Topological theories and automata

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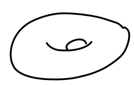
based on joint work with Mee Seong Im (in progress)

Universal construction of topological theories

start with invariants of closed n-dimensional manifolds $\quad M \longmapsto \alpha(M) \in R$



$\alpha(M_1 \sqcup M_2) = \alpha(M_1)\alpha(M_2)$ multiplicative

$\alpha(\emptyset_n) = 1$

R - commutative ring

$\supseteq$

(today) commutative semiring

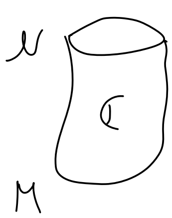
state spaces for $(n-1)$-dimensional objects

$\alpha(N)$ - state space (R-module).

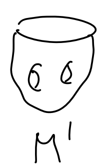
start with $Fr(N)$ - free R-module on $\{[M] \mid \partial M = N\}$

$Fr(N) \times Fr(N) \longrightarrow R$

bilinear pairing



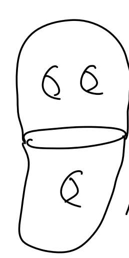
$\partial M = N$



$\partial M' = N$

$([M], [M'])_N = \alpha(\overline{M'}M)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad \cap$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad R$

closed n-manifold



$\alpha(\overline{M}'M)$

symmetric bilinear form. If keeping track of orientation, may require $\alpha(-M) = \overline{\alpha(M)}$, for some involution $\bar{\ }$ on R.

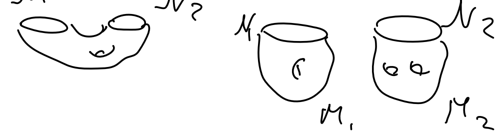
Define $\mathcal{L}(N) = \mathrm{Fr}(N) / \ker((\,,\,)_N)$

$x \in \ker((\,,\,)_N)$ if $\forall y$ $(x,y)_N = 0$

$\mathcal{L}(N)$ is the state space of N, R-module

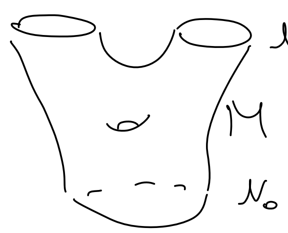
$\mathcal{L}(N_1) \otimes \mathcal{L}(N_2) \longrightarrow \mathcal{L}(N_1 \sqcup N_2)$ usually injective, often not surjective.

(lax tensor structure)



Functoriality:

M induces an R-linear map

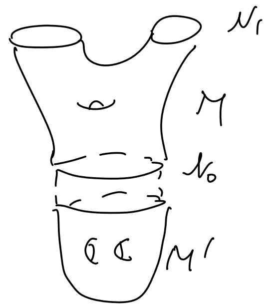


$\partial M = (-N_0) \sqcup N_1$

$[M_1]: \mathcal{L}(N_0) \longrightarrow \mathcal{L}(N_1)$
$[M_2]: \mathcal{L}(N_1) \longrightarrow \mathcal{L}(N_2)$

$[M]: \mathcal{L}(N_0) \longrightarrow \mathcal{L}(N_1)$

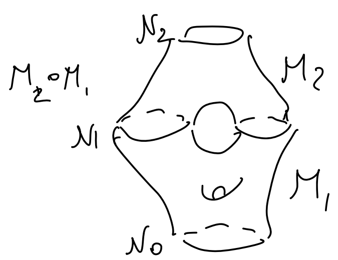
$\quad\quad\;\; [M'] \longrightarrow [MM']$



compose M' with M along N_0

well-defined, R-linear

$[M_2 \circ M_1] = [M_2] \circ [M_1]$ respects composition



get a functor

$(n-1)$-manifold $N \longrightarrow \mathcal{L}(N)$
n-manifold $M \longrightarrow [M]: \mathcal{L}(N_0) \to \mathcal{L}(N_1)$

$\mathrm{Cob}_n \longrightarrow R\text{-mod}$

Cat. of n-dim cobordisms

R-modules

call it a topological theory

lax $\otimes$ structure $\Rightarrow$ not a TQFT in Atiyah's sense (for most α).

$\mathcal{L}(N_1 \sqcup N_2) \supset \mathcal{L}(N_1) \otimes \mathcal{L}(N_2)$

no isomorphism

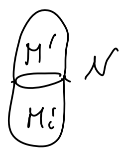
Interesting case: $\alpha(N)$ is a finite-rank R-mod, $\forall N$.

Skein relations

$\lambda_i \in R$

$$\sum_i \lambda_i [M_i] = 0 \text{ in } \alpha(N) \text{ iff } \forall$$

$$\sum_i \lambda_i \, \alpha(M' M_i) = 0 \in R$$



Today we discuss case $n=1$, manifolds carry 0-dim defects (labelled dots),

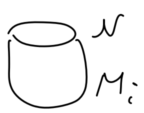
R is a commutative semiring (Boolean semiring) addition multiplication

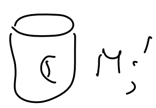
$\mathbb{B} = \{0, 1 \mid 1+1=1\}$ no subtraction

Redefine state space $\alpha(N)$ (for a semiring R) $\lambda_i, \mu_j \in R$

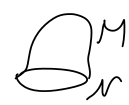
$$\sum_i \lambda_i [M_i] = \sum_j \mu_j [M_j'] \text{ iff for closure by any } M$$

Cannot subtract!





For $\mathbb{B}$ can assume all $\lambda_i = 1$, $\mu_j = 1$



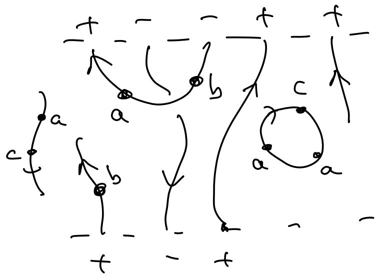
$$\sum_i \lambda_i \, \alpha(M M_i) = \sum_j \mu_j \, \alpha(M M_j')$$

Cannot cancel
$[M_1] + [M_2] = [M_1] + [M_3]$
$\not\Downarrow$
$[M_2] = [M_3]$

Pick finite set Σ (labels of dots).

1) Category C_Σ - objects sequences of $+, -$ (oriented 0-manifolds).

Morphisms - Σ-decorated oriented cobordisms



morphism from $(+-+)$ to $(+--++)$

composition is concatenation.

Cobordisms may 'end' in the middle $\Rightarrow$ 2 types of boundary points: top/bottom (outer) and inner/floating.

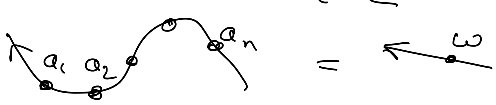
$a, b, c \in \Sigma$

(intersections are virtual)

Closed morphism is a hom from $\emptyset_0$ to $\emptyset_0$ (everything is floating)

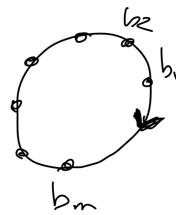
Connected components:

$a_i \in \Sigma$



$\omega = a_1 a_2 \ldots a_n$ word

$\omega \in \Sigma^*$ free monoid on Σ



$b_1 \ldots b_m$ circular word

$\omega_1 \omega_2 \sim \omega_2 \omega_1$ rotational equivalence

$\omega_1, \omega_2 \in \Sigma^*$

each such must evaluate to 0 or 1 (el's of $\mathbb{B}$)

$$\alpha_I : \overset{\text{words}}{\Sigma^*} \longrightarrow \mathbb{B} \qquad \alpha_\circ : \overset{\text{circular words}}{\Sigma^* / _{\text{rotations}}} \longrightarrow \mathbb{B}$$

language L_I — circular language $L_\circ$

$\omega \in L_I$ iff $\alpha_I(\omega) = 1$ — $\omega \in L_\circ$ iff $\alpha_\circ(\omega) = 1$

$$\alpha = (\alpha_I, \alpha_\circ) = (L_I, L_\circ)$$

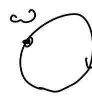
language — rotationally-invariant language.

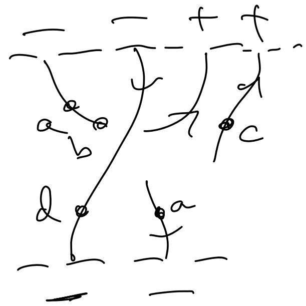
Language L is a subset of Σ^* (L is a set of words in alphabet Σ)

$L_I, L_\circ$ are, in general, unrelated.

2) Category C'_α (intermediate category)

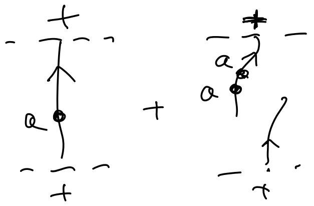
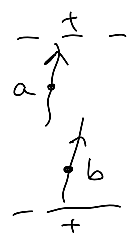
a) add relations to evaluate floating 1-manifolds

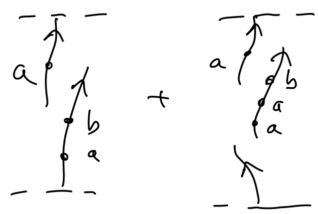
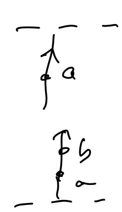
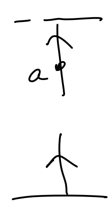
 $= \alpha_I(\omega)$  $= \alpha_0(\omega)$



Morphisms reduce to those w/o floating components.

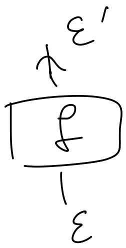
b) allow $\mathbb{B}$-(semi)linear combinations of morphisms. $\mathbb{B} = \{0,1 \mid 1+1=1\}$

$x =$  $y =$ 

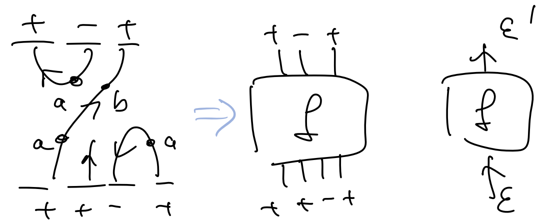
$yx =$  $=$  $+ \underset{\{0,1\}}{\underset{\cap}{\alpha_I(baa)}}$ 

3) Mod out by the universal construction to get C_α

$$C_\Sigma \xrightarrow[\substack{\text{evaluate}\\ \text{floating cobs,}\\ \mathbb{B}\text{-lin. combinations}}]{} C'_\alpha \xrightarrow[\substack{\text{equivalence}\\ \text{rel'n}}]{} C_\alpha$$



schematic notation for decorated 1-cobordism



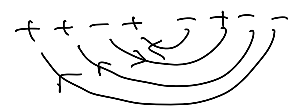
$\varepsilon' = (+-+)$

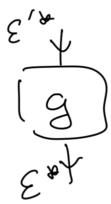
$\varepsilon = (++-+)$

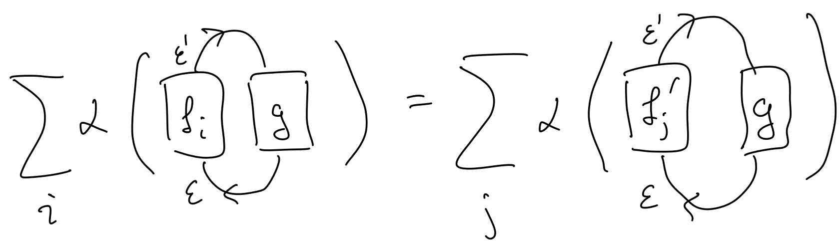
Quotient category C_α (equivalent to universal construction)

$$\sum_i \boxed{f_i} = \sum_j \boxed{f_j'} \quad \text{in } \mathrm{Hom}_{C_\alpha}(\varepsilon, \varepsilon') \text{ iff}$$

$\varepsilon = (++-+) \Rightarrow \varepsilon^* = (-+--)$



$\forall g$  :

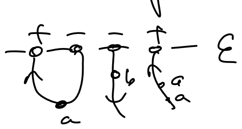


equality in $\mathbb{B}$.

C_α : objects are ε, sequences of $+,-$

Morphisms : $\mathbb{B}$-semilinear combinations of decorated 1-cobordisms. Floating components evaluated via α + universal construction quotient.

State space: $A(\varepsilon) := \mathrm{Hom}_{C_\alpha}(\emptyset_0, \varepsilon)$



$\mathbb{B}$-lin combinations / bilin form

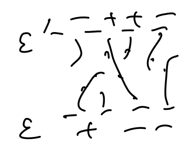
- - - - - - $\emptyset_0$

C_α - rigid symmetric monoidal $\mathbb{B}$-semilinear category

rigid: cups/caps + isotopy

$\alpha = (\alpha_I, \alpha_0) = (\angle_I, \angle_0)$

Call α <u>Cob-regular</u> if all hom spaces $\mathrm{Hom}_{C_\alpha}(\varepsilon, \varepsilon')$ are finite (= finitely-generated)



(Finite) $\mathbb{B}$-semimodule M $\quad a+b$

$\mathbb{B} = \{0, 1\}$ $\qquad 1+1 = 1 \Rightarrow a+a = a$ $\qquad +$ is idempotent

$0 \cdot a = 0$

M - idempotent abelian monoid under $+$

Partial order on M: $a \leq b \Leftrightarrow a+b = b$

$M \Longleftrightarrow$ semilattice with 0 $\qquad a \vee b = \sup(a,b) = a+b$

least upper bound of a, b.

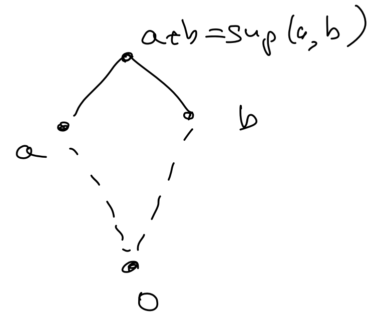
(poset) $\qquad a \vee b = b \vee a$

$(a \vee b) \vee c = a \vee (b \vee c)$

$a \vee 0 = a = 0 \vee a$

$a \leq b \Leftrightarrow a \vee b = b$

$\mathbb{B}$-semimodules $\Longleftrightarrow$ semilattices with 0.



Examples of semimodules

	x_1	x_2	x_3	x_4
y_1	1	1	0	1
y_2	1	0	1	1
y_3	0	1	1	1

$x_1 + x_2 = x_1 + x_3 = x_2 + x_3 = x_4$

M, x_1, x_2, x_3 generators, relations

M is dual to the semimodule generated by rows

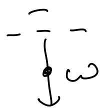
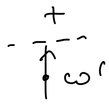
	x_1	x_2
y_1	1	1
y_2	1	0

$x_1 + x_2 = x_1$ $\qquad$ no cancellation

$M = \langle x_1, x_2 \mid x_1 + x_2 = x_1 \rangle$

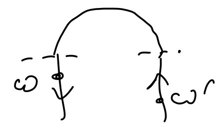
$M = \{0, x_2, x_1\}$.

Given α, look at $A(+)$, $A(-)$

$A(+)$

$A(-) \times A(+) \longrightarrow \mathbb{B}$



$\alpha_I(\omega\,\omega')$

$A(-)$ spanned by these diagrams

$\omega \times \omega' \longrightarrow \alpha_I(\omega\omega')$

$\langle \omega| \in A(-)$, modulo relations

$\mathcal{L}_I$ internal language

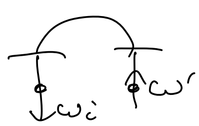
$\langle \omega| :=$ 

$\langle \emptyset| :=$  empty word

$A(-) =$ spanned by $\langle \omega|$ subject to relations

$\sum_i \langle \omega_i| = \sum_j \langle \omega_j|$ iff $\forall$ way to close up & evaluate $\forall$ word ω'

$\sum_i \alpha(\omega_i \omega') = \sum \alpha(\omega_j \omega')$



$\xrightarrow{\alpha} \alpha_I(\omega_i \omega')$

$A(-)$ is a $\mathbb{B}$-semimodule with an action of Σ^*

<u>Example</u> $L_I = \{ \omega \in \{a,b\}^* \mid$ 2nd to last letter is $b \}$

abaaba
bbba $\in L_I$
aabb

Matrix of bilinear pairing

	$\downarrow$	$a\downarrow$	$b\downarrow$	$aa\downarrow$	$ab\downarrow$	$ba\downarrow$	$bb\downarrow$
$\uparrow$	0	0	0	0	1	0	1
$a\uparrow$	0	0	1	0	0	1	1
$b\uparrow$	0	0	1	0	0	1	1
$aa\uparrow$	0	0	0	0	0	0	0
$ab\uparrow$	0	0	0	0	0	0	0
$ba\uparrow$	1	1	1	1	1	1	1
$bb\uparrow$	1	1	1	1	1	1	1
	x	x	y	x	z	y	w

$w = y+z$

$A(-) = \langle x, y, z \mid \begin{matrix} x+y=y \\ x+z=z \end{matrix} \rangle$

$A(-) = \{0, x, y, z, y+z\}$

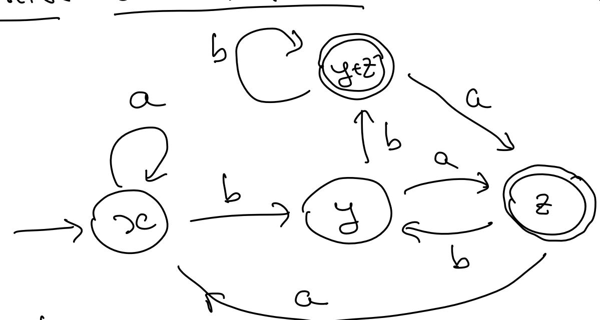
Let $Q = \{ \langle \omega |, \ \omega \in \Sigma^* \} \subset A(-)$

$\langle \omega | = \overline{\omega \downarrow}$

x, y, z are <u>irreducible</u> el's

$u \neq 0$, $u = a+b \Rightarrow a=u$ or $b=u$

Q gives rise to the minimal <u>deterministic</u> <u>finite</u> <u>automaton</u> that accepts language L_I



$\rightarrow\bigcirc$ initial state

$\circledcirc$ accepting states

irreducible el's of $A(-)$ are always in Q.

$0 \leftarrow$ not in automaton, but in $A(-)$

In general, $Q \subset A(-)$ can be a much smaller subset. In above example, $0 \notin Q$ since L_I does not have <u>unrecoverable</u> words ω : $\omega\omega' \notin L_I \ \forall \omega'$

minimal DFA for L_I sits inside $A(\sim)$
as set $Q = \{ \langle \omega | \mid \omega \in \Sigma^* \}$

<u>Review</u>: regular languages & finite state automata.

Deterministic FSA for alphabet Σ is a (Q, δ, q_{in}, Q_t)

Q - finite set of states

$q_{in} \in Q$ - initial state

$\delta : Q \times \Sigma \longrightarrow Q$ — in state q, read letter $a \in \Sigma$ $\Rightarrow$ go to state $\delta(q,a)$
transition function

Q_t - set of accepting states $Q_t \subset Q$

$q \xrightarrow{a} \delta(q,a)$

$q \xrightarrow{b} \delta(q,b)$

given a word $\omega = a_1 a_2 \ldots a_n$,

start in q_{in}

$\underset{=q_0}{q_{in}} \xrightarrow{a_1} \underset{=q_1}{\delta(q_{in}, a_1)} \xrightarrow{a_2} q_2 \cdots \xrightarrow{a_n} q_n$

$q_{i+1} = \delta(q_i, a_{i+1})$

if $q_n \in Q_t$, $\omega \in L$

if $q_n \notin Q_t$, $\omega \notin L$

L is regular if it's accepted by some finite state automaton.

Another characterization of regular languages:

Smallest set of languages that

(a) contains all finite languages

(b) closed under sum (union) $L_1 + L_2 = L_1 \cup L_2$ and $\{ \omega \mid \omega \in L_1 \text{ or } \omega \in L_2 \}$

product $L_1 L_2 = \{ \omega_1 \omega_2 \mid \omega_1 \in L_1, \omega_2 \in L_2 \}$

(c) with L contains $L^* = \emptyset + L + LL + \ldots$ - all concatenations of words in L.
(star closure of L) $L^* = \sum_{n=0}^{\infty} L^n$

$\emptyset$ empty sequence

Thm L_I is regular $\Leftrightarrow$ $\exists$ a deterministic FSA for L_I $\Leftrightarrow$ $A(-)$ (or $A(\epsilon)$) is a finite $\mathbb{B}$-semimodule.

Nondeterministic FSA: (Q, δ, Q_{in}, Q_t)

states ↑ transition ↑

$Q_{in} \subset Q$ subset init states

$Q_t \subset Q$ subset accepting states

$$Q \times \Sigma \xrightarrow{\delta} P(Q) \leftarrow \text{powerset of } Q$$

from a state can go to any state in

$w \in L$ if for some

$w = a_1 \dots a_n$

$q_0 \dots q_n$

$$q \xrightarrow{a} \cdot \qquad q \xrightarrow{a} \cdot \qquad \delta(q,a) \subset Q$$

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \to \dots q_i \xrightarrow{a_{i+1}} q_{i+1} \longrightarrow \dots \to q_n$$

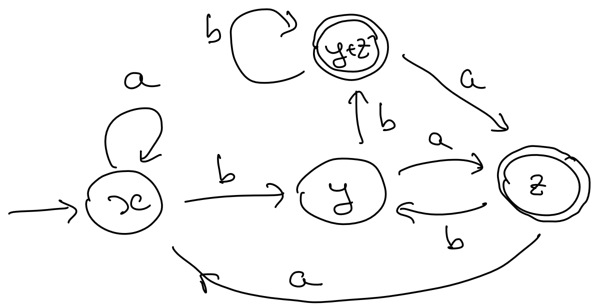
$q_0 \in Q_{in}$, $q_1 \in \delta(q_0, a_1)$, $q_{i+1} \in \delta(q_i, a_{i+1})$, $q_n \in Q_t$ accepting

Thm (Classic)

L regular $\Leftrightarrow$ L accepted by some DFSA $\Leftrightarrow$ L accepted by some NFSA.

Nondeterministic FSA for a language L can have exponentially smaller states than the smallest deterministic FSA for L

$L_I = (a+b)^* b (a+b)$, see earlier



$A(-) = \langle x, y, z \mid x+y=y,\ x+z=z \rangle$

$\nwarrow$ deterministic FA

$m_a : A(-) \to A(-)$

$\langle w| \mapsto \langle wa|$

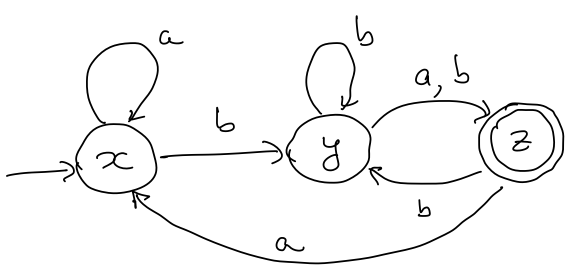
$$
\begin{tikzcd}
A(-) \arrow[loop above, "m_a, m_b"] & \mathbb{B}^J \arrow[l, two heads] \arrow[loop above, "\tilde{m}_a, \tilde{m}_b"]
\end{tikzcd}
$$

free module, $J \supset \mathrm{irr}(A(-))$

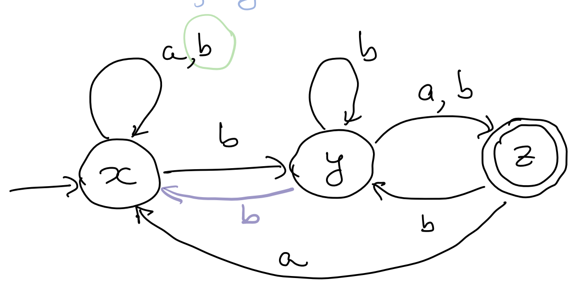
$\langle \emptyset| \in A(-)$, $\quad \langle \tilde{\emptyset}| \in \mathbb{B}^J$

$A(-) \supset Q$ states of min DFA

$J = \{x, y, z\}$



$x+y=y \qquad x+y+z=y+z$



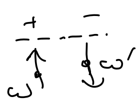
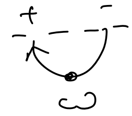
For min # of states, take $J = \mathrm{irr}(A(-))$

If want unique q_{in}, add $\langle \emptyset|$ to J if it is not irreducible.

Minimal NFA is not unique, in general

Nonuniqueness is due to many ways of lifting action of Σ from $A(-)$ to $\mathbb{B}^J$.

Beyond $A(-)$, $A(+)$

$A(+-)$: 2 types of diagrams  

and their $\mathbb{B}$-linear combinations.

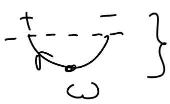
$A(+-) \supset A(+) \otimes A(-)$ usually a proper embedding

$\otimes$ is a bit tricky with semimodules (usually want an explicit realization, i.e. via embedding into free semimodules).

No neck-cutting, in general, and gets complicated.

We don't know how to write down a set of defining relations in C_α for some simple pairs (L_I, L_0) even for one-letter languages

<u>Example</u> Let $L_0 = \emptyset$ (empty language, no words)  $\xrightarrow{\alpha} 0$ $\forall \omega$

Then $\{$  $\}_\omega$ is the <u>syntactic monoid</u> of L_I. (subset of $A(+-)$)

In standard Modern Algebra course we spend a semester studying (finite) groups, but rarely cover (finite) monoids. Possible reasons:

(Finite) groups have better structural theory deep connections to number theory, topology, etc.

Symmetries vs non-invertible maps.

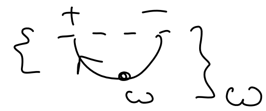
<u>Another reason:</u> they are secretly studied in CS courses

Each language L gives rise to its syntactic monoid:

$\Sigma^*/\sim$ $\qquad \omega_1 \sim \omega_2$ iff $\forall$ words x, y

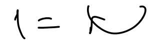
$x\omega_1 y$, $x\omega_2 y$ are either both in L or not in L.

Prop L is regular $\Longleftrightarrow$ its syntactic monoid is finite.

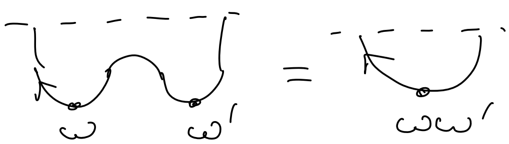
Prop Syntactic monoid $(L) \subset A(+-)$ as 

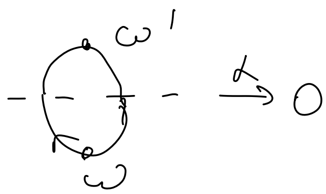
if $L_0 = \emptyset$

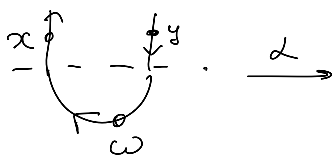
$A(+-)$ is a unital semiring.



associative unital multiplication







$\alpha_I(x\,\omega\,y)$

$A(+-) \supset$ semiring spanned by 

$\supset$ syntactic monoid (L_I).

if $L_0 \neq \emptyset$, only get a surjective map onto syntactic monoid.

M finite $\mathbb{B}$-semimodule $\Leftrightarrow$ M finite semilattice with 0 $\Leftrightarrow$ M finite lattice

$0,\ a+b$

$a+a=a$

comm, assoc.

$0,\ \underset{a\vee b}{\sup(a,b)} = a+b$

$\underset{a\wedge b}{\inf(a,b)} := \sum\limits_{\substack{c\le a\\ c\le b}} c$

$L(M)$ - lattice of M

$L(M)=M$ as sets

$1=\sum\limits_{a\in M} a$

$a\wedge b=b\wedge a$

$(a\wedge b)\wedge c = a\wedge(b\wedge c)$

$a\wedge(a\vee b)=a$

$a\vee(a\wedge b)=a$

$\mathbb{B}$ - fmod. Category of finite $\mathbb{B}$-semimodules

$\mathrm{Hom}(M,N)$ $\quad f: M\to N$

$f(0)=0,\ f(a+b)=f(a)+f(b)$

A retract of a free module

$$M \xrightarrow{i} \mathbb{B}^n \xrightarrow{p} M \qquad pi=id_M$$

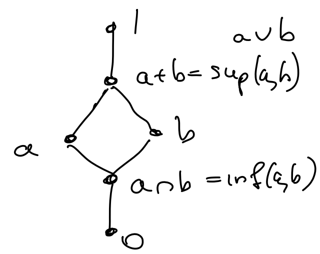
usually not a direct summand

$$M=\{x_1,x_2 \mid x_2=x_1+x_2\} \qquad \underset{3 \text{ elements}}{M} \xrightarrow{i} \underset{4 \text{ elements}}{\mathbb{B}^2} \xrightarrow{p} M$$

$i:\ x_1\mapsto \binom{1}{0},\ x_2\mapsto\binom{1}{1}$

$p:\ \binom{1}{0}\mapsto x_1$

$\binom{0}{1}\mapsto x_2$

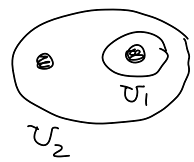


Thm (Hofmann, Mislove, Stralka 1974) TFAE:

1) M is projective in $\mathbb{B}$-fmod
2) M is injective in $\mathbb{B}$-fmod
3) M is a retract of a free module
4) Lattice of M is distributive: $\quad a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, $\quad a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$

Birkhoff Thm

5) M is the lattice of open sets of a finite top space X

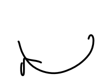


$U_1+U_2=U_2$

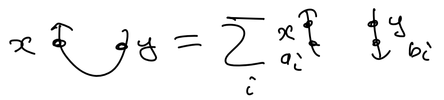
Also, M-projective $\Rightarrow \exists$ morphism $\mathbb{B}\to M\otimes M^*$ s.t. ...

(usually, $M^*\otimes M \longrightarrow \mathrm{Hom}(M,M)$ is not an isomorphism even for finite M)

Given a regular $\mathcal{L} = (L_I, L_0)$, assume $A(-)$ is a distributive lattice (a projective $\mathbb{B}$-module)

Then can write "" $:= \sum_{i=1}^{n} a_i \otimes b_i$ $\quad a_i \in A(+)$, $b_i \in A(-)$

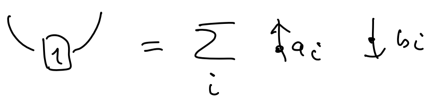
meaning



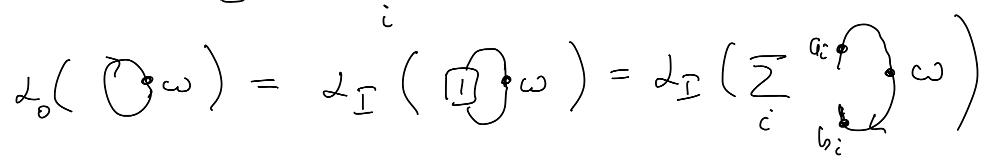
(ignoring L_0)

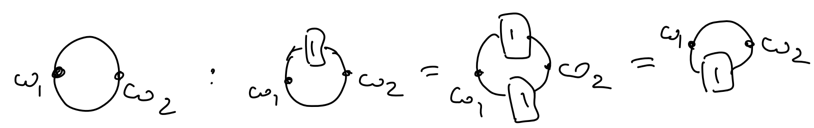
$$\alpha(xy) = \sum_i \alpha(xa_i)\,\alpha(yb_i)$$

Define



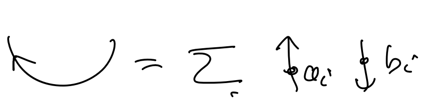
"decomposition of identity"





$L_I \rightsquigarrow L_0$

Then



holds in C_α (only for this α_0).

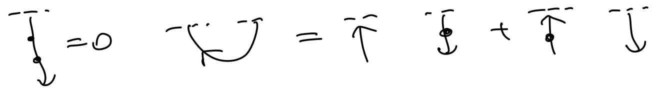
$$A(\varepsilon\varepsilon') = A(\varepsilon) \otimes A(\varepsilon') \qquad \text{(isom, not just inclusion)}$$

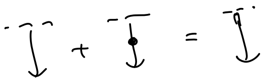
<u>Prop</u> Let L_I be a regular language and assume $A(-)$ is a distributive lattice. Then, for L_0 as above, α gives rise to a $\mathbb{B}$-valued TQFT, with tensor product decomposition for state spaces.

<u>Remark</u> To manipulate $M \otimes N$, embed them in free modules $M \hookrightarrow \mathbb{B}^m$, $N \hookrightarrow \mathbb{B}^n$. Then $M \otimes N \hookrightarrow \mathbb{B}^{nm}$

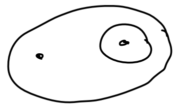
(Gratzer, Lakser, Quackenbush, 1981)

<u>Example</u> $L_I = \{\emptyset, a\}$



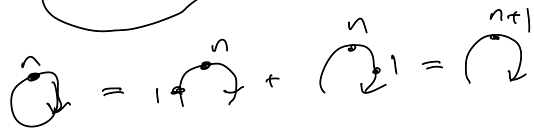


$A(-)$ distributive lattice



$A(+)$ dual (semi) lattice

$L_0 = \{\emptyset\}$



$A(+--+) \simeq A(+) \otimes A(-) \otimes A(-) \otimes A(+)$

With assumptions as before, $A(-)$, $A(+)$ carry commutative multiplication

$ab := a \wedge b \quad (\inf(a,b))$

$a(b+c) = ab + ac$

Natural step into 2D. Convert

into

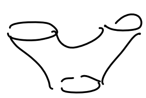
$a, b \in \Sigma$

$c, d \in A(-)$

$a, b \in \Sigma$

multiplication,

no



orientation reversal along seam.

THANK YOU!