1/31/2022

Annular link Floer homology and $gl(1|1)$

(with A. Manion and M. Wong)

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Kh $\downarrow$ Jones poly

graded χ.

HFK $\downarrow$ Alexander poly

① **A TQFT approach to link polynomials.**

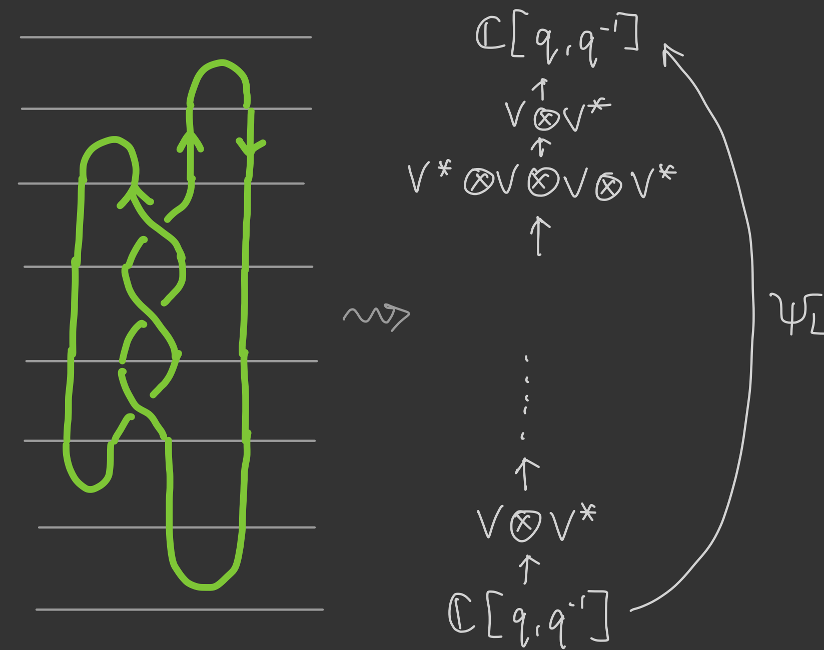
. Reshetikhin, Turaev, Viro

Δ_K: Murakami, Rozansky-Saleur, Reshetikhin

<u>Idea:</u>

- fix a quantum group $U_q(\mathfrak{g})$ and a representation V of $U_q(\mathfrak{g})$.
- Decompose link into elem. pieces
- Associate • $\otimes$ of V, V^* to cuts. $\uparrow = V$ $\downarrow = V^*$
  • maps of vs. to tangles
- Compose maps, get

$$\Psi_L : \mathbb{C}[q,q^{-1}] \longrightarrow \mathbb{C}[q,q^{-1}]$$
$$1 \longmapsto \Psi_L(1) \text{ - link invt.}$$

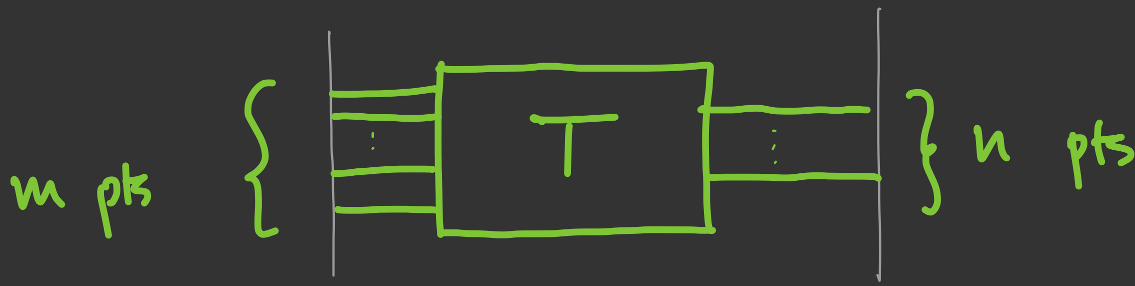


<u>Ex:</u>

Jones poly	$\mathfrak{g} = \mathfrak{sl}_2$	2-dim'l vector rep.	U
Alexander poly	$\mathfrak{g} = \mathfrak{gl}_{1\|1}$		V (sort of...)

<u>Goal:</u> "Categorify" this construction

Tangle Floer homology (P.-Vértesi '14)



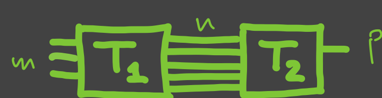
A_m $\quad$ $CT(T)$ $\quad$ A_n

dg bimodule

Thm: (P.-Véresi)

① $CT(T)$ is an invariant of T.

②.

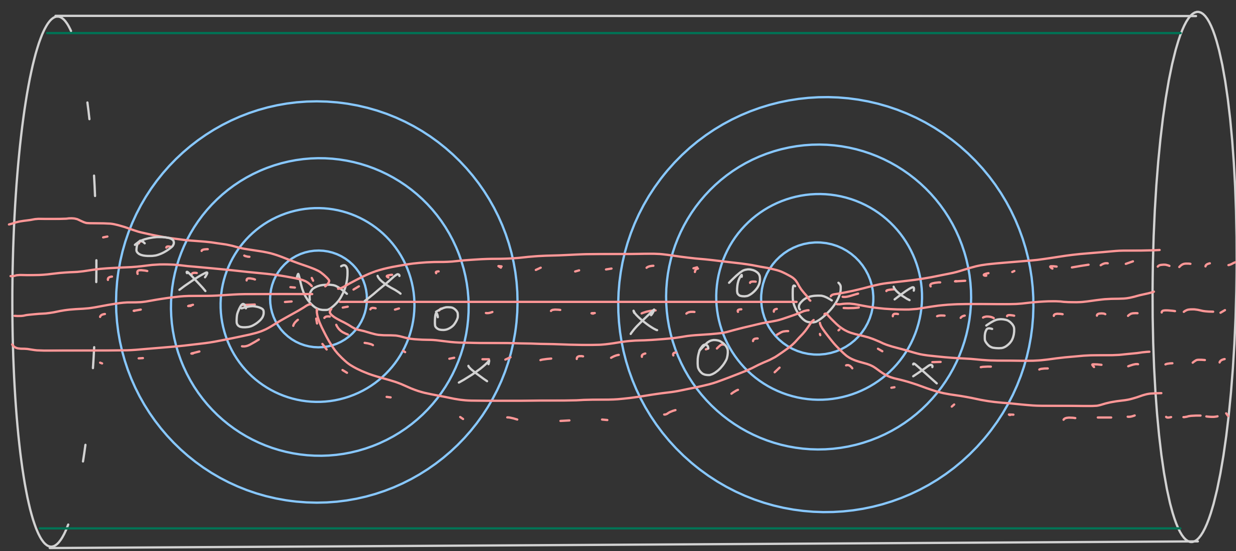


$$CT(T_1) \tilde{\otimes}_{A_n} CT(T_2) \simeq CT(T_1 \circ T_2)$$

③. $CT(L) \simeq HFL(L) \otimes (\mathbb{F}_2 \oplus \mathbb{F}_2[1])$.

CT(T) Given a tangle T, represent by a multipointed bordered Heegaard diagram

$\Sigma(\alpha, \beta, X, O)$

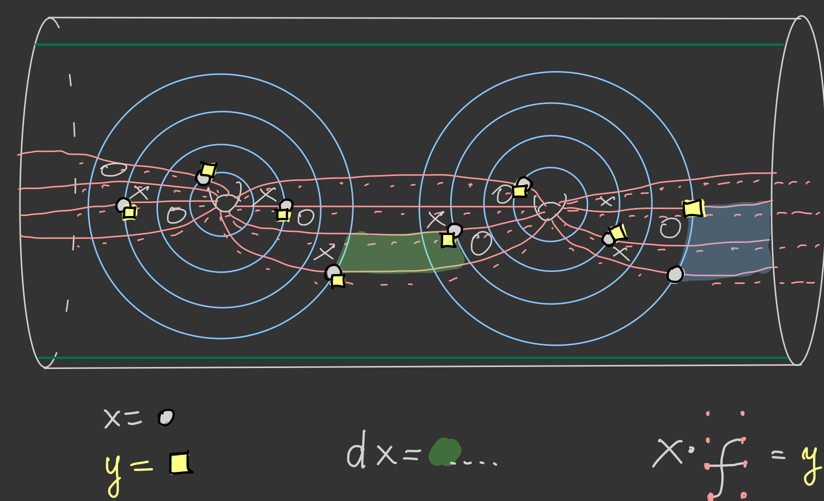


· Count hol. curves in $\Sigma \times I \times \mathbb{R}$ with certain asymptotics.

Ingredients:

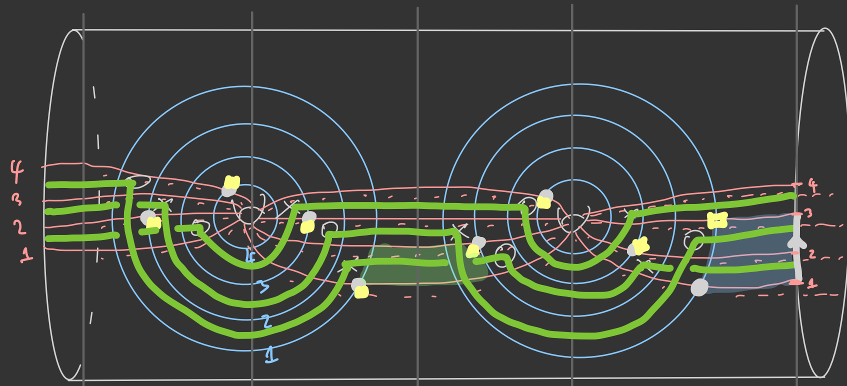
'06 Lipshitz - "Cyl. reform. of HF"

'08 Lipshitz-Ozsváth-Thurston "Bordered HF"



· Record $\partial\Sigma$ data of curves as algebra action.

CT(T)

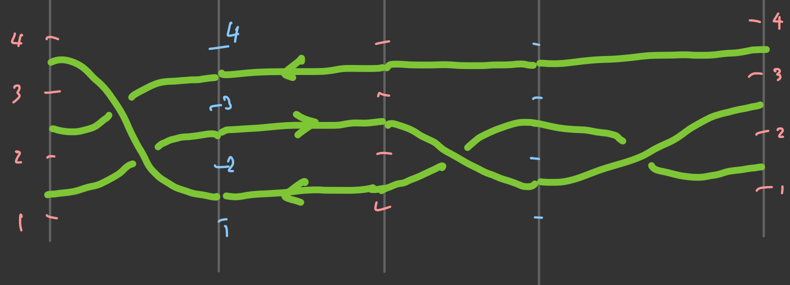


T: { connect ① to X away from β. "above Σ".

connect X to O away from α. (and to $\partial\Sigma$ ...) "below Σ".

$dx = \dots$

$x \cdot$  $= y$



A_m: $\boxed{a} \circ \boxed{b} = \boxed{a|b}$ / relns

$\partial \boxed{a} = \sum_{a'\text{: smooth one x-ing in } a} \boxed{a'}$ / relns

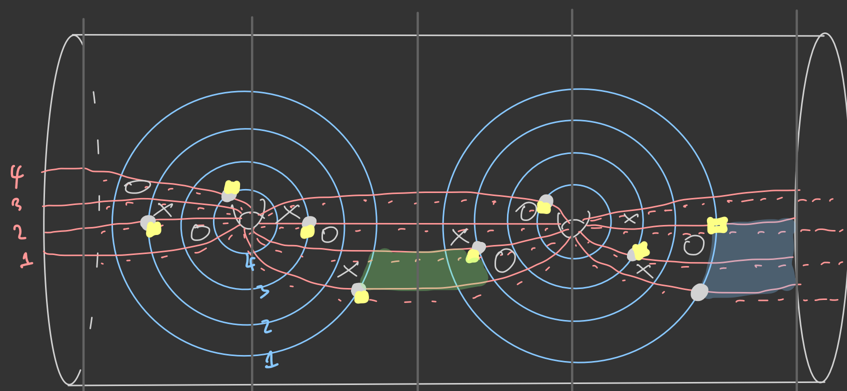
relns:

 $= 0$

 $= 0$

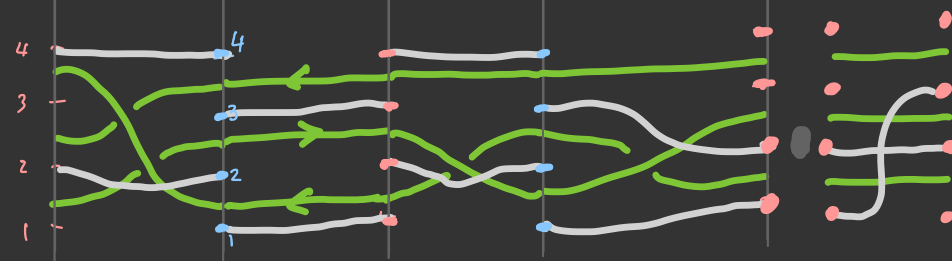
 $= 0$

CFT(T)



$dx = \ldots$

$x \cdot$  $= y$



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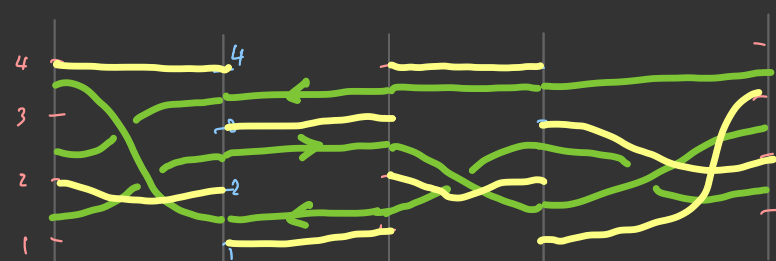

CFT & $U_q(\mathfrak{gl}_{1|1})$

$V_{p_i} := \begin{cases} V & \text{if } p_i = + \\ V^* & \text{if } p_i = - \end{cases}$

Thm: (Ellis-P.-Vértesi '15)

> CFT categorifies $RT_{\mathfrak{gl}_{1|1}}$

$V_{\underline{p}} := V_{p_1} \otimes V_{p_2} \otimes \cdots \otimes V_{p_n}$.

More concretely:

① $K_0(A(P))$ - free $\mathbb{Z}[q,q^{-1}]$-module w/ basis $\{ [A(P)e_S] \mid S \subset \{0,1,\dots,|P|\} \}$

basis for $V_{\underline{p}} \otimes L(\lambda_{\underline{p}})$ $\updownarrow$ $\{ [A(P)e_S] \mid S \subset \{0,1,\dots,|P|\} \}$

so $K_0(A(P)) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{C}(q) \cong V_{\underline{p}} \otimes L(\lambda_P)$

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②

$$\begin{array}{ccc} A(P)\text{-Mod} & \xrightarrow{CT(T)\,\tilde{\otimes}\,-} & A(P')\text{-Mod} \\ \downarrow K_0 & \downarrow & \downarrow \\ V_P \otimes L(\lambda_m) & \xrightarrow{RT_{U_q(\mathfrak{gl}_{1|1})}(\vec{T}^{\lambda_P})} & V_{P'} \otimes L(\lambda_n) \end{array}$$

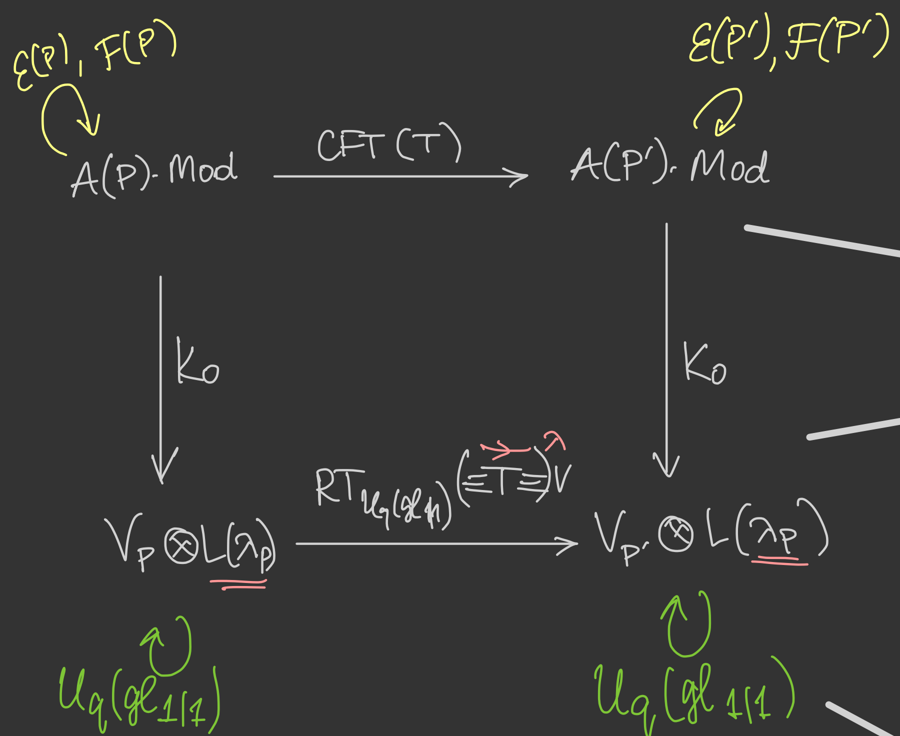
---

③ We construct functors on $A(P)$-Mod corresponding to the action of $E, F \in U_q(\mathfrak{gl}_{1|1})$ on $V_P \otimes L(\lambda_P)$.

Recall: $U_q(\mathfrak{gl}_{1|1}) = \langle E, F, q^{(a,b)} \mid (a,b) \in \mathbb{Z}^2 \rangle / \sim$

We construct:

$\mathcal{E}, \mathcal{F}$ bimodules (which have Heegaard diagrammatic descriptions) s.t.



$$\mathcal{E} \,\tilde{\otimes}\, CFT(T) \simeq CFT(T) \,\tilde{\otimes}\, \mathcal{E}$$

$$\mathcal{F} \,\tilde{\otimes}\, CFT(T) \simeq CFT(T) \,\tilde{\otimes}\, \mathcal{F}$$

$$\mathcal{E} \tilde{\otimes} \mathcal{E} \simeq 0 \qquad \mathcal{F} \tilde{\otimes} \mathcal{F} \simeq 0$$

$$\mathcal{E} \tilde{\otimes} \mathcal{F} \xrightarrow{f} A \longrightarrow \mathcal{F} \tilde{\otimes} \mathcal{E} \longrightarrow \mathcal{E} \tilde{\otimes} \mathcal{F}[1]$$

$$E \, RT(\overrightarrow{T}) = RT(\overrightarrow{T}) \, E$$

$$F \, RT(\overrightarrow{T}) = RT(\overrightarrow{T}) \, F$$

$$E^2 = 0 \qquad F^2 = 0$$

$$EF + FE = 1$$

Question: $HFL(K) \circlearrowleft \mathfrak{gl}_{1|1}$?

Answer: In the *annular* setting, yes!

(cf. Grigsby - A. Licata - Wehrli for $SKh(\hat{T})$ & $\mathfrak{sl}_2$)

Annularization

2-category $\mathcal{C}$:
- $\mathrm{Ob}(\mathcal{C})$: DGAs
- $1\text{-}\mathrm{Mor}_{\mathcal{C}}$: bimodules
- $2\text{-}\mathrm{Mor}_{\mathcal{C}}$: bimodule homomorphisms

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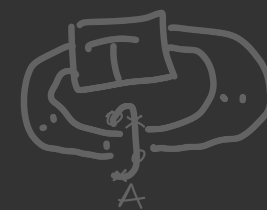
CH_*: $1\text{-}\mathrm{Mor}(A(P), A(P))$ $M \longmapsto CH_* M$ - chain cx.

$2\text{-}\mathrm{Mor}({}_A M_A, {}_A N_A)$ $f \longmapsto CH_*(f)$



$\overline{T}$

Thm (P-Vértesi '15) $HH_*(CT(T)) \cong AHFL(\overline{T}) = HFL(\widehat{T} \cup A)$

Thm (Manion-P.-Wong, in progress)

A: Bimods $\mathcal{E}, \mathcal{F} \rightsquigarrow e, f \circlearrowright \widetilde{CFL}(\widehat{T} \sqcup A)$ s.t.
(chain maps)

1. $e^2 = 0 = f^2$
2. $ef + fe = \mathbb{I}$
3. e, f commute with candidate tangle cobordism maps. !!

B: The actions $e, f \circlearrowright \widetilde{CFL}(\widehat{T} \sqcup A)$ correspond to "combinatorial basepoint actions" by 2 of the 4 basepoints for A.

Pf/Construction:

Idea: Use a construction on ! bicategories $hTr(\mathcal{C})$

Horizontal trace

Def (Beliakova-Habiro-Lauda-Zivkovic '14)

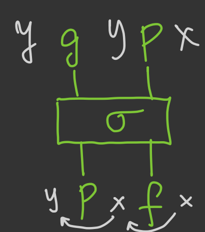
The **horizontal trace** of a bicategory $\mathcal{C}$ is the category $hTr(\mathcal{C})$ with

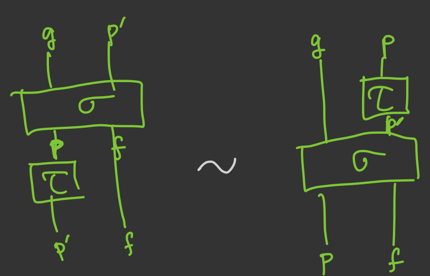
$Ob(hTr(\mathcal{C}))$: 1-endomorphisms $f: x \to x$, $x \in Ob(\mathcal{C})$

$Mor_{hTr(\mathcal{C})}(f: x \to x,\ g: y \to y)$: classes $[p, \sigma]$ s.t. $p: x \to y$ · 1-morphism in $\mathcal{C}$

$\sigma: p \circ f \Rightarrow g \circ p$ - 2-morphism in $\mathcal{C}$

Graphically: $[p, \sigma] =$

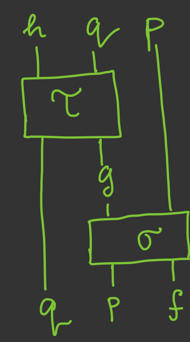




$\in$

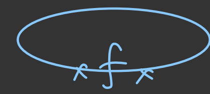
$Mor_{hTr(\mathcal{C})}(f, g)$

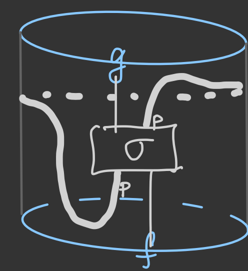
Composition: $[q, \tau][p, \sigma] =$



Identity morphism: $1_f = [1_x, 1_f]$.

From $hTr(C)$ to Ch

$[f: x \to x] \in Ob(hTr(C))$ ⇝ 

$[p, \sigma] \in Mor_{hTr(C)}(f, g)$ ⇝ 

What do ∪ , ∩ "mean"?

If p has a left dual $^*p$, then we can think of ∪ as coev, ∩ as ev.

$^*f \in C(y, x)$ is a **left dual** of $f \in C(x, y)$ if we have

2-morphisms

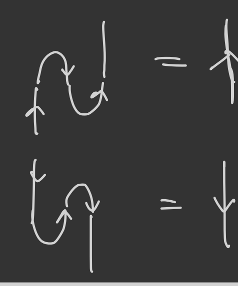
$$f \circ {}^*f \overset{\varepsilon}{\Rightarrow} id_y \qquad id_x \overset{\eta}{\Rightarrow} {}^*f \circ f$$

s.t.

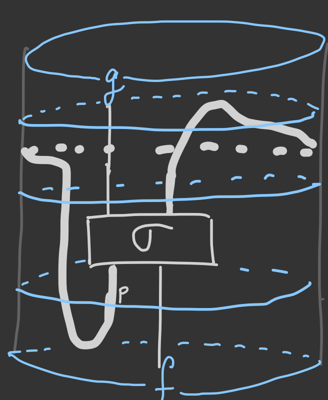
$$f \xrightarrow{1_f \circ \eta} f \circ {}^*f \circ f \xrightarrow{\varepsilon \circ 1_f} f = 1_f$$

$$^*f \xrightarrow{\eta \circ 1_{^*f}} {}^*f \circ f \circ {}^*f \xrightarrow{1_{^*f} \circ \varepsilon} {}^*f = 1_{^*f}$$

Graphically



If so



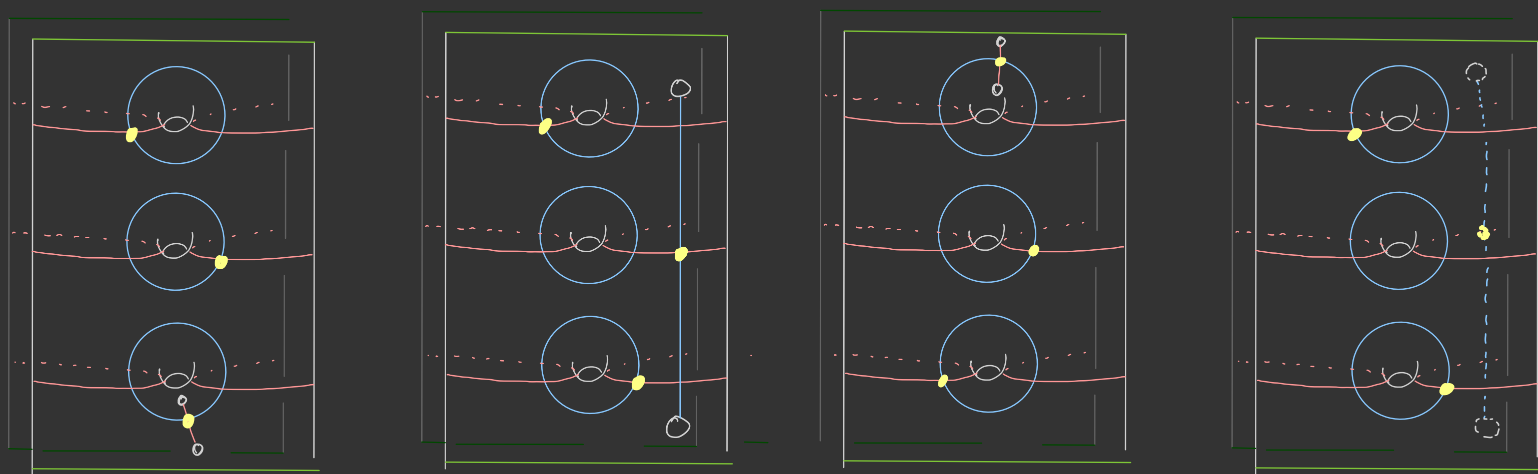
$$CH_*(g)$$
$$\uparrow (id_g \otimes \varepsilon)$$
$$CH_*(g p {}^*p)$$
$$\uparrow \theta$$
$$CH_*({}^*p g p)$$
$$\uparrow (id_{p^*} \otimes \sigma)$$
$$CH_*({}^*p p f)$$
$$\uparrow (\eta \otimes id_f)$$
$$CH_*(f)$$

Get a well-def'd map T to Ch

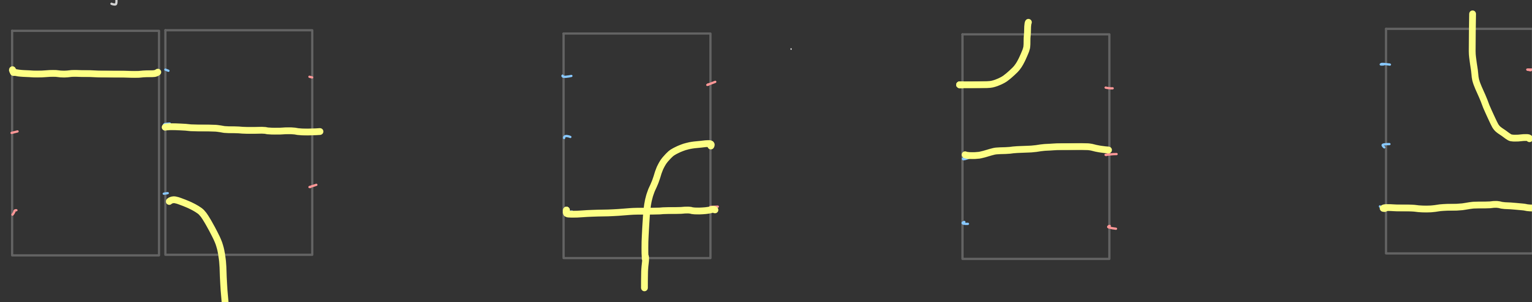
Turns out $\mathcal{E}$, $\mathcal{F}$ have left adjoints!

Prop. (Manion–P.–Wong): ① ${}^*\mathcal{E} = \mathcal{F}$. ② ${}^{***}\mathcal{F} = \mathcal{E}$.

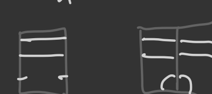
Pf: • We define explicitly bimodules:



$\mathcal{E} = {}^{***}\mathcal{F}$ $\qquad$ $\mathcal{F} = {}^*\mathcal{E}$ $\qquad$ ${}^*\mathcal{F}$ $\qquad$ ${}^{**}\mathcal{F}$



• We define maps $\eta_{\mathcal{E}}, \varepsilon_{\mathcal{E}}$ diagrammatically. e.g. $\eta_{\mathcal{E}}: \mathrm{Id}_n \to \mathcal{F}\mathcal{E}$



… prove they satisfy adjunction rules.

Thm (Manion–P.–Wong):

> $\mathcal{E}, \mathcal{F}, {}^{*}\mathcal{F}, {}^{**}\mathcal{F} \rightsquigarrow e, f, {}^{*}f, e^{*} \circlearrowright \widetilde{CFL}(\hat{T} \sqcup A)$ s.t.
>
> (${}^{**}\mathcal{F} = \mathcal{E}^{*}$; $e, f, {}^{*}f, e^{*}$ are chain maps)
>
> ① $e^2 = 0 = f^2$
>
> ② $ef + fe = \mathbb{I}$
>
> ③ e, f commute with candidate tangle cobordism maps.

Pf: Step 1:

We prove certain properties of the 2-morphisms related to $\mathcal{E}, \mathcal{F}$:

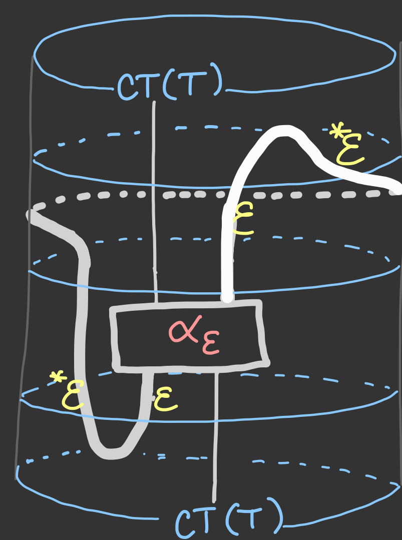
① $\mathcal{E} \tilde{\otimes} \mathcal{E} \xrightarrow{\simeq} 0 \qquad \mathcal{F} \tilde{\otimes} \mathcal{F} \xrightarrow{\simeq} 0$

② $\mathcal{E} \tilde{\otimes} \mathcal{F} \longrightarrow A \longrightarrow \mathcal{F} \tilde{\otimes} \mathcal{E} \longrightarrow \mathcal{E} \tilde{\otimes} \mathcal{F}[1]$

③ $\mathcal{E} \tilde{\otimes} CT(T) \underset{\simeq}{\xrightarrow{\alpha_{\mathcal{E}}}} CT(T) \tilde{\otimes} \mathcal{E}$

$\mathcal{F} \tilde{\otimes} CT(T) \xrightarrow{\simeq} CT(T) \tilde{\otimes} \mathcal{F}$

Step 2:



$$
\begin{array}{c}
CH_*(CT(T)) \simeq CFL(\hat{T} \sqcup A) \\
\uparrow \\
CH_*(CT(T) \tilde{\otimes} \mathcal{E} \tilde{\otimes} {}^{*}\mathcal{E}) \\
\uparrow \\
CH_*({}^{*}\mathcal{E} \tilde{\otimes} CT(T) \tilde{\otimes} \mathcal{E}) \\
\uparrow \\
CH_*({}^{*}\mathcal{E} \tilde{\otimes} \mathcal{E} \tilde{\otimes} CT(T)) \\
\uparrow \\
CH_*(CT(T)) \simeq CFL(\hat{T} \sqcup A)
\end{array}
\Bigg\} =: e
$$

Thm (Manion-P.-Wong):

> The actions $e, f, {}^*f, e^* \circlearrowright \widetilde{CFL}(\hat{T} \sqcup A)$ correspond to the basepoint actions by the 4 basepoints for A.

$ef + fe = 1$

$\rightsquigarrow$

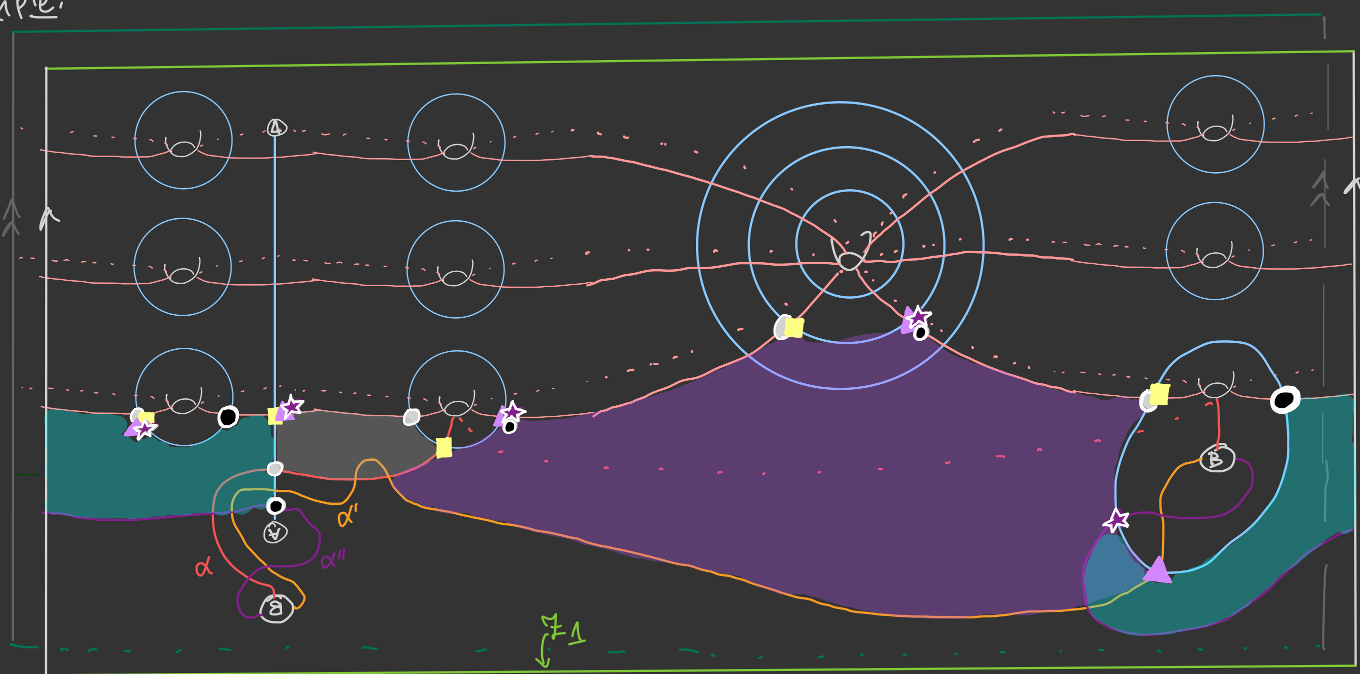
cf. $\partial_z\partial_w + \partial_w\partial_z \sim \mathrm{Id}$,

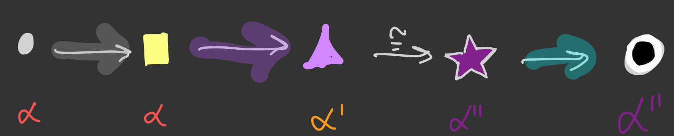
e.g. Baldwin-Levine-Sarkar

Pf: The diagrammatically-defined maps that compose to e correspond to certain polygons on a Heegaard diagram.

Encode all relevant bimodules on a common Heegaard diagram and count!

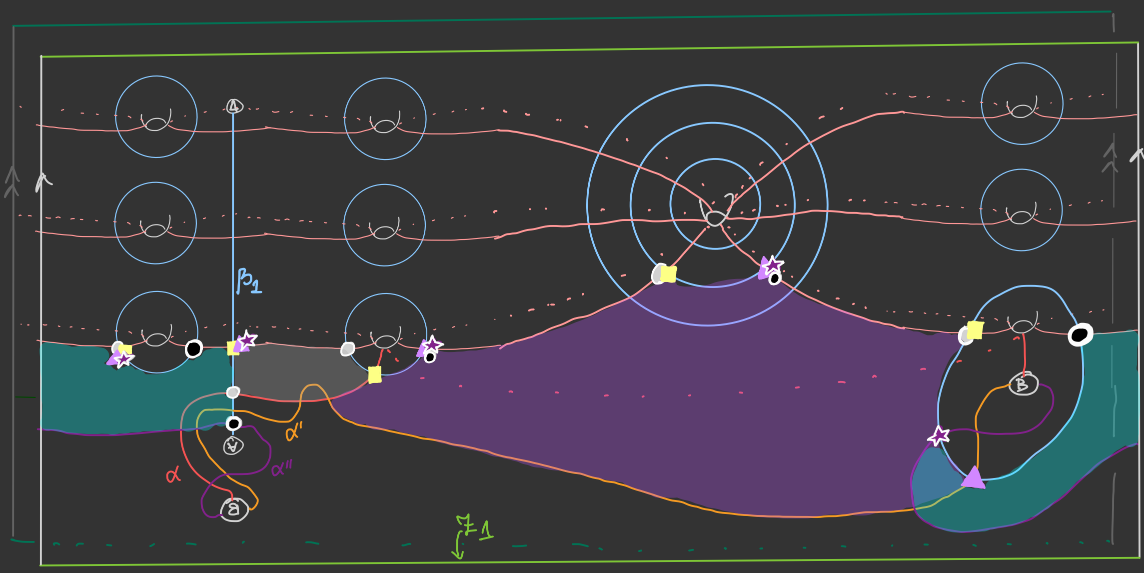
Example:

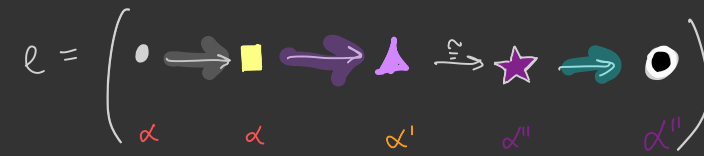


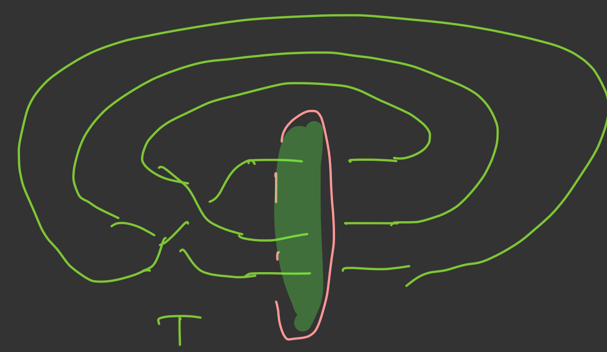
$e =$ 

Thm (MPW) The actions $e, f, {}^*f, e^* \circlearrowright \widetilde{CFL}(\hat{T} \sqcup A)$ correspond to the basepoint actions by the 4 basepoints for A.

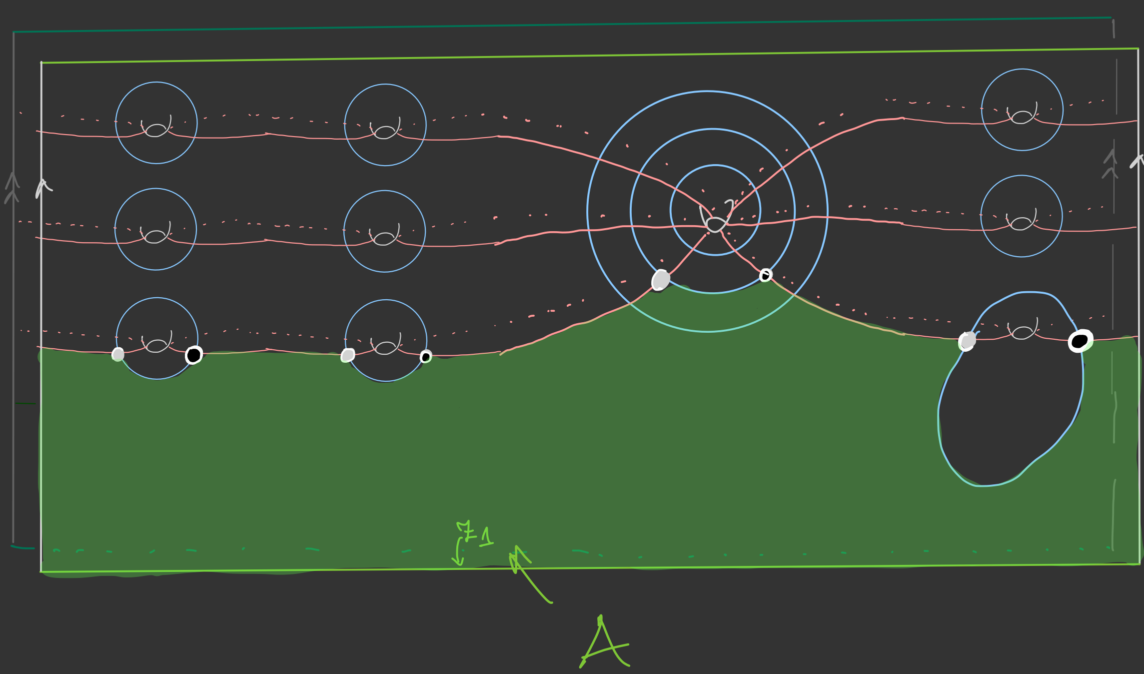
Pf:







$\mathcal{H}$ for $\hat{T} \sqcup A$.



$CH_*(CT(T)) \simeq \widetilde{CFL}(\hat{T} \sqcup A)$

Merci vielmal!