⟷ twisted sectors



$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

$$\mathcal{C}_{g_1} \boxtimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2}$$

F.Costantino, S.G., P.Putrov

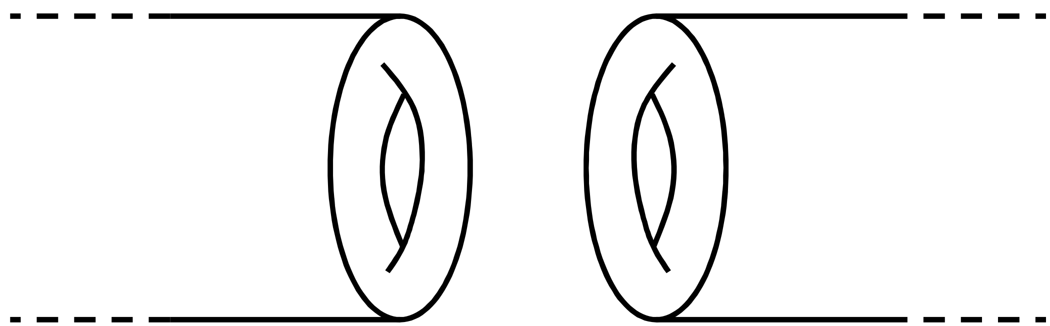
Similarly, 3d $H^2(\,\cdot\,,\mathbb{Z})$-decorated TQFTs

$$G = \mathbb{Z}$$

$$b \in H^2(\Sigma \times S^1, G) \cong H^2(\Sigma,\mathbb{Z}) \oplus \underbrace{H_1(\Sigma,\mathbb{Z})}$$

$\mathcal{H}(\Sigma)$ graded by

$$\mathrm{Hom}(H_1(\Sigma,\mathbb{Z}),U(1)) \cong H^1(\Sigma,U(1))$$



F.Costantino, S.G., P.Putrov

0-form symmetry

$G \Longrightarrow \mathcal{H}(\Sigma)$ decorated by $H^1(\Sigma, G)$
graded by $H^2(\Sigma, \widehat{G})$

1-form symmetry

$G \Longrightarrow \mathcal{H}(\Sigma)$ decorated by $H^2(\Sigma, G)$
graded by $H^1(\Sigma, \widehat{G})$

Application

BCGP $\Longrightarrow \mathcal{H}(\Sigma)$ decorated by $H^1(\Sigma, U(1))$
0-form $U(1)$ graded by $H^2(\Sigma, \mathbb{Z})$

GPPV $\Longrightarrow \mathcal{H}(\Sigma)$ decorated by $H^2(\Sigma, \mathbb{Z})$
1-form $\mathbb{Z}$ graded by $H^1(\Sigma, U(1))$

F.Costantino, S.G., P.Putrov

Fourier transform (gauging, “orbifolding”) of TQFTs

$$\widehat{G} \quad \mathrm{TQFT}/G$$

(d-2-n)-form symmetry

$$B \in H^{d-1-n}(M_d; \widehat{G})$$

n-form symmetry

$$A \in H^{n+1}(M_d; G)$$

$$Z_{\mathrm{TQFT}/G}(M_d, B) \propto \sum_A e^{i(B,A)} Z_{\mathrm{TQFT}}(M_d, A)$$

$$e^{i(-,-)}: \quad H^{d-1-n}(M_d; \widehat{G}) \times H^{n+1}(M_d; G) \to H^d(M_d; U(1)) \cong U(1)$$

F.Costantino, S.G., P.Putrov

Fourier transform (gauging, "orbifolding") of TQFTs

$$\widehat{G} \circlearrowright \text{TQFT}/G$$

(d-2-n)-form symmetry

$B \in H^{d-1-n}(M_d; \widehat{G})$

n-form symmetry

$A \in H^{n+1}(M_d; G)$

swaps gradings and decorations on $\mathcal{H}(\Sigma)$

__Theorem:__ Assuming convergence, extends to cobordisms.

F.Costantino, S.G., P.Putrov

Fourier transform (gauging, “orbifolding”) of TQFTs

$$\widehat{G} \curvearrowright \mathrm{TQFT}/G$$

(d-2-n)-form symmetry n-form symmetry

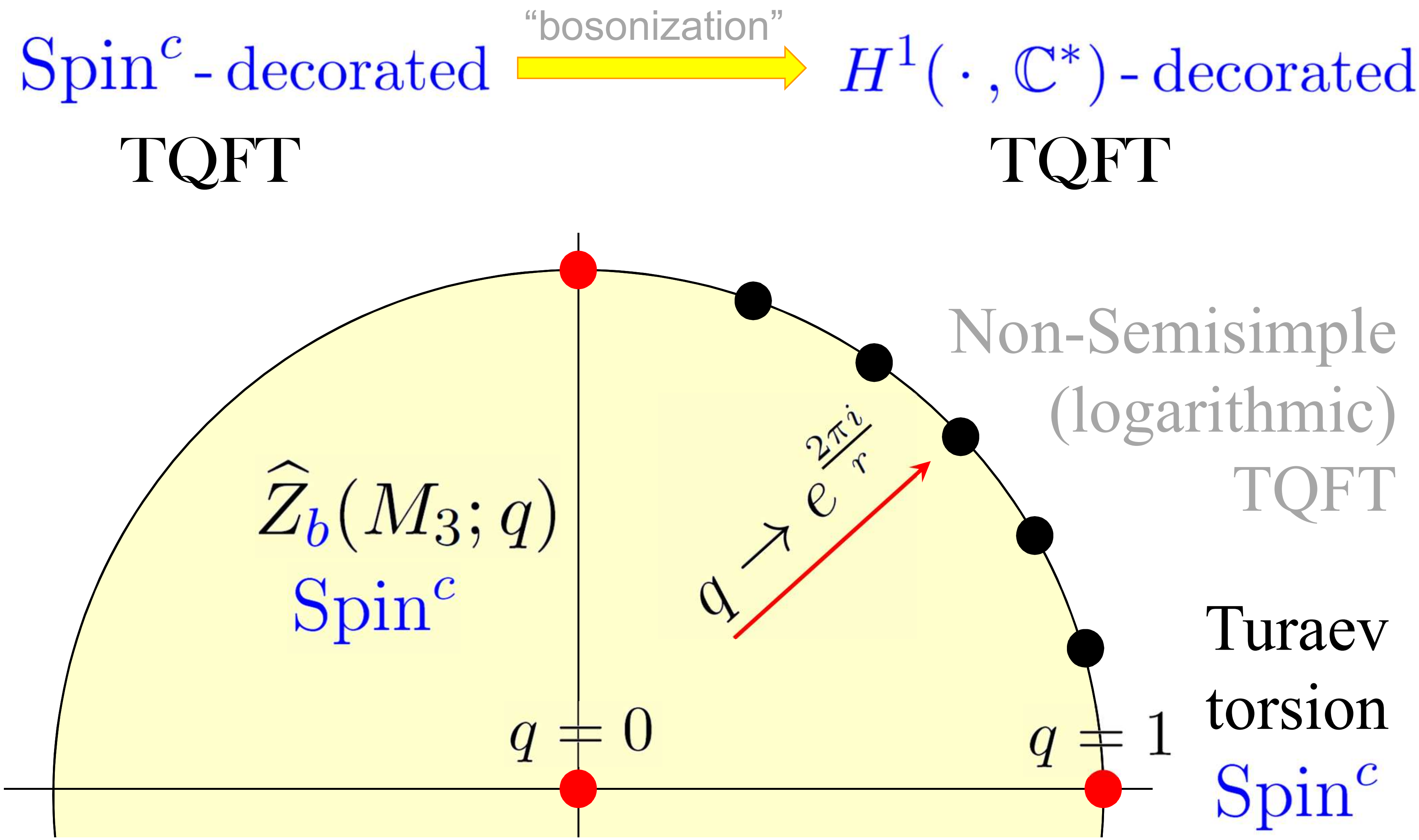
<u>Theorem:</u>

$$\boxed{\mathrm{TQFT}/G/\widehat{G} = \mathrm{TQFT}}$$

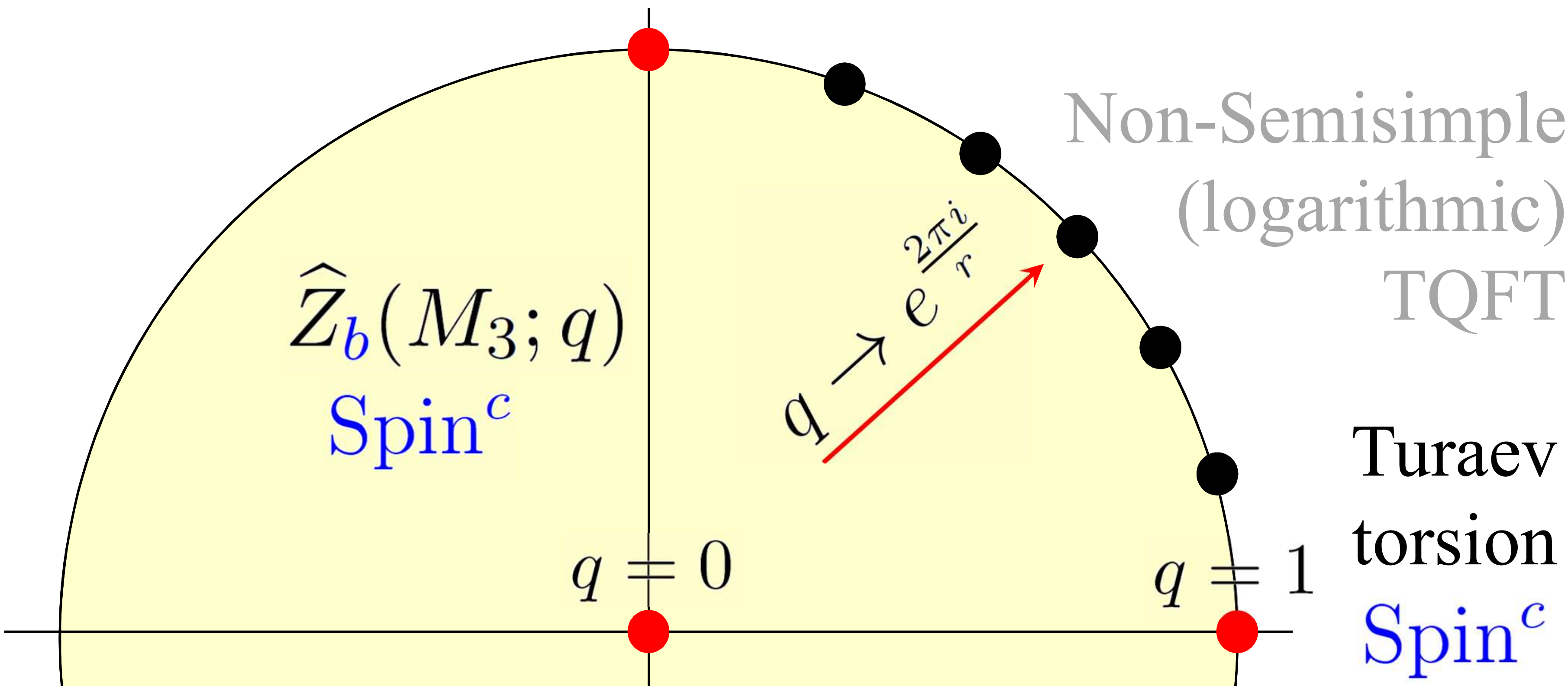
$$\text{cf.} \quad \widehat{G} = \mathrm{Hom}(G, U(1))$$

$$G = \mathrm{Hom}(\widehat{G}, U(1))$$

$$N_r(M_3,\omega) \;=\; \sum_b c^{\mathrm{CGP}}_{\omega,b}\,\widehat{Z}_b(M_3,q)\Big|_{q\to e^{\frac{2\pi i}{r}}}$$



“r-wrapping” (reduction of grading mod r) $\overset{?}{\Longleftrightarrow}$ Hopfological algebra

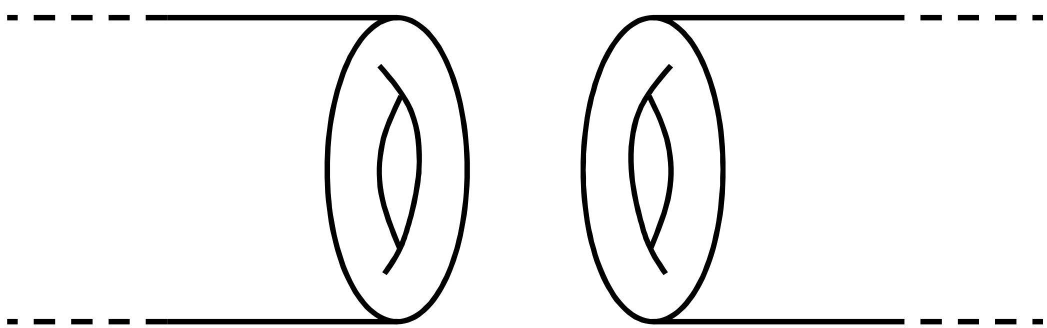


Surgery formulae:

WRT :

$$\left(\frac{k}{2}\right)^{g-1} \sum_{n=1}^{k-1} \left(\sin\frac{\pi n}{k}\right)^{2-2g}$$

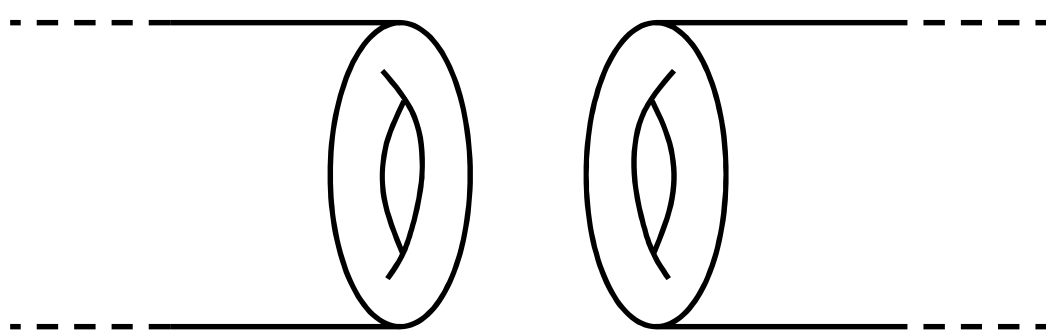
$$\sum_n \oint_{|x|=1} \frac{dx}{2\pi i x} \longleftrightarrow \sum_{(m,n)\in\mathbb{Z}^2}$$



Surgery formulae:

$$q \to e^{\frac{2\pi i}{r}} : \quad \mathcal{H}(T^2) = \mathbb{C}\left[\frac{\Lambda}{W \times r\Lambda^\vee}\right] \Rightarrow \sum_{n=1}^{r}$$

$$\text{generic } q : \quad \mathcal{H}(T^2) = \mathbb{C}\left[\frac{\Lambda \times \Lambda^\vee}{W}\right] \Rightarrow \sum_{(m,n)\in\mathbb{Z}^2}$$



Where else do we find decorated TQFTs and Spin-TQFTs ?

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

"Rozansky-Witten invariants via formal geometry"

"Rozansky-Witten invariants via Atiyah classes"

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

answers. In general, the analysis by cutting and summing over physical states is likely to be quite subtle if X is not compact, roughly because there is a continuum of almost Q-invariant states starting at zero energy. In the presence of such a continuum, formal arguments claiming to show a reduction to the Q-cohomology are hazardous at best. But if X is compact, the spectrum is discrete, and one will get a quite straightforward formalism involving a sum over finitely many physical states.

(cf. eq. (5.8)). If X is non-compact, the continuous spectrum starting at zero energy obstructs a reduction to a description with a finite-dimensional space of physical states. We therefore consider only compact X, such as $X = \mathrm{K3}$, to obtain the surgery formulas.

Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

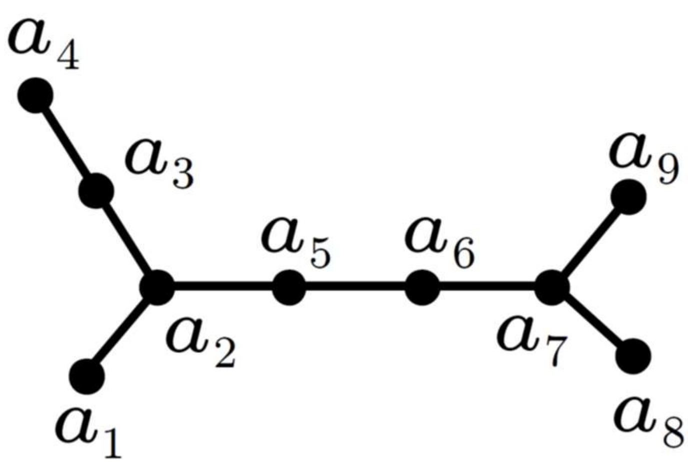
E. Witten

$$\mathcal{H}(\Sigma_g) \;=\; \bigoplus_{q=0}^{\dim_{\mathbb{C}} X} H^q_{\bar\partial}(X, (\wedge^* V)^{\otimes g})$$

$$= \begin{cases} \displaystyle\bigoplus_{l=0}^{2n} H^{0,l}(X)\,, & g=0 \quad (\Sigma_g = S^2) \\ \displaystyle\bigoplus_{l,m=0}^{2n} H^{l,m}(X)\,, & g=1 \quad (\Sigma_g = T^2) \\ \vdots & \end{cases}$$

$$U(1)_x \circlearrowright X = \text{Coulomb branch, …}$$

tri-holomorphic



$b \in \mathrm{Spin}^c(M_3)$

$$Z_{\mathrm{RW}[X]}(M_3) = \sum_{m_i \in \mathbb{Z}} \int \prod_{i \in \text{vertices}} \frac{dx_i}{2\pi i x_i} x_i^{\sum_j Q^{ij} m_j} g(x_i)^{\deg(i)-2} x_i^{b_i}$$

$$\frac{1}{g^2(x)} = Z_{\mathrm{RW}[X]}\left(S^1 \times S^2\right) = \text{Hilbert series of } X$$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

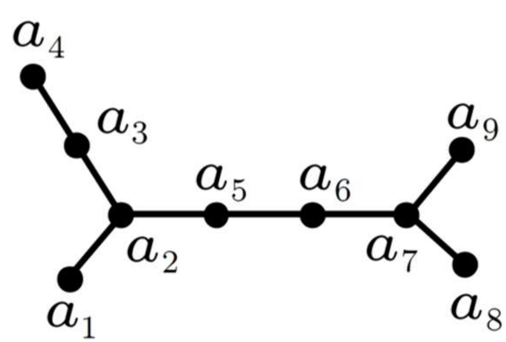
LATTICE COHOMOLOGY AND q-SERIES INVARIANTS OF 3-MANIFOLDS

ROSTISLAV AKHMECHET, PETER K. JOHNSON, AND VYACHESLAV KRUSHKAL

Definition 4.1. Fix a commutative ring $\mathcal{R}$. A family of functions $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n \geq 0}$ is *admissible* if

(A1) $F_2(0) = 1$ and $F_2(r) = 0$ for all $r \neq 0$.
(A2) For all $n \geq 1$ and $r \in \mathbb{Z}$,

$$F_n(r+1) - F_n(r-1) = F_{n-1}(r).$$



Theorem 5.10. *For any admissible family of functions F, the weighted graded root is an invariant of the 3-manifold $Y(\Gamma)$ equipped with the* spin^c *structure $[k]$.*

cf. A.Nemethi

3d analogue of Vafa-Witten invariants

$$Z_{VW}^{(v)}(M_4, q) = \sum_n q^n \chi\left(\mathcal{M}^{\text{inst}}_{c_2=n,\, c_1=v}(M_4)\right)$$

- Physical definition (at last, in principle) for general M_4
- Mathematical construction for Kahler M_4
- Decorated by $v \in H^2(M_4; \pi_1(G))$
- Reproduce characters of VOA[M_4]

Thanks for listening.

Questions?