Khovanov homology and rational unknotting

joint w. Iltgen & Marino, 2110.15107

Thm For all knots $K \subset S^3$, $\lambda(K) \leq u_q(K)$

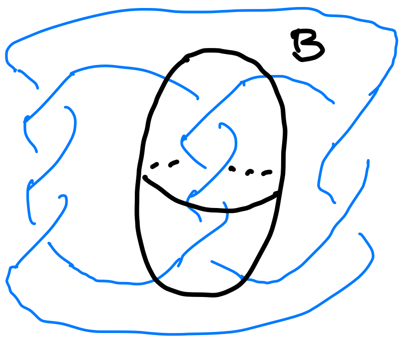
$$\begin{array}{ccc} \lambda(K) & \leq & u_q(K) \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\leq$} \\ u_x(K) & \leq & u(K) \end{array}$$

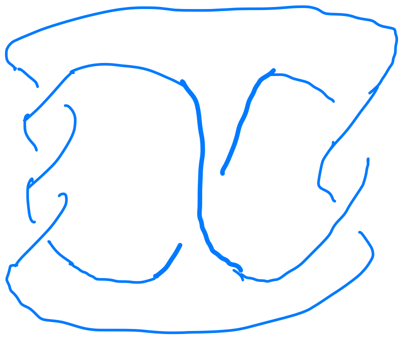
Alishahi-Dowlin '17: $u_x(K) \leq u(K)$

Def Knots K, J are related by a proper rational replacement if $\exists$ 3-ball $B \subset S^3$ st $K \cap (S^3 \setminus B) = J \cap (S^3 \setminus B)$ and $K \cap B$, $J \cap B$ are $\underbrace{\text{rational tangles}}_{\text{obtained from })(\text{ by gluing half twists to right \& bottom}}$ with same $\underbrace{\text{connectivity}}_{\text{which endpoint connected to which: })(,\ \asymp,\ \times}$

Ex

K





J

Def $u_q(K) :=$ min # proper rational repl. relating K to U

$u(K) :=$ min # replacements  relating K to U

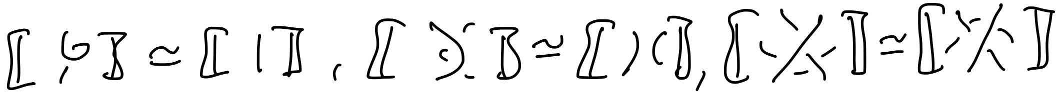
Plan u_q, KH, λ, Pf of Thm

Bar-Natan's KH for tangles.

$2n$-ended diagram $D \longmapsto$ Complex $[\![D]\!]$ over C_{2n}

Obj(C_{2n}) generated by crossingless $2n$-ended diagrams

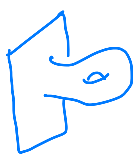
$C_{2n}(D_1, D_2)$ generated by cobordisms mod relations



$[\![D]\!]$ glues well $\Rightarrow$ $[\![T]\!]/_{\text{homotopy}}$ is a tangle invariant

C_2 equivalent to cat. of f.g. free shifted $\mathbb{Z}[G]$-mod

 $\longmapsto \mathbb{Z}[G]$

 $\longmapsto G$

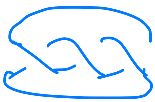
$$\text{Knot } K \xrightarrow{\ [\![K \text{ cut open}]\!]\ } \text{Complex } [\![K]\!] \text{ over } \mathbb{Z}[G]$$

Thm (Naot '07) * $[\![K]\!]$ is universal, ie it determines all other versions of KH (arbitrary Frobenius algebras, coefficients, reduced/unreduced)

* $[\![K]\!] \cong$ reduced chain complex from Frobenius algebra $\mathbb{Z}[X,G]/(X^2-GX)$ over $\mathbb{Z}[G]$

Def $\lambda(K) := \lambda(K,U)$

$$\lambda(K,J) := \min\Big\{ n \geq 0 \;\Big|\; \exists \text{ ungraded chain maps } [\![K]\!] \underset{g}{\overset{f}{\rightleftarrows}} [\![J]\!] \text{ st } fg \simeq G^n \cdot \mathrm{id}_{[\![J]\!]},\ gf \simeq G^n \cdot \mathrm{id}_{[\![K]\!]} \Big\}$$

Ex $K =$  $\Rightarrow \lambda(K) = 1$

$$\begin{tikzcd}
{[\![K]\!]} \arrow[r, phantom, "\simeq"] & \mathbb{Z}[G] \arrow[r] \arrow[d, bend left, "g=G"] & 0 \arrow[r] & \mathbb{Z}[G] \arrow[r, "G"] & \mathbb{Z}[G] \arrow[l, bend left, "1"', "\text{homotopy}"] \\
{[\![U]\!]} \arrow[r, phantom, "\simeq"] & \mathbb{Z}[G] \arrow[u, bend left, "f=1"] & & &
\end{tikzcd}$$

Tangle T with one endpoint marked as basepoint

$\longmapsto [\![T]\!]$ in $C^{\bullet}_{2n}$ (C_{2n} with $\mathbb{Z}[G]$ action)

Def $\lambda(T,T') := \min \{ n \geq 0 \mid$

$\exists$ ungraded chain maps $[\![T]\!] \underset{g}{\overset{f}{\rightleftarrows}} [\![T']\!]$

st $fg \simeq G^n \cdot \mathrm{id}_{[\![T]\!]}$, $gf \simeq G^n \cdot \mathrm{id}_{[\![T']\!]} \}$

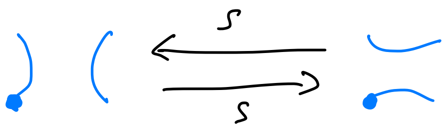
Pf of Thm (1) Show $\lambda(T,T') \leq 1$ for

T, T' rational tangles with same connectivity.

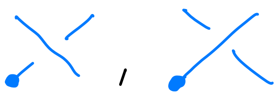
(2) Glue: $\lambda(R \cup T, R \cup T') \leq \lambda(T,T')$ $\square$

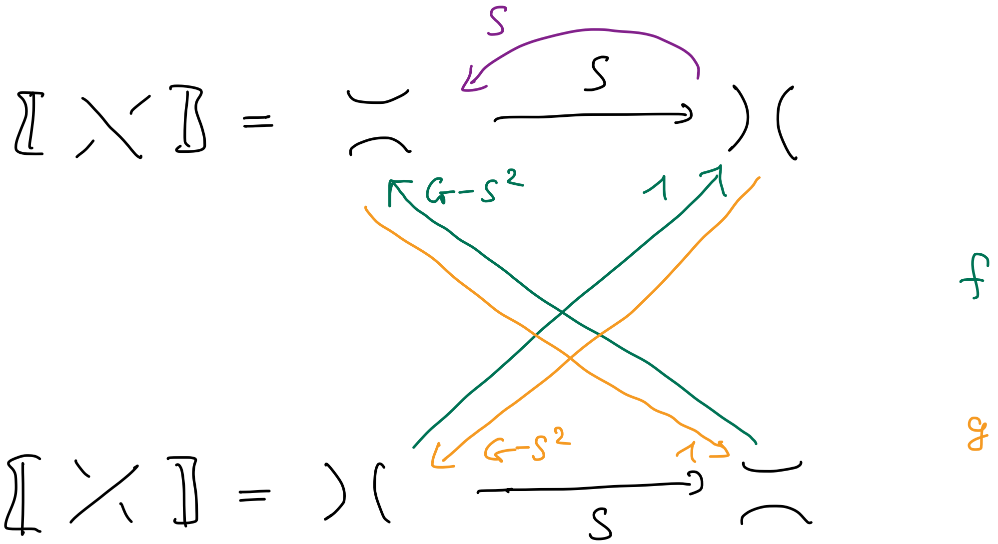
Thm (Kotelskiy-Watson-Zibrowius '19)

$C^{\bullet}_4$ equivalent to $\mathbb{Z}[G]$-enriched cat. gen. by



modulo $S^3 = GS$

Ex $\lambda($  $) = 1$



$$fg = (\underbrace{G - S^2}_{\simeq 0}) \cdot \mathrm{id}_{[\![\times]\!]}$$

$$\simeq G \cdot \mathrm{id}_{[\![\times]\!]}$$

gf analogous $\square$