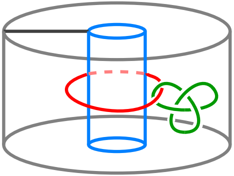
2-Verma modules and link homologies

Pedro Vaz (Université catholique de Louvain)



$$\begin{tikzcd} \mathcal{C}_{\smiley} \arrow[rr] \arrow[d, twoheadrightarrow, "K_0"'] & & \mathcal{M}(\lambda) \arrow[d, twoheadrightarrow, "K_0"'] \\ \mathfrak{g} \arrow[rru] \arrow[rr] & & M(\lambda) = \mathfrak{g} \otimes_{\mathfrak{b}} \mathbb{C}_\lambda \end{tikzcd}$$

J.W.W. $\mathcal{P}\{$Abel Lacabanne, Grégoire Naisse$\}$

RDL(Les Diablerets, Feb. '22) : A conference in $\{0 \subset \mathbb{k}^2 \subset \mathbb{k}^3\}$

Topology ∩ RT ∩ Verma modules

Beyond their interest to RT, Verma modules were recently found to have applications to topology :

- braid grp reps : Burau, Lawrence–Krammer–Bigelow (Jackson–Kerler '11),
  - categorification : Dupont–Naisse '21,
- HOMFLYPT invariants (Naisse–V. '17),
  - categorification : Naisse–V. '17,
- Jones invariants for links in a solid torus (Iohara–Lehrer–Zhang '18),
  - categorification : Lacabanne–Naisse–V. '20.

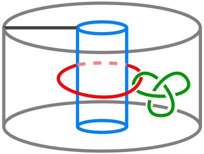
K_0(Today's specials) :

➔ Links in H_1 • Jones poly of type B • blob algebra/Verma modules

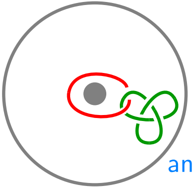
Today's specials :

➔ 2-Rep.Theory • 2-Verma modules • 2-blob algebra.

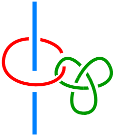
Suppose you have a link in a solid torus



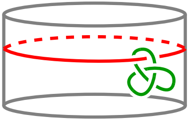
This gives rise to three different kinds of link diagrams :



annular



flagpole

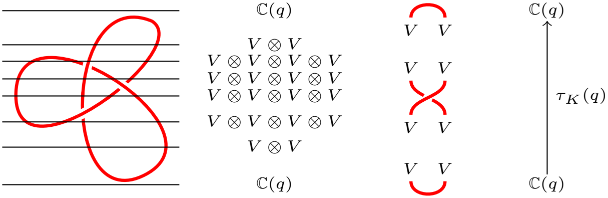


cylinder

Extending invariants like Jones's $\mathfrak{sl}_2$-link polynomial from S^3 to the solid torus results in the Jones poly. of type B (Geck–Lambropoulou '97).

ILZ's blob algebra and the flagpole Jones invariant

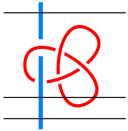
The main idea of WRT is to construct quantum link polynomials via a 0+1 TQFT. Consider (quantum) $\mathfrak{sl}_2$ and its 2-dim irrep $V = \mathbb{C}_q^2 := \mathbb{C}^2(q)$. Since $\mathfrak{sl}_2$ is a Hopf algebra its category of f.d. reps is monoidal. It is even braided...



☞ Operator-invariant of tangles !

$\mathcal{Q}$: What about for links in the solid torus ?

WRT for links in a solid torus : mind the flagpole !



?
?

$\mapsto$?

One possible solution : Vermas (∞-dimensional reps).

$$
\begin{tikzcd}
\cdots \arrow[r, bend right, "{[\lambda;k+1]}"'] & m_{k+1} \arrow[l, bend right, "{[k+2]}"'] \arrow[r, bend right, "{[\lambda;k]}"'] & m_k \arrow[l, bend right, "{[k+1]}"'] \arrow[loop below, "\lambda q^{-2k}"] & \cdots & m_2 \arrow[r, bend right, "{[\lambda;1]}"'] & m_1 \arrow[l, bend right, "{[2]}"'] \arrow[r, bend right, "{[\lambda;0]}"'] & m_0 \arrow[l, bend right, "{[1]}"'] \arrow[loop below, "\lambda"]
\end{tikzcd}
$$

F

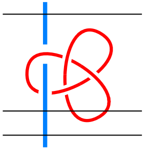
K

E

$$[\lambda;k] = \frac{\lambda q^{-k} - \lambda^{-1} q^{k}}{q - q^{-1}}, \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [\mathrm{E}, \mathrm{F}] = \frac{\mathrm{K} - \mathrm{K}^{-1}}{q - q^{-1}}$$

This is the universal Verma module for $\mathfrak{sl}_2$, irred. over the field $\Bbbk(\lambda, q)$.

Vermas and SW duality : the blob algebra



$$M(\lambda) \xrightarrow{\zeta_K(q,\lambda)} M(\lambda)$$

The blob algebra (Martin–Saleur '94)

Iohara–Lehrer–Zhang '18 : $\mathrm{End}_{\mathfrak{sl}_2}(M(\lambda) \otimes (\mathbb{C}^2_{q,\lambda})^{\otimes r}) = \mathcal{B}_r(\lambda, q)$.

- Generators :  ,  and 

- Relations : planar isotopies,  $= -(q + q^{-1})$,

 $= (\lambda q^2 + \lambda^{-1})$  $- q^2$     $= -(\lambda q + \lambda^{-1} q^{-1})$

Does it generalize ?...yes

- $\mathfrak{p} \subseteq \mathfrak{gl}_m$ (standard) parabolic with Levi $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$,
- $M^{\mathfrak{p}}(\Lambda)$ a (universal) parabolic Verma module.

Lacabanne-V. '20

$\mathcal{H}_{\underline{m}}(d,n) \simeq \mathrm{End}_{\mathfrak{gl}_m}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes d})$ is the 2-row quotient of an Ariki–Koike algebra.

Particular cases (d is the number of blocks in $\mathfrak{p}$) :

- $\mathfrak{p} = \mathfrak{gl}_{m \geq n}$: $\mathcal{H}_{\underline{m}}(d,n) \simeq$ Hecke algebra of type A ($m = 2$: TL).
- $\mathfrak{p}$: $m \geq nd$ and $m_i \geq n$ for all i : $\mathcal{H}_{\underline{m}}(d,n) \simeq$ Ariki–Koike.
- $\mathfrak{p}$: $d = 2$ and $m_1, m_2 \geq n$: $\mathcal{H}_{\underline{m}}(d,n) \simeq$ Hecke algebra of type B with unequal and algebraically independent parameters.
- $\mathfrak{p} = \mathfrak{b}$: $\mathcal{H}_{\underline{m}}(d,n) \simeq \mathcal{B}_{\mathsf{gen}}(d,n) \cong$ generalized blob $\leftrightsquigarrow$ new presentation !

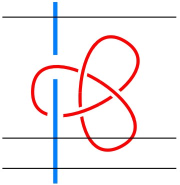
Categorification is coming

We had promised categorification !

we'd like to see ...

... a categorification of the Jones polynomial for links in the solid torus in the flagpole picture via a categorical action of the blob algebra.

- The first step would be to replace the vector spaces $M(\lambda) \otimes (\mathbb{C}^2_{q,\lambda})^{\otimes r}$ by categories, on which
- the commuting 2-actions of $\mathfrak{sl}_2$ and of the blob algebra are realized via (endo-)functors.



$\mathcal{C}(M(\lambda) \otimes V \otimes V)$

$\mathcal{C}(M(\lambda))$

$$\begin{array}{c} \mathcal{C}(M(\lambda)) \\ \Big\uparrow {\scriptstyle \mathcal{F}(\zeta_K(q,\lambda))} \\ \mathcal{C}(M(\lambda)) \end{array}$$

Tensoring without Vermas : KLRW algebras

Khovanov–Lauda–Rouquier–Webster '08-'10

Categorifications of tensor products of f.d. irreps are given through (cyclotomic) KLRW algebras, which are certain diagrammatic algebras.

These are **algebraic categorifications** : (certain) categories of modules over certain algebras on which $\mathfrak{g}$ acts via (certain) endofunctors.

☞ We are interested in (quantum) $\mathfrak{g} = \mathfrak{sl}_2$. The approach consists of :

- replacing weight spaces by categories,

and

- defining functors E, F, $\mathrm{K}^{\pm 1}$ that move between weight spaces and "satisfy" the $\mathfrak{sl}_2$-relations :

$$[\mathrm{E}, \mathrm{F}] = \frac{\mathrm{K} - \mathrm{K}^{-1}}{q - q^{-1}}.$$

Khovanov–Lauda, Rouquier & Webster's catg's

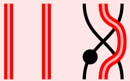
Fix a field $\Bbbk$.

☞ The following exists equally for $\mathfrak{g}$ of symmetrizable type and with the red strands labeled by dominant integral $\mathfrak{g}$-weights.

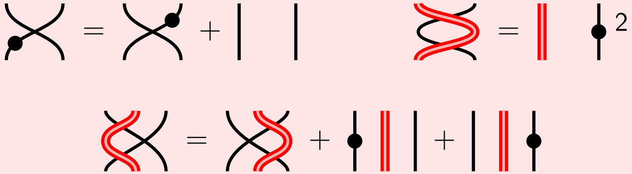
Definition

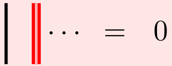
The KLRW algebra T^r is the graded, associative, unital $\Bbbk$-algebra generated by isotopy classes of braid-like diagrams

- Strands can either be black or red (there are r reds)
  - red strands cannot intersect each other.
  - black strands can cross red strands and each other and they can carry dots.
- Multiplication is concatenation.

For example,  (Stendhal diagrams)

Generators are required to satisfy local relations. For example :





Cyclotomic

The cyclotomic condition makes T^r f.d. Without it we have an affine algebra : call it T^r_{aff}.

Categorical $\mathfrak{sl}_2$-action

Adding a black strand at the right of a diagram from T^r gives rise to functors of *induction* (called F) and *restriction* (called E) on a category $T\text{-}mod_g$ of modules over $T = \oplus_{r\geq 0} T^r$. The functor K is defined as a grading shift.

These functors have very nice properties...

Theorem *(Webster '10) :*

▷ They are *biadjoint* (a.k.a. Frobenius)

▷ the composites EF and FE satisfy $\oplus$ *decompositions* lifting the $[\ ,\ ]$ (on f.d. reps K acts as a polynomial dividing $q-q^{-1}$). On $T^r\text{-}mod_g$:

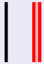
$$\mathrm{EF} \simeq \mathrm{FE} \oplus_{p(r)} \mathrm{Id} \quad \left(\text{or} \quad \mathrm{FE} \simeq \mathrm{EF} \oplus_{p(r)} \mathrm{Id} \right)$$

▷ Moreover, $K_0(T^r) \cong V^{\otimes r}$ (as $\mathfrak{sl}_2$)-modules)

☞ $\exists M > 0$ such that F^M and E^M act as zero on $T{-}mod_g$

Categorification of $\otimes$s with a Verma

The idea is to see T^r as a dg-algebra with zero differential and "integrate" the cyclotomic condition



$$\cdots \; = \; 0$$

into a *dg-algebra* $(\mathcal{T}^r, \partial)$, together with a isomorphism $H(\mathcal{T}^r, \partial) \simeq T^r$.

A similar idea was carried out in the 70' : *minimal models* in $\mathbb{Q}$-homotopy theory (Sullivan,...)

This is how it goes :

- To construct such an algebra we note that T^r_{aff} acts on T^r.
- Writing a free resolution of T^r as a module over T^r_{aff} one gets a DGA $(\mathcal{T}^r, \partial)$ whose homology is T^r.

Testing the idea...

We intend to categorify the rational fraction $\frac{\lambda q^{-k}-\lambda^{-1}q^{k}}{q-q^{-1}}$ (which we see as a power series).

- We know that a categorification of multiplication by $[n]$ is $\mathbb{Q}[X]/X^n$ (secretly this is $H(G_1(n))$) via grading shifts of some id. functor $\oplus_{[n]}\mathrm{Id}$

- But $\mathbb{Q}[X]/X^n$ is a module over $\mathbb{Q}[X]$ (secretly this is $H(G_1(\infty))$) for which

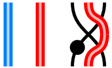
$$\mathbb{Q}[X]/X^n \longleftarrow \mathbb{Q}[X] \xleftarrow{X^n} \mathbb{Q}[X]$$

gives as a free resolution (grading shifts involved !).

- We can write this as a DGA $(\mathbb{Q}[X,\omega]/\omega^2, \partial)$ with $\partial X = 0$, $\partial\omega = X^n$, which has homology $\mathbb{Q}[X]/X^n$.

- Tensoring M with $\mathbb{Q}[X,\omega]/\omega^2$ gives ... $\frac{\lambda q^{-k}-\lambda^{-1}q^{k}}{q-q^{-1}}[M]$ (☺ hooray !).

One can give a diagrammatic presentation of $(\mathcal{T}^r, \partial)$

(Stendhal with a blue)  

New generator !
homological deg 1

dg-enhancement of cyclotomic KLRW algebras

The differential "turns" $\mathcal{T}^r$ into the f.d. algebra T^r.

When no differential is present : new λ-grading.

Now just **forget** *there is a differential.*

To define an $\mathfrak{sl}_2$-categorical action we use the map that adds a vertical black strand at the right of a diagram from $\mathcal{T}^r$: this defines functors F and E as before.

- They are not biadjoint, and
- $\exists M$: $\mathrm{E}^M(\mathcal{T}\text{-}mod) = 0$, but **no such** M **exists** for F.

Categorification of tensor products with a Verma

Theorem (Lacabanne–Naisse–V. '20)

These functors fit in a SES

$$0 \to \mathrm{EF} \longrightarrow \mathrm{FE} \longrightarrow \oplus_p \mathrm{Id} \to 0,$$

Bringing ∂ back into the picture we can define analogous of the functors F and E on $\mathcal{D}_{dg}(\mathcal{T}, \partial)$

This results in a SES of complexes whose resulting LES in homology recovers Webster's result for $\otimes$ of f.d. reps.

Theorem (Lacabanne–Naisse–V. '20)

There are isomophisms of $\mathfrak{sl}_2$-modules

$$\mathbf{K}_0^{\Delta}(\mathcal{T}^r, 0) \cong M(\lambda) \otimes V^{\otimes r},$$

$$\mathbf{K}_0^{\Delta}(\mathcal{T}^r, \partial) \cong V^{\otimes (r+1)}.$$

A categorical blob action

There are endo functors on $\mathcal{T}$-mod categorifying the blob algebra action : needs A-infinity stuff

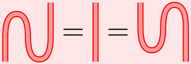
⚠ On our order to prove the (categorical) blob relations one needs to go to the world of A_∞-bimodules.

As Webster, we define the cup B_i and the cap $\overline{B}_i$ functor for $1 < i \leq r-2$. They are defined diagrammatically : 

Proposition (Lacabanne–Naisse-V. '20)

There are quasi-isomorphisms of A_∞-bimodules

$$\mathcal{T}^r \xrightarrow{\simeq} \bar{B}_{i\pm 1} \otimes^{\mathrm{L}}_{\mathcal{T}} B_i,$$



$$q(\mathcal{T}^r)[1] \oplus q^{-1}(\mathcal{T}^r)[-1] \xrightarrow{\simeq} \bar{B}_i \otimes^{\mathrm{L}}_{\mathcal{T}} B_i.$$

 $= -q - q^{-1}$

We can define a double braiding functor Ξ in the same spirit : a certain diagram modulo relations.

Proposition (Lacabanne–Naisse-V. '20)

1. The functor $\Xi : \mathcal{D}_{dg}(\mathcal{T}^r, 0) \to \mathcal{D}_{dg}(\mathcal{T}^r, 0)$ is an autoequivalence, with inverse given by $\Xi^{-1} := \mathrm{RHOM}_T(X, -)$.

2. There are natural isomorphisms of functors

$$\mathrm{E} \circ \Xi \cong \Xi \circ \mathrm{E},$$
$$\mathrm{E} \circ \mathsf{B}_i \cong \mathsf{B}_i \circ \mathrm{E},$$

and

$$\mathrm{E} \circ \overline{\mathsf{B}}_i \cong \overline{\mathsf{B}}_i \circ \mathrm{E}.$$

(similarly for F in the place of E).

Theorem (Lacabanne–Naisse–V. '20)

1. There is a quasi-isomorphism of functors :

$$\mathrm{Cone}\Big(\lambda q^2\Xi[1] \to q^2\,\mathrm{Id}[1]\Big)[1] \xrightarrow{\simeq} \mathrm{Cone}\Big(\Xi\circ\Xi \to \lambda^{-1}\Xi\Big).$$

☞ This corresponds to λq^2  $- q^2$  $=$  $- \lambda^{-1}$ 

2. There is a quasi-isomorphism of A_∞-bimodules :

$$\lambda q(\mathcal{T}^r)[1] \oplus \lambda^{-1}q^{-1}(\mathcal{T}^r)[-1] \xrightarrow{\simeq} \bar{B}_1 \otimes^{\mathrm{L}}_{\mathcal{T}} X \otimes^{\mathrm{L}}_{\mathcal{T}} B_1.$$

☞ This corresponds to $-(\lambda q + \lambda^{-1}q^{-1})$ $=$ 

Link homology : so far so good ... so what ?

- As Webster, we can define functors for the (type A) braid generators.

- A link diagram with a flagpole then gives a functor from $\mathcal{D}_{dg}(\mathcal{T}^0, 0)$ to itself, categorifying the Jones invariant.

At the time being we cannot tell much about its properties...

- At the time being we have a link homology for links in the solid torus coming from commuting categorical actions of $\mathfrak{sl}_2$ and the blob algebra on $\mathcal{D}_{dg}(\mathcal{T}^r, 0)$:

$$\begin{tikzcd}
& \mathcal{D}_{dg}(\mathcal{T}^r, 0) \arrow[d, "K_0"] & \\
\mathfrak{sl}_2 \arrow[ur, dashed, "x \mapsto \mathcal{F}_x"] \arrow[r] & M(\lambda) \otimes V^{\otimes r} & \mathcal{B}_r(q, \lambda) \arrow[ul, dashed, "b \mapsto \mathcal{F}_b"'] \arrow[l]
\end{tikzcd}$$

Some hurdles to cross

- If you want to formalize this diagram, or if you aim at categorifying the blob algebra as an algebra of operators on $\mathcal{D}_{dg}(\mathcal{T}^r, 0)$, you might prefer instead to (the bottom arrow is a homomorphism)

$$
\begin{tikzcd}
 & \mathrm{Fun}(\mathcal{V}) \arrow[d, "K_0"] & \text{Category} \\
A \arrow[ur, dashed, "a \mapsto \mathcal{F}_a"] \arrow[r] & \mathrm{End}_{\Bbbk}(V) & \text{vec. space}
\end{tikzcd}
$$

and ask what is $\mathrm{END}_{\mathfrak{sl}_2}(\mathcal{V}) \subset \mathrm{Fun}(\mathcal{V})$ and so on...
- Diagrams like the above work perfectly if the $\mathcal{F}_a$'s are exact functors acting on an additive or abelian category. We can then ask natural questions like irreducibility, $\otimes$s, etc...

In our case, $\mathcal{F}_a$'s are triangulated functors.
Categories of triangulated functors are in general not triangulated :
$\rightsquigarrow$ $\mathrm{Fun}(\mathcal{V})$ is not a good choice here.

Still some hurdles to cross

One possible solution is to work on a DG-enrichment of our triangulated category $\mathcal{D}_{dg}(\mathcal{T}^r, 0)$, which is triangulated.

- Our functors lift to DG-functors and we can ask for the smallest triangulated 2-category containing them.

- Notions like irreducibility become tricky (sometimes modding out by an (invariant ideal) of morphisms ruins the triangulated structure...).

DG-lifts of triangulated functors not always exist.

A further step would be to construct a 2-category $\mathcal{C}_{\smiley}(\mathfrak{g})$ such that

$$\begin{tikzcd} \mathcal{C}_{\smiley}(\mathfrak{g}) \arrow[r, "\text{2-functor}"] \arrow[d, "K_0"'] & \mathcal{C}^{\triangle}(\mathcal{V}) \arrow[d, "K_0"] \\ \mathfrak{g} \arrow[r, "\text{functor}"] \arrow[ur, dashed] & \mathrm{End}_{\Bbbk} V \end{tikzcd}$$

Some questions are best not left unanswered : a blob 2-category

One can give a definition of a blob 2-category as a certain $(\infty, 2)$-category :

- The objects r are the dg-categories $\mathcal{D}_{dg}(\mathcal{T}^r, 0)$
- The $\mathrm{Hom}(\mathrm{r}, \mathrm{r}')$ are (Lurie's dg nerves of) dg-categories of certain subcategory of dg-functors $\mathcal{D}_{dg}(\mathcal{T}^r, 0) \to \mathcal{D}_{dg}(\mathcal{T}^r, 0)$ generated by all compositions of Ξ, B_i and $\overline{B}_i$, and the identity functor whenever $r = r'$.

Theorem (Lacabanne–Naisse–V. '20)

There is an isomorphism of categories

$$\mathbf{K}_0^{\Delta}(\mathfrak{B}) \cong \mathcal{B}.$$

rk : $\mathcal{B} = \oplus_{r,r' \geq 0} \mathcal{B}(r, r')$, $\mathcal{B}(r, r)$ being the blob algebra $\mathcal{B}_r(\lambda, q)$

Thank you for the attention!

Schur–Weyl sunshine

Links in H_1

$\mathcal{C}^{\triangle}(\mathcal{D}_{dg}(\mathcal{T}^r))$

$\mathfrak{B}(r,r)$