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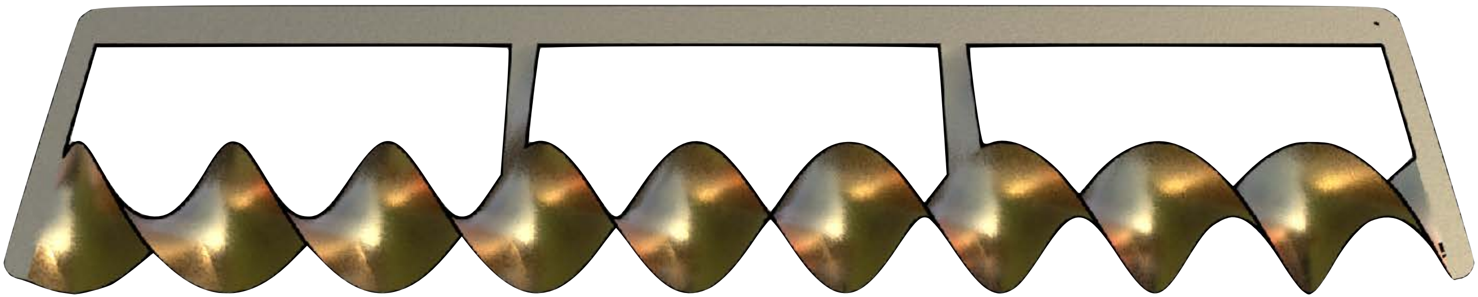
NON-ORIENTABLE COBORDISMS AND KHOVANOV HOMOLOGY



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Views expressed are those of the PI and not the National Science Foundation.

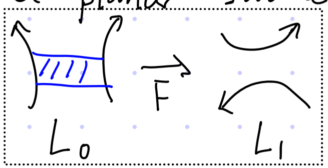
PLAN FOR THE TALK

1. Construction of the mixed invariant.
   A. Basics of non-orientable surfaces.
   B. Equivariant Khovanov homology.
   C. Functoriality under non-orientable cobordisms.
   D. Admissible cuts and the mixed invariant.
2. Applications of Khovanov homology and the mixed invariant.
   E. More properties of the mixed invariant.
   F. Surprising surfaces on the trefoil and Sundberg-Swann's pair.
   G. Exotic pairs from Hayden-Sundberg's examples.
   H. Some open questions

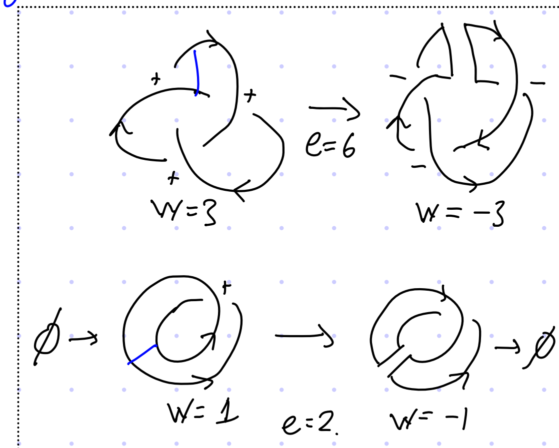

Khovanov homology for
Embedded surfaces ...
...in space.
Textures and background from BlenderKit

NON-ORIENTABLE SURFACE BASICS

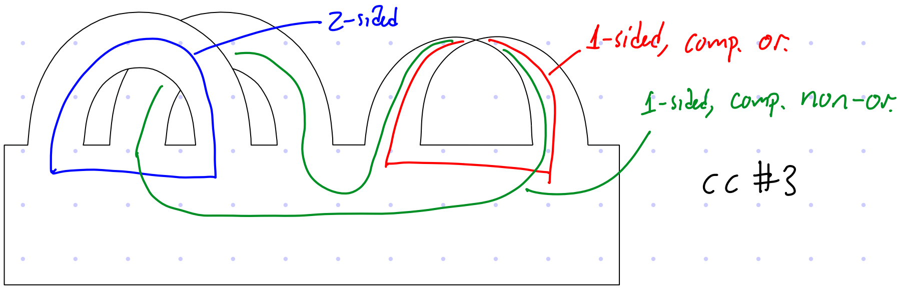
- $F = (\overbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}^{c}) \setminus (D^2 \sqcup \cdots \sqcup D^2)$ has <u>crosscap number</u> c. $cc\#(\sqcup F_i) = \sum_i cc\#(F_i)$
- If $F \subset [0,1] \times S^3$, $\partial F = -\{0\} \times L_0 \cup \{1\} \times L_1$, choose a generic vector field v on F s.t. $v|_{\{i\} \times L_i}$ is Seifert framing. Signed count of zeroes of v is the <u>normal Euler number</u> $e(F)$. (Makes sense even though F nonorientable.)
- For a planar saddle



$e(F) = w(L_0) - w(L_1)$.
Additive.



- Non-orientable surfaces have 1-sided and 2-sided curves.
- Curves can be <u>complement-orientable</u> or <u>complement-nonorientable</u>.



cc#3

Surfaces are <u>not</u> assumed connected!

EQUIVARIANT KHOVANOV HOMOLOGY

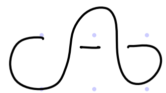
Over $\mathbb{Q}[T]$, ← Lee TQFT

$V = \mathbb{Q}[X,T]/(X^2 = T)$,

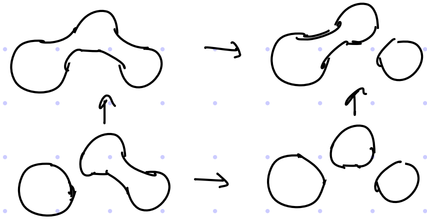
$\Delta : V \to V \otimes V$

$\Delta(1) = 1 \otimes X + X \otimes 1$,

$\Delta(X) = X \otimes X + T\, 1 \otimes 1$



Knot diagram L ⇝



Cube of Res. ⇝

$$\begin{tikzcd} V \arrow[r, "\Delta"] & V^{\otimes 2} \\ V^{\otimes 2} \arrow[u, "m"] \arrow[r, "\mathrm{Id} \otimes \Delta"] & V^{\otimes 3} \arrow[u, "m \otimes \mathrm{Id}"'] \end{tikzcd}$$

Khovanov Cube

$V^{\otimes 2} \to V \oplus V^{\otimes 3} \to V^{\otimes 2}$

Khovanov complex $C^-(L)$ over $\mathbb{Q}[T]$

$C^-(\emptyset) = \mathbb{Q}[T]$.

$C^-(U) = \mathbb{Q}[T] \oplus \mathbb{Q}[T]$

$\partial : C^-_{i,j}(L) \to C^-_{i+1,j}(L)$

T has bigrading $(0,-4)$

Let $C^\infty(L) = T^{-1} C^-(L)$ $\qquad C^\infty(\emptyset) = \mathbb{Q}[T, T^{-1}]$

$C^+(L) = C^\infty(L)/C^-(L)$

$\widetilde{C}(L) = C^-(L)/T\,C^-(L)$ usual Khovanov complex.

$0 \to C^-(L) \to C^\infty(L) \to C^+(L) \to 0$

⇝ long exact sequence

$\cdots \to \mathcal{H}^-(L) \to \mathcal{H}^\infty(L) \to \mathcal{H}^+(L) \to \mathcal{H}^-(L) \to \cdots$

Let $\mathcal{H}^{red}(L) = \ker(\mathcal{H}^-(L) \to \mathcal{H}^\infty(L))$

$= \operatorname{coker}(\mathcal{H}^\infty(L) \to \mathcal{H}^+(L))$.

Story works similarly for the Bar-Natan deformation.

9				d
7				
5			$c \xrightarrow{\partial}$	Td
3	b			
1	a		$Tc \xrightarrow{\partial}$	$T^2 d$
-1	Tb			
-3	Ta		$T^2 c \xrightarrow{\partial}$	$T^3 d$
-5	$T^2 b$			
	0	1	2	3

$C^-(3_1)$

FUNCTORIALITY OF KHOVANOV HOMOLOGY

Thm. [Jacobsson, Khovanov, Bar-Natan, Morrison–Wedrich–Walker, Ballinger, L–Sarkar] Given a (possibly non-orientable) cobordism $F \subset [0,1]\times S^3$, $\partial F = -\{0\}\times L_0 \cup \{1\}\times L_1$, there is an induced map $\mathcal{H}^\bullet(F): \mathcal{H}^\bullet_{i,j}(L_0) \to \mathcal{H}^\bullet_{i-\frac{e}{2},\, j+\chi-\frac{3e}{2}}(L_1)$, $\bullet \in \{+,-,\infty,\wedge\}$, well-defined up to sign, s.t. $\mathcal{H}^\bullet(Id)=\pm Id$, $\mathcal{H}^\bullet(F'\circ F) = \pm \mathcal{H}^\bullet(F')\circ \mathcal{H}^\bullet(F)$, and $F\sim F' \Rightarrow \mathcal{H}^\bullet(F)=\pm \mathcal{H}^\bullet(F')$

(if $\exists$ diffeo $\varphi: [0,1]\times S^3 \circlearrowleft$, $\varphi|_\partial = Id$, $\varphi(F)=F'$)

Pf. (outline)

- Up to isotopy, every surface represented by a movie of R. moves, births, deaths, saddles.

- For invariance in $[0,1]\times\mathbb{R}^3$, check invariance under Carter–Saito's movie moves.
- For invariance in $[0,1]\times S^3$, also check MWW's sweep-around move. $\square$

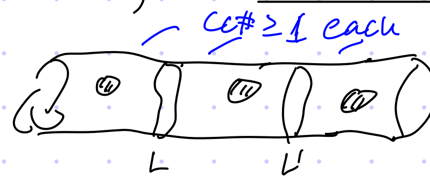
Remark For non-orientable cobordisms I don't know how to fix sign ambiguity.

Thm. [Rasmussen] If F is non-orientable then $\mathcal{H}^\infty(F): \mathcal{H}^\infty(L_0)\to \mathcal{H}^\infty(L_1)$ vanishes.

($\mathbb{Q}$-coeffs. relevant here.)

ADMISSIBLE CUTS

Def. For a cobordism $F: L_0 \to L_1$ w/ crosscap number ≥ 2, an admissible cut is a decomposition $F = F_1 \circ_L F_0$, F_0, F_1 non-orientable



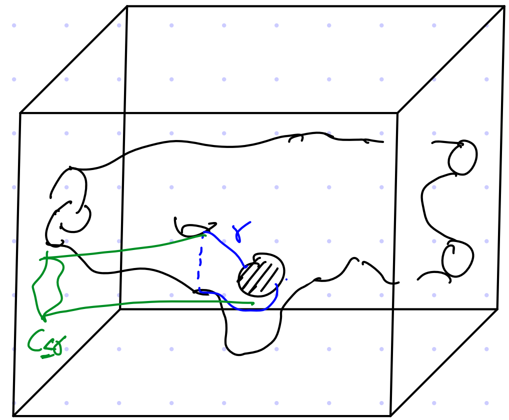
- Two admissible cuts L, L' are equivalent if diffeomorphic or disjoint

Thm. (a) If F has cc # ≥ 2 an admissible cut exists.

(b) If F has cc # ≥ 3 all admissible cuts are equivalent.

pf. (sketch)

(a). Choose a complement-nonorientable 1-sided curve γ in F.



- Let $C_{\leq\gamma} = \{(t,p) \in [0,1] \times S^3 \mid (s,p) \in \gamma \text{ for some } s > t\}$

$\mathcal{U} = \text{nbd}\,(C_{\leq\gamma} \cup \{0\} \times S)$ Perturb s.t. $C_{\leq\gamma} \cap F = \gamma$.

- Decomposition along $\partial\mathcal{U}$ is an admissible cut.

(b) Any admissible cut is equivalent to one as in (a). Disjoint $\gamma, \eta \Rightarrow$ equivalent cuts.

So, use connectedness of the complement-nonorientable 1-sided curve complex. $\square$

THE MIXED INVARIANT

Decompose $F = F_1 \circ_L F_0$ by an admissible cut.

$$\begin{tikzcd}
& \mathcal{H}^{\infty}(L) \arrow[d] \arrow[dr, "0\ (\text{Rasmussen})"] & \\
& \mathcal{H}^{+}(L) \arrow[d] \arrow[r] & \mathcal{H}^{+}(L_1) \\
& \mathcal{H}^{red}(L) \arrow[d] \arrow[ur, dashed] & \\
\mathcal{H}^{-}(L_0) \arrow[ur, dashed] \arrow[r, "\mathcal{H}^{-}(F_0)"'] \arrow[dr, "0\ (\text{Rasmussen})"'] \arrow[uurr, bend left, "\Phi(F)\text{, the mixed invariant.}"] & \mathcal{H}^{-}(L) \arrow[d] & \\
& \mathcal{H}^{\infty}(L) &
\end{tikzcd}$$

<u>Thm</u>. If F has crosscap number ≥ 3 then $\Phi(F)$ is independent of the choice of admissible cut.

<u>pf.</u> (sketch)

Easy from functoriality + equivalence of all admissible cuts. $\square$

PROPERTIES AND APPLICATIONS

Photo by Tierra Mallorca on Unsplash

PROPERTIES OF THE INVARIANTS

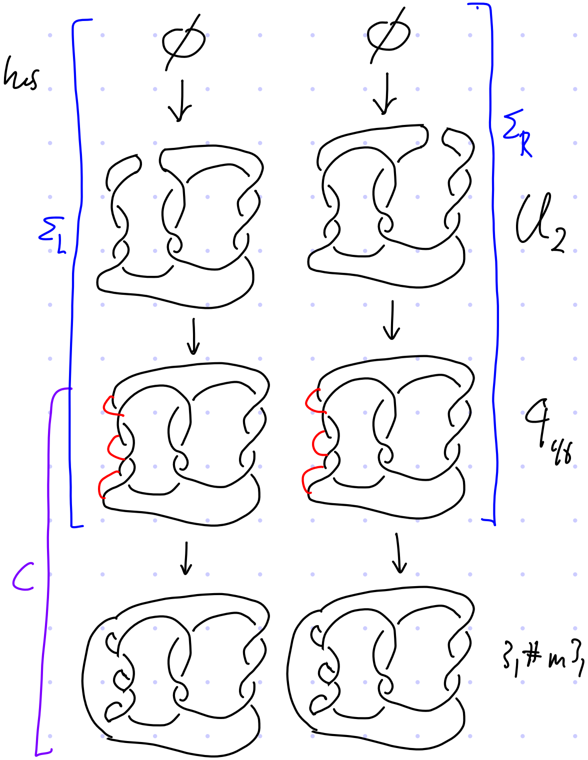
1. The map $\mathscr{H}^-(F)$ shifts $(\mathrm{gr}_h, \mathrm{gr}_q)$ by $(-e/2, \chi - 3e/2 \cancel{- 2s})$. The mixed invariant $\Phi(F)$ shifts $(\mathrm{gr}_h, \mathrm{gr}_q)$ by $(-1 - e/2, \chi - 3e/2 \cancel{- 2s})$.
2. If F is closed, non-orientable then $\mathscr{H}^-(F) = 0$. If connected with crosscap number ≥ 3 then $\Phi(F) = 0$.
3. For F non-orientable:

   $\# T^2$ attach 1-handle.

   1. If F is a standard stabilization then $\mathscr{H}^-(F) = \Phi(F) = 0$.
   2. If F is any stabilization then $\mathscr{H}^-(F) = 0$.
   3. If $F = F' \# F''$ with F'' closed, non-orientable then $\mathscr{H}^-(F) = 0$. If F' has crosscap number ≥ 2 then $\Phi(F) = 0$.
   4. Both $\mathscr{H}^-$ and Φ are unchanged by connected sums with S^2s.

SURPRISING SURFACES FROM SUNDBERG-SWANN

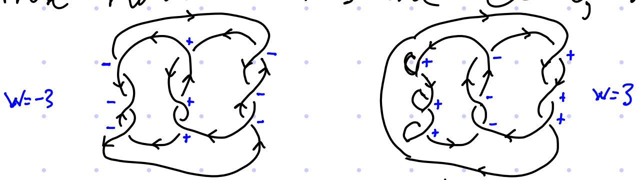
<u>Thm.</u> [Sundberg-Swann] Khovanov homology distinguishes this pair of slice disks for 9_{46}.



<u>pf.</u>
They actually show $\hat{\mathcal{H}}(C\circ\Sigma_L)=0$, $\hat{\mathcal{H}}(C\circ\Sigma_R)\neq 0$. □

<u>Cor.</u> $C\circ\Sigma_L \not\simeq C\circ\Sigma_R$.

These have cc # 3 and $e=-6$, bdy $3_1 \# m(3_1)$:



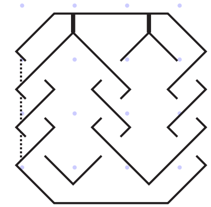
<u>Cor.</u> $C\circ\Sigma_R$ is not a stabilization or crosscap stabilization.

<u>Cor.</u> $\Phi(C\circ\Sigma_R)\neq 0$. $\Phi(C\circ\Sigma_L)=0$.

<u>pf.</u> $\mathcal{H}^+ \to \hat{\mathcal{H}}$.

$\partial\,\Phi(C\circ\Sigma_R) = \hat{\mathcal{H}}(C\circ\Sigma_R)\neq 0$.

Second statement by inspecting form of $\mathcal{H}^+$ & using $\partial\circ\Phi=0$. □



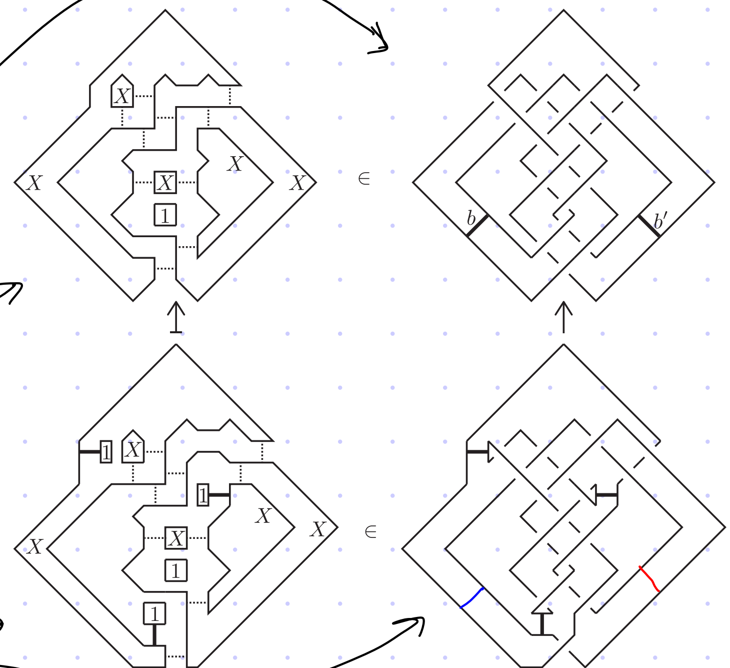
EXOTIC PAIRS

<u>Def.</u> $F, F' \subset [0,1]\times S^3$ are an <u>exotic pair</u> if $\exists$ homeo $\varphi: [0,1]\times S^3 \circlearrowleft$, $\varphi|_\partial = id$, $\varphi(F)=F'$ but no such diffeo.

<u>Thm.</u> [Hayden–Sundberg] These slice disks are exotic.

<u>pf.</u>
Image of this class distinguishes them.
(Plus Conway-Powell + a π_1 computation.) $\square$

<u>Cor.</u> [L-Sarkar] These $(\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2 \setminus D^2)_S$ are exotic. $(e=-6)$



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<u>pf.</u>
Image of this class distinguishes them.
(So does Φ.) $\square$

SOME OPEN QUESTIONS

1. Are there any exotic pairs detected by $\mathbf{\Phi}$ but not $\mathscr{H}^-$?
2. Does $\mathbf{\Phi}$ distinguish some closed, disconnected surfaces?
   $\mathbb{R}P^2 \sqcup \mathbb{R}P^2 \sqcup \mathbb{R}P^2$?
3. Does $\mathbf{\Phi}$ distinguish the stabilizations of some Möbius bands?
4. Can you turn *any* exotic pair of slice disks [distinguished by $\mathscr{H}$] into an exotic pair of non-orientable cobordisms [distinguished by $\mathscr{H}$ or $\mathbf{\Phi}$]?
5. Is there a precise relationship between $\mathbf{\Phi}$ and the Seiberg-Witten invariant of the branched double cover?

Thanks for listening!