arXiv:2107.08117

A SKEIN RELATION FOR SINGULAR SOERGEL BIMODULES

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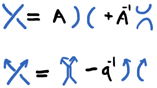
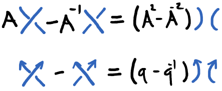
RDLH / [K-OS]

3rd February 2022

I Intro/Review
II Skein relation
III Proof

Two kinds of skein relations?





Type P
- cube of resolution
- simplification algo
- use webs

Type I
- no webs?
- signs on both sides

Goal today:

A type I skein relation for Λ-colored HOMFLYPT/$\mathfrak{gl}_N$, categorified

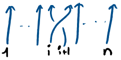
Why care?
- skein algebras
  e.g. Eugene's talk
- inductive arguments
  e.g. Anton's talk
  . . .

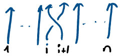
- hard to compute
- families of colored link invts.
  may be relevant for manifold invts.
  . . .

I Intro

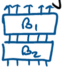
I.1 $Br_n :=$ braid group on n strands

$$= \left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & 1\le i<n-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \end{array} \right\rangle$$

$\sigma_i \longleftrightarrow$ 

$\sigma_i^{-1} \longleftrightarrow$ 

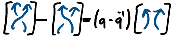
group operation $\longleftrightarrow$ stacking

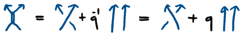
$\beta_1 \beta_2 \longleftrightarrow$ 

I.2

$H_n :=$ Hecke algebra of type A_{n-1}

$$= \mathbb{Z}[q^{\pm 1}]Br_n \Big/ \big\langle \underbrace{\sigma_i - \sigma_i^{-1} = (q-q^{-1})\cdot 1}\big\rangle$$

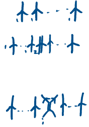


Alternative generators: 

$$b_i = \sigma_i + q^{-1} = \sigma_i^{-1} + q$$

I.3 Soergel bimodules. Fix $n \geq 0$.

cheat sheet for bimodules



- $R := \mathbb{C}[x_1, \dots, x_n]$ $\quad \deg(x_i) := 2$

  $\cup$
- $R^{S_i} :=$ invariants under $x_i \leftrightarrow x_{i+1}$

  $= \mathbb{C}[x_1, \dots, x_i + x_{i+1}, x_i x_{i+1}, \dots, x_n]$
- $B_i := q^{-1} R \otimes_{R^{S_i}} R$

  ↖ grading shift down

$SBim_n :=$ full subcat of graded R-R-bimodules
- containing R, B_i $1 \leq i \leq n-1$
- closed under $\otimes_R$, $\oplus$, $\overset{\subseteq}{\oplus}$, grading shift.

– a graded, $\mathbb{C}$-linear (not abelian), monoidal category

$$K_0(SBim_n) \cong H_n$$

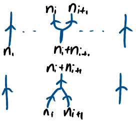
More generally: $SSBim_n :=$ full sub-2-cat of graded bimodules

"singular"

- built from

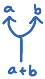
$${}_{R^{S_{n_1}\times\cdots\times S_{n_m}}}\left(R^{S_{n_1}\times\cdots\times S_{n_m}}\right)_{R^{S_{n_1}\times\cdots\times S_{n_i+n_{i+1}}\times\cdots\times S_{n_m}}}$$

$$q^{-\bullet}\;{}_{R^{S_{n_1}\times\cdots\times S_{n_i+n_{i+1}}\times\cdots\times S_{n_m}}}\left(R^{S_{n_1}\times\cdots\times S_{n_m}}\right)_{R^{S_{n_1}\times\cdots\times S_{n_m}}}$$



- closed under $\otimes$, $\oplus$, $\overset{\subseteq}{\oplus}$, grading shift.

– a graded, $\mathbb{C}$-linear (not abelian) 2-category

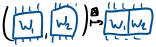
Example 

$$\cong \frac{\mathrm{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{D})}{\big\langle e_k(\mathbb{D}) - \sum_{i+j=k} e_i(\mathbb{A})e_j(\mathbb{B}) \,\big|\, k \geq 0\big\rangle}$$

$|\mathbb{A}| = a$, $|\mathbb{B}| = b$, $|\mathbb{D}| = a+b$

↖ forces $f(\mathbb{D}) = f(\mathbb{A} \cup \mathbb{B})$
$\forall$ sym fcns f.

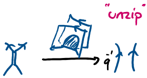
Remark $\boxtimes := \otimes_{\mathbb{C}}$ $\quad SBim_n \times SBim_m \to SBim_{m+n}$
$\quad SSBim_n \times SSBim_m \to SSBim_{m+n}$



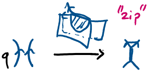
turns $\bigoplus_{n\geq 0} SBim_n$ and $\bigoplus_{n\geq 0} SSBim_n$ into monoidal 2-categories.

I.4 Rouquier complexes

$$T_i := \left(0 \longrightarrow \underline{B_i} \longrightarrow q^{-1} R \right)$$
$$1\otimes 1 \longmapsto 1$$



$$T_i^{-1} := \left(qR \longrightarrow \underline{B_i} \longrightarrow 0 \right)$$
$$1 \longmapsto x_i\otimes 1 - 1\otimes x_{i+1}$$



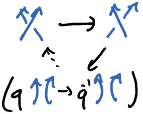
Theorem (Rouquier) The complexes constructed from T_i, T_i^{-1} via $\otimes_R$ satisfy the braid relations up to canonical homotopy equivalence.

Aside: HH_* of Rouquier cx computes triply-graded Khovanov-Rozansky link homology of braid closure.

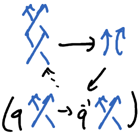
Type I skein relation:

$$\mathrm{Cone}\left(\begin{tikzcd} 0 \arrow[r] & \underline{B_i} \arrow[r] & q^{-1}R \\ qR \arrow[r] & \underline{B_i} \arrow[r] \arrow[u, leftarrow, "id"] & 0 \end{tikzcd}\right) \simeq \left(qR \xrightarrow{x_i - x_{i+1}} q^{-1}R \right)$$

get distinguished triangles

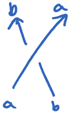


equiv:

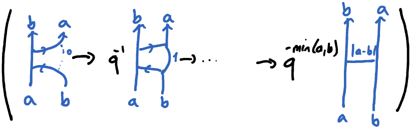


II Rickard complexes, skein relation

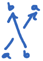
II.1 Rickard (Chuang-Rouquier) complexes



$\longmapsto C_{a,b} :=$



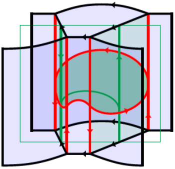
& similarly for

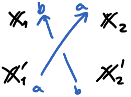


.

- chain complexes of singular Soergel bimodules.
  $a=b=1$: $C_{1,1}$ = Rouquier cx
- from categorical $\mathfrak{sl}_2$ actions

differential — as foam
— in thick $\mathfrak{sl}_2$ calculus

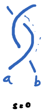


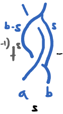
- satisfy braid relations
  (Cautis-Kamnitzer-Licata, Webster-Williamson..):
- HH_* categorifies Λ-colored HOMFLYPT
- 

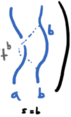
  for any symmetric function f:
  actions: $f(\mathbb{X}_1) \sim f(\mathbb{X}_2')$
  $f(\mathbb{X}_2) \sim f(\mathbb{X}_1')$
  "dot sliding"
- a skein relation for $C_{a,b}$ is expected to involve $\min(a,b)+2$ terms.

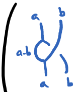
II.2 The skein relation

Theorem (HRW 21) For any $a,b \in \mathbb{Z}_{\geq 0}$ there exists a homotopy equiv. of one-sided twisted complexes:

$tw_\alpha\Big($ 

$\to \cdots q^{s(b-1)} t^s$ 

$\dashrightarrow q^{b(b-1)} t^b$ 

$\Big) \simeq q^{b(a-b-1)} t^b \, tw_B\Big($ 

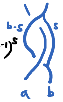
$\otimes \Lambda^\bullet \mathbb{k}^b \Big)$

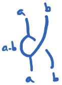
"complex of threaded digons"
twist increases s
possible long arrows omitted

Koszul complex for the action of
$h_k(\mathbb{X}_2-\mathbb{X}_2') = \sum_{i+j=k} h_i(\mathbb{X}_2)(-1)^j e_j(\mathbb{X}_2')$
Web ≃ 0 unless $b \leq a$

⇝ complex of threaded digons detects thickness of thread.

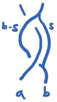
Corollary (HRW 21) upon decategorifying

$\sum_{s=0}^{b} (-q^{b-1})^s$ 

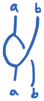
$= (-1)^b q^{b(a-b-1)} \prod_{i=1}^{b} (1-q^{2i})$ 

in HOMFLYPT skein theory
a.k.a Schur algebroid

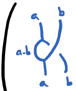
Notation: fix $a,b \in \mathbb{Z}_{\geq 0}$

- 

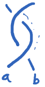
=: MCCS^s (merge crossing split)

- 

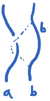
=: MCS (merge crossing split)

- $tw_B\Big($ 

$\otimes \Lambda^\bullet \mathbb{k}^b \Big)$ =: KMCS (koszul)

MCCS^0 = 

full twist

MCCS^b = 

identity

Skein relation in compact form:

$$tw_\alpha\Big(\bigoplus_{s=0}^{b} q^{s(b-1)} t^b \, \mathrm{MCCS}^s\Big) \simeq q^{b(a-b-1)} t^b \, \mathrm{KMCS}$$

Proof strategy ① simplify RHS as much as possible
KMCS → $\overline{\mathrm{KMCS}}$

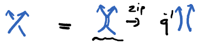
② show $\overline{\mathrm{KMCS}}$ has a natural filtration with subquotients denoted
$\overline{\mathrm{MCCS}}^s$ $s = 0 \cdots b$ (omitting shifts)

③ show $\overline{MCCS}^s \simeq MCCS^s$

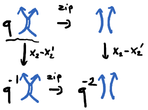
- ③a $\overline{MCCS}^0$ is a minimal complex for the full twist as conjectured by Beliakova – Habiro
  - ③b $s = 1 \ldots b$.

Example: toy case $a=b=1$

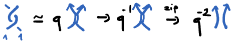
① $MCS =$



② $q^{-1} t \, KMCS =$



③ $MCCS^0 =$



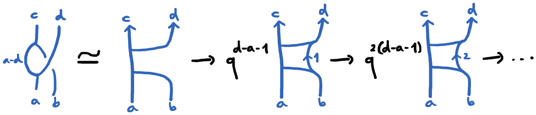
$= \overline{MCCS}^0$

$MCCS^1 =$

$= \overline{MCCS}^1$

III Proof of the shein relation

Lemma Let $a,b,c,d \in \mathbb{Z}_{\geq 0}$, $a+b=c+d$, then:

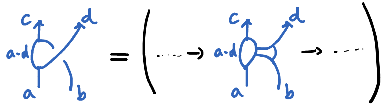


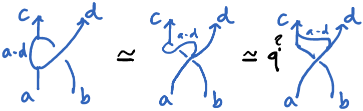
RHS
- is finite, indecomposable, differentials unique up to $\neq 0$ scalars,
- call it a shifted Rickard complex
- notation ${}_{(c,d)}C_{(a,b)}$
- ${}_{(b,a)}C_{(a,b)} = C_{a,b}$, ${}_{(c,d)}C_{(a,b)} \simeq 0$ if $a<d$
- for $a=c, b=d$:

Define:

$$MCS \simeq {}_{(a,b)}C_{(a,b)} =: \overline{MCS}$$

Proof of Lemma:

brute force:  ≃ .... simplify ....

better:  ≃ $\underset{\text{calculus KLMS Cautis}}{\text{U(sl}_2\text{) thick}}$ $_{(c,d)}C_{(a,b)}$

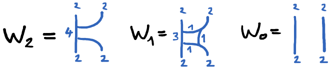
□

Definition Let X be a dg $\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}_2')$-module, $|\mathbb{X}_2|=|\mathbb{X}_2'|=b$.
Consider $\Lambda := \Lambda[\xi_1,\dots,\xi_b]$ with $\deg(\xi_i) = q^{2i}t^{-1}$ and define:

$$KX := \mathrm{tw}_{\sum_{i=1}^{b} h_i(\mathbb{X}_2-\mathbb{X}_2')\otimes\xi_i^*}(X\otimes\Lambda)$$

Corollary: $KMCS \simeq K\overline{MCS} =: \overset{\text{Defines}}{\overline{KMCS}}$ ① ✓ simplified RHS

Example $a=b=2$

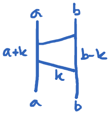
$\overline{MCS} = W_2 \to W_1 \to W_0$ $W_2 =$ $W_1 =$ $W_0 =$ 

$\overline{KMCS}$:

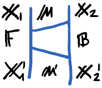
$$\begin{tikzcd}
 & W_2\otimes\xi_1\xi_2 \arrow[rrr] \arrow[dr] \arrow[ddl] & & & W_1\otimes\xi_1\xi_2 \arrow[rrr] \arrow[dr] \arrow[ddl] & & & W_0\otimes\xi_1\xi_2 \arrow[dr] \arrow[ddl] & \\
 & & W_2\otimes\xi_2 \arrow[rrr] \arrow[ddl] & & & W_1\otimes\xi_2 \arrow[rrr] \arrow[ddl] & & & W_0\otimes\xi_2 \arrow[ddl] \\
W_2\otimes\xi_1 \arrow[rrr] \arrow[dr] & & & W_1\otimes\xi_1 \arrow[rrr] \arrow[dr] & & & W_0\otimes\xi_1 \arrow[dr] & & \\
 & W_2\otimes 1 \arrow[rrr] & & & W_1\otimes 1 \arrow[rrr] & & & W_0\otimes 1 & \\
 & & & & & & & & \\
 & & & & & & & & \overline{\mathrm{MCS}}_{2,2}\otimes\Lambda \arrow[l, leftrightarrow] \arrow[uu, leftrightarrow]
\end{tikzcd}$$

② The filtration on $\overline{KMCS}$

$\overline{KMCS}$ has Koszul columns



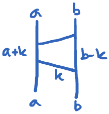
$\otimes \Lambda[\xi_1, \dots, \xi_b] \qquad \delta^v(\xi_i) = h_i(\mathbb{X}_2 - \mathbb{X}_2')$



base change to exterior variables

$$g_j^{(k)} := \sum_{i=1}^{j} (-1)^{i-1} e_{j-i}(\mathbb{M}) \otimes \xi_i$$

new presentation for k-th column:



$\otimes \underbrace{\Lambda[g_1^{(k)}, \dots, g_b^{(k)}]}_{}$ $\qquad \delta^v(g_i^{(k)}) = e_i(\mathbb{M}) - e_i(\mathbb{M}')$

partition set of var's

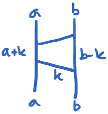
$\{g_1^{(k)} \dots g_k^{(k)}\} \quad \{g_{k+1}^{(k)}, \dots, g_b^{(k)}\}$

this is $\equiv 0$ for $i > k$

since $|\mathbb{M}| = |\mathbb{M}'| = k$

Decompose the column:

$P_{k,e,s} := q^{\bullet} t^{\bullet}$



$\otimes \overset{e}{\Lambda}[g_1^{(k)}, \dots, g_k^{(k)}] \otimes \overset{s}{\Lambda}[g_{k+1}^{(k)}, \dots, g_b^{(k)}]$

Proposition

$$q^{\bullet} t^{\bullet} \overline{\mathrm{KMCS}} \cong \mathrm{tw}_{\delta^v + \delta^h + \delta^c} \bigoplus_{0 \le e \le k \le b-s} P_{k,e,s}$$

with $\delta^v : P_{k,e,s} \to P_{k,e-1,s}$

$\delta^h : P_{k,e,s} \to P_{k-1,e,s}$

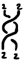
$\delta^c : P_{k,e,s} \to P_{k-1,e-1,s+1}$

so s defines a filtration with subquotients

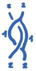
$$q^{\bullet} t^{\bullet} \overline{\mathrm{MCCS}}^{s} := P_{\bullet,\bullet,s} := \mathrm{tw}_{\delta^v + \delta^h} \bigoplus_{0 \le e \le k \le b-s} P_{k,e,s}$$

e.g.:

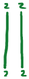
$\overline{\mathrm{MCCS}}^{0} \simeq$



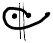
$\overline{\mathrm{MCCS}}^{1} \simeq$

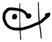


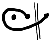
$\overline{\mathrm{MCCS}}^{2} =$



$$\begin{tikzcd}
 & P_{\bullet,\bullet,0} & & & P_{\bullet,\bullet,1} & & P_{\bullet,\bullet,2} & & \\
 & \mathbf{q}^{-2}\underline{W_2\cdot\zeta_1^{(2)}\zeta_2^{(2)}\cdot 1} \arrow[rrr, "\chi_0^+"] \arrow[dr, "e_1-e_1'"] \arrow[ddl, "e_2'-e_2"'] & & & \mathbf{q}^{-3}W_1\cdot\zeta_1^{(1)}\cdot\zeta_2^{(1)} \arrow[rr, "\chi_0^+"] \arrow[dr, "e_1'-e_1"] & & \mathbf{q}^{-4}W_0\cdot 1\cdot\zeta_1^{(0)}\zeta_2^{(0)} & & \delta^v \\
 & & \mathbf{q}^{-2}W_2\cdot\zeta_2^{(2)}\cdot 1 \arrow[rrr, "\chi_0^+"] \arrow[dr, "\chi_1^+"] \arrow[ddl, "e_2-e_2'"] & & & \mathbf{q}^{-3}W_1\cdot 1\cdot\zeta_2^{(1)} \arrow[rr, "\chi_0^+"] \arrow[dr, "\chi_1^+"] & & \mathbf{q}^{-4}W_0\cdot 1\cdot\zeta_2^{(0)} & \delta^h \\
\mathbf{q}^{-2}W_2\cdot\zeta_1^{(2)}\cdot 1 \arrow[rrr, "\chi_0^+"] \arrow[dr, "e_1-e_1'"] & & & \mathbf{q}^{-3}W_1\cdot\zeta_1^{(1)}\cdot 1 \arrow[rrr, "\chi_0^+"] \arrow[dr, "e_1'-e_1"] & & & \mathbf{q}^{-4}W_0\cdot 1\cdot\zeta_1^{(0)} & & \delta^c \\
 & \mathbf{q}^{-2}W_2\cdot 1\cdot 1 \arrow[rrr, "\chi_0^+"] & & & \mathbf{q}^{-3}W_1\cdot 1\cdot 1 \arrow[rrr, "\chi_0^+"] & & & \mathbf{q}^{-4}W_0\cdot 1\cdot 1 & \\
 & \underbrace{\qquad}_{P_{2,\bullet,\bullet}} & & & \underbrace{\qquad}_{P_{1,\bullet,\bullet}} & & & \underbrace{\qquad}_{P_{0,\bullet,\bullet}} &
\end{tikzcd}$$

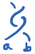






③ Identifying the subquotients

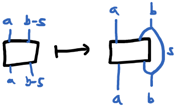
$\overline{MCCS}^0$ is the image of (a version of) the Beliakova-Habiro categorified ribbon element of quantum $\mathfrak{sl}_2$ in dg singular SBim.

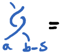
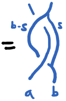
Conjecturally $\simeq$  $= MCCS^0$

Theorem $\overline{MCCS}^0 \simeq MCCS^0$ (3a) ✓

↳ Proof outline below.

Corollary $\overline{MCCS}^s \simeq MCCS^s$ (3b) ✓

↳ Proof: use dg functor 

which sends  $= MCCS^0_{a,b-s} \longmapsto MCCS^s_{a,b} =$ 

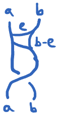
and check $\overline{MCCS}^0_{a,b-s} \longmapsto \overline{MCCS}^s_{a,b}$

using foams. □

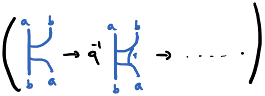
Proof outline for Theorem ($\overline{MCCS}^0 \simeq MCCS^0 =$ full twist)

- the rows of $\overline{MCCS}^{\circ}$

$$R_e := tw_{\delta^h} \bigoplus_{e\le k\le b} P_{k,e,0} \simeq q^{\bullet}t^{\bullet}$$



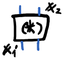
- $\overline{MCCS}^{\circ} \circ$

is a complex 

$=: (*)$

- Recognition principle

$(*) \cong C_{b,a} \iff (*)$ is indecomposable

$\iff \forall r>0 : h_r(\mathbb{X}_2 - \mathbb{X}_1')$ acts nullhomotopically on 

$\Uparrow$

Construct explicit dot-sliding homotopies on $\overline{KMCS}$ and its subquotients + dot sliding for

$\square$

Remark [HRW 21b]: y-ified colored homology generalizing work of Gorsky-H and Cautis-Lauda-Sussan:

$\Longrightarrow$ Type I colored skein relation y-ifies term-wise!

Application: Hopf link invariant & splitting map in y-ified colored homology.