$\mathfrak{gl}_0$ IS THE MIDDLE MAN
LECTURE NOTES FOR "CATEGORIFICATION IN LOW DIMENSIONAL TOPOLOGY"

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ABSTRACT. These notes are meant to cover the two first lectures of the mini-course with the same title given in Bochum in July 2025 for the summer school *Categorification in Low Dimensional Topology*. The aim of the minicourse is to give detail on a spectral sequence from the reduced triply graded homology to Knot Floer homology constructed in [BPRW25]. In these two first lectures, we aim to introduce $\mathfrak{gl}_0$-homology which is yet another knot homology theory which plays a key role for the construction of the spectral sequence.

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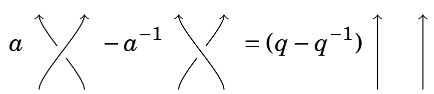
These notes are probably in a very imperfect shape. If you spot a mistake, have a question or a comment, please send an email to louis-hadrien.robert@uca.fr. I will make my best to correct mistake and keep this document up to date at the following URL: https://lrobert.perso.math.cnrs.fr/Talk/bochum.pdf

1. INTRODUCTION

The HOMFLY-PT polynomial is a link invariant satisfying the skein relation:

(1)



They are two variants for this invariant, the normalized one for which the value of the unknot is 1 and the un-normalized one for which it is $\frac{a-a^{-1}}{q-q^{-1}}$. In particular, the HOMFLY-PT polynomial is in general not a polynomial.

Polynomial	Homology
Jones	Khovanov
$\mathfrak{gl}_N$	Khovanov–Rozansky
Alexander	Knot Floer
HOMPLY-PT	Triply graded

TABLE 1. Classical categorifications

When one specializes the HOMPLY-PT polynomial by setting $a = q^2$ one obtains the (normalized or un-normalized) Jones polynomial. More generally the specialization $a = q^N$ gives the $\mathfrak{gl}_N$ polynomial. Setting $a = 1$ in the normalized HOMFLY-PT polynomial gives the Alexander polynomial.

All these links invariants have been categorified (sometimes more than once) by link homology theory. Table 1 sums up the most classical caltegorifications.

The specializations which relate the HOMPLY-PT polynomial to the $\mathfrak{gl}_N$ invariants have an incarnation at the homological level: they translate into spectral sequences (due to Rasmussen [Ras15]). The aim of the whole lecture is to explain that the same happens for the Alexander polynomial: there is a spectral sequence from the reduced triply graded homology to Knot Floer homology. The existence of all these spectral sequences was conjectured in [DGR06].

This spectral sequence has not (yet ?) a nice definition as Rassmussen's ones. In fact it is a composition of two spectral sequences. As we shall see $\mathfrak{gl}_0$-homology (a new knot homology theory) play a central role in this composition. The aim of this first two lectures is to give a definition of $\mathfrak{gl}_0$ homology.

1.1. **Quantum integers.** For k in $\mathbb{Z}$, define $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}} = \sum_{i=1}^{k} q^{-k-1+2i} \in \mathbb{Z}\left[q, q^{-1}\right]$, if $k \geq 0$, define $[k]! = \prod_{i=1}^{k}[i] \in \mathbb{Z}\left[q, q^{-1}\right]$, with the usual convention that an empty product is equal to $1 \in \mathbb{Z}\left[q, q^{-1}\right]$. Finally, if $n, a \in \mathbb{Z}$, define

$$\begin{bmatrix} n \\ a \end{bmatrix} = \begin{cases} \prod_{k=1}^{a} \dfrac{[n+1-k]}{[k]} & \text{if } a \geq 0, \\ 0 & \text{ortherwise.} \end{cases}$$

Exercise 1.
(1) Establish a Pascal-like relation for quantum binomials.
(2) Deduce that quantum binomials are indeed Laurent polynomials (symmetric in q and q^{-1}).
(3) What are their degrees?

2. Annular combinatorics

2.1. **Vinyl graphs.** In these lectures, knots and links are always presented as braid closures. This is because many notions need this annular presentation of knots. Generalizing what is discussed here to a more general setting seems difficult but potentially very interesting.

Let us fix $\mathscr{A} = \{z \in \mathbb{C}^2 \text{ s.t. } 1 \leq |z| \leq 2\}$ be the standard annulus. We will think of braid diagrams as being drawn in $\mathscr{A}$ and braid closure as subsets of $\mathscr{A} \times [0,1]$. There is a canonical map $\pi_{\mathbb{S}^1} : \mathscr{A} \to \mathbb{S}^1$. An oriented curve $\gamma : [0,1] \to \mathscr{A}$ is *directed* if the map $\pi_{\mathbb{S}^1} \circ \gamma$ is nowhere singular and the (oriented) tangent bundle of $\pi_{\mathbb{S}^1} \circ \gamma$ agrees with that of $\mathbb{S}^1$.

Definition 2.1. A *vinyl graph* Γ is a finite $\mathbb{N}$-edge-labeled (by a *thickness function*, usually denoted by $\mathtt{t}$), oriented trivalent graph[1] embedded in the interior of $\mathscr{A}$ such that:

- Every (oriented) edge is directed.
- A flow condition is respected at every vertex: the sum of the in-going thicknesses equals the sum of the out-going thicknesses.

The sum of thicknesses of edges intersection a generic ray is constant and is called the *index* of Γ. A vinyl graph is *elementary* if thickness function is $\{1,2\}$ valued (and there is no circle of thickness 2).

In a vinyl graph there are two kind of vertices: the *split* vertices (two out-going edges) and the *merge* vertices (two in-going edges). At each vertex v there are two *thin* half-edges denoted ($l(v)$ and $r(v)$ for left and right) and one *thick* half-edge denoted $b(v)$ (for big). Define $d(v) = \mathtt{t}(l(v))\mathtt{t}(r(v))$ and $w(v) = \begin{bmatrix} \mathtt{t}(b(v)) \\ \mathtt{t}(l(v)) \end{bmatrix}$.

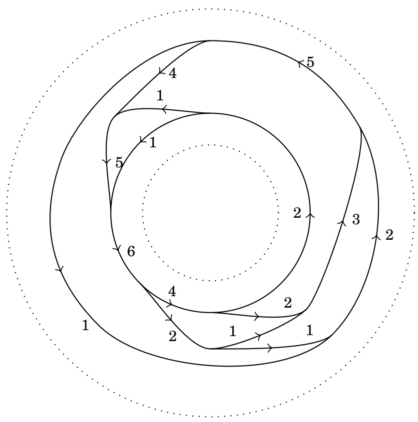
Exercise 2. For any vinyl graph Γ, prove that:

$$\sum_{v\in\mathrm{split}(\Gamma)} d(v) = \sum_{v\in\mathrm{merge}(\Gamma)} d(v) \qquad \text{and} \qquad \prod_{v\in\mathrm{split}(\Gamma)} w(v) = \prod_{v\in\mathrm{merge}(\Gamma)} w(v).$$

The first one is a consequence of the second, but might be easier to start with.

Hint: *For the first one, can parallelize edges according to their thickness and observe the collection of curves arising this way.*

Example 2.2. The following vinyl graph has index 7.



Remark 2.3. It is convenient but not essential to allow vinyl graphs to have edges of thickness 0. However such edges should be considered as non-existent. In other words, we identify vinyl graph with 0-thick edges with the same vinyl graph with these edges removed (and bi-valent remaining vertices smoothed out).

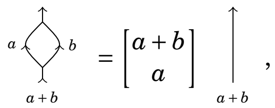
2.2. **Some skein modules.**

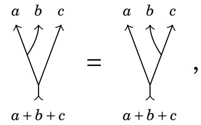
Definition 2.4. For any non-negative integer k denote Skein_k the torsion free[2] $\mathbb{Z}[q,q^{-1}]$-module generated by vinyl graphs of index k and modded out by ambient

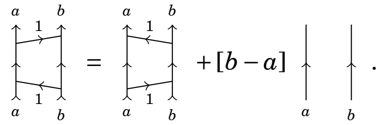
[1] Parallel edges are allowed, as well as vertex-less loops.

[2] We mean here that the torsion part that might occur after applying the relation is modded out.

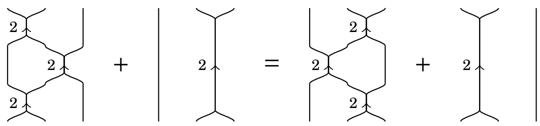
isotopy and by the following local relations (and their mirror images):

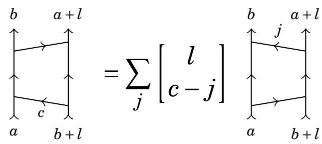
(2) 

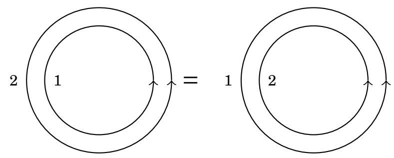
(3) 

(4) 

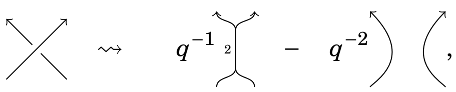
Exercise 3. Prove that the following relations hold in Skein_k:

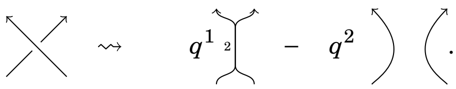
(5) 

(6) 

(7) 

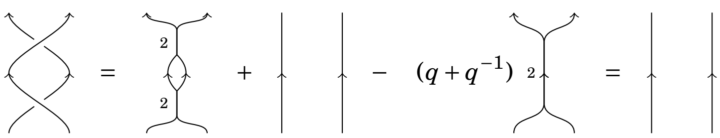
For any braid closure diagram D of index D, define $\iota(D)$ to be the element of Skein_k computed using the following two local rules:

(8) 

(9) 

Proposition 2.5. *The element $\iota(D)$ depends only the braid closure D represents, i.e. it is invariant by the braid relations and by conjugation in the braid group.*

Proof. Invariance by conjugation and by far commutation is immediate. Let us deal with the second Reidemeister move:

(10) 

Exercise 4. Prove R3, i.e. that $\iota(D) = \iota(D')$ using (5) □

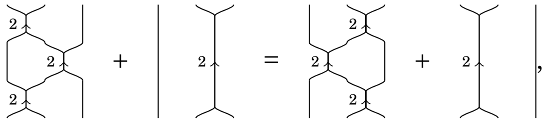
Theorem 2.6 ([QR18, Lemma 5.2])**.** *The module* Skein_k *has a basis given by collections of circles* $(\mathbb{S}_\lambda)_{\lambda \vdash k}$.

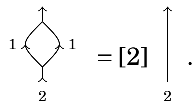
The proof is given by an effective algorithm.

Conjecture 2.7 ([QR18, Conjecture 5.4] partly proved in [Hik24])**.** Any vinyl graph decomposes positively (and unimodally) in the basis $(\mathbb{S}_\lambda)_{\lambda\vdash k}$.

If one is only interested in (uncolored) braid closure, one can also consider a slightly small skein module.

Definition 2.8. For any non-negative integer k denote $\mathrm{Skein}_k^{\mathrm{el}}$ the torsion free $\mathbb{Z}[q,q^{-1}]$-module generated by elementary vinyl graphs of index k and modded out by ambient isotopy and by the following local relations:

(11) 

(12) 

Theorem 2.9 (Turaev 88)**.** *The module* $\mathrm{Skein}_k^{\mathrm{el}}$ *has a basis given by collections of positively curled unknots* $(U_\lambda)_{\lambda\vdash k}$.

Corollary 2.10. *The module* $\mathrm{Skein}_k^{\mathrm{el}}$ *has a basis given by disjoint union of chains of of dumbells* $(\mathbb{D}_\lambda)_{\lambda\vdash k}$.

Corollary 2.11. *The module* $\mathrm{Skein}_k^{\mathrm{el}}$ *is a submodule of* Skein_k.

Let us a fix $\star = \frac{19}{10} \in \mathscr{A}$.

Definition 2.12. A *pointed* vinyl graph is a vinyl graph Γ a for which $\star$ is in the interior of an outermost edge of Γ of thickness 1.

Definition 2.13. For any non-negative integer k, denote $\mathrm{Skein}_{k,\star}^{\mathrm{el}}$ the torsion free $\mathbb{Z}[q,q^{-1}]$-module generated by vinyl graphs of index k and modded out by ambient isotopy (preserving the base point) and by the local relations (11) and (12) (away from the base point):

Proposition 2.14. *The skein module* $\mathrm{Skein}_{k,\star}^{\mathrm{el}}$ *has a basis given by disjoint unions of chains of dumbells.*

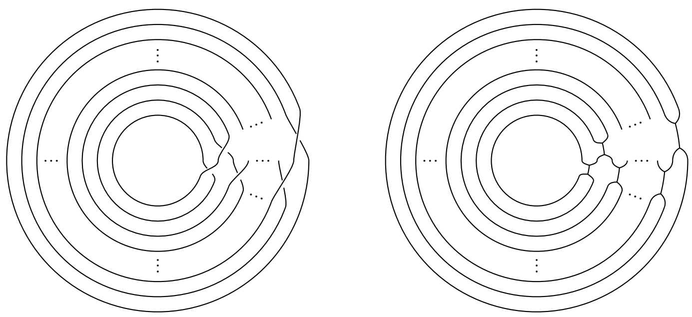


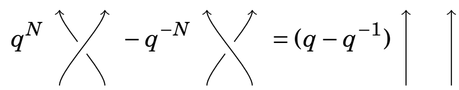
FIGURE 1. On the left, a positively curled unknot in the solid torus. On the right a chain of dumbells.

One defines morphisms $\mathbb{Z}[q,q^{-1}]$-morphisms from $\mathrm{Skein} := \bigoplus_{k\in\mathbb{N}} \mathrm{Skein}_k$ to $\mathbb{Z}[q,q^{-1}]$ by defining the values for each collection of circles. Note that these values might not behave multiplicatively with respet to disjoint union (but they might). Since braid closures are naturally seen as elements of Skein, one may wonder when the images of braid closures by these morphisms provide link invariants.

Let N be a positive integers and consider $\psi_N : \mathrm{Skein} \to \mathbb{Z}[q,q^{-1}]$ which is defined multiplicatively on collection of circles and which associates with a circle of thickness k the Laurent polynomial $\begin{bmatrix} N+k-1 \\ k \end{bmatrix}$.

Proposition 2.15. *The morphism ψ_N provides an invariant of framed links. It can be renormalized by multipliying the results by $q^{(1-N)\mathtt{w}(\beta)}$ where $\mathtt{w}(\beta)$ is the writhe of the braid β, that is the number of positive crossings minus the number of negative ones. The renormalized invariant satisfies the skein relation*

(13)



and is equal to $[N]$ on the unknot. It is therefore the $\mathfrak{gl}_N$ polynomial invariant (i.e. Jones polynomial for $N = 2$ and the trivial invariant equal to 1, for $N = 1$)[3].

Sketch of the proof. Since we already know that braid closures are naturally seen as elements in Skein, the only thing to inspect is the behavior under stabilization. The place where we stabilize, behave then like a base point. Proposition 2.14 is then very useful to work, but can be improved to get collection of circles with one of them having a digon on which the base point stands. Working out the images under ψ_N is then very easy.

Exercise 5. (1) Fill in the gaps of this proof.
(2) What happens for negative N?

The statement about the skein relation is trivial. □

Proposition 2.16. *For any vinyl graph Γ,*

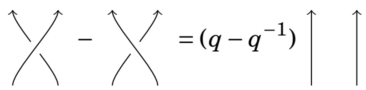
$$\psi_1(\Gamma) = \prod_{v \in merge(\Gamma)} w(v) =: w(\Gamma). \tag{14}$$

Exercise 6. Prove Proposition 2.16.

Let $\psi_0 : \mathrm{Skein}^{\mathrm{el}} \to \mathbb{Z}[q,q^{-1}]$ be the $\mathbb{Z}[q,q^{-1}]$-linear map defined on collection of collection of chain of dumbells to be 1 if the chain is connected, and 0 otherwise.

Proposition 2.17. *The morphism ψ_0 provides an invariant of framed links. It can be renormalized by multipliying the results by $q^{\mathtt{w}(\beta)}$. The renormalized invariant satisfies the skein relation*

(15)



and is equal to 1 *on the unknot. It is therefore the Alexander polynomial.*

Exercise 7. Prove Proposition 2.17. This is similar, but substantially easier, than for Proposition 2.15.

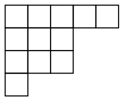
3. State spaces and morphisms

The aim of this section is to categorify ψ_1 and ψ_0.

[3] Depending on your preferred convention for this skein relation you might need to change q for q^{-1} and links for their mirror images.

3.1. **Symmetric polynomials.** Let $\mathbf{X} = (x_1, \dots, x_a)$ be a finite alphabet of a letters. The symmetric group S_a acts on $\mathbb{Q}[\mathbf{X}]$ by permuting the letters. A symmetric polynomial is an element of $\mathbb{Q}[\mathbf{X}]$ which is invariant under this action. The ring $\mathbb{Q}[\mathbf{X}]$ is graded by setting $\deg(x)_\bullet = 2$ and so is $\mathbb{Q}[\mathbf{X}]^{S_a}$.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_a \geq 0)$ be a partition. It is convenient to depict partitions by so called Young (or Ferrer) diagrams. The correspondence is as follow:

(16) $(5,3,3,1) \quad \leftrightsquigarrow$ 

The *Schur polynomial* $s_\lambda(\mathbf{X})$ is defined as the quotient:

$$s_\lambda := \frac{\det\left(\left(x_i^{\lambda_j + a - j - 1}\right)_{1 \leq i,j \leq a}\right)}{\det\left(\left(x_i^{a-j-1}\right)_{1 \leq i,j \leq a}\right)} \tag{17}$$

The numerator being anti-symmetric, it is divisible by the Vandermonde determinant so that it is indeed a polynomial.

Schur polynomials are fascinating... but studying them is not the purpose of these lectures. We only recall a few fact which will be useful to us (but strongly recommend [Mac15]).

Proposition 3.1. *Let $\mathbf{Y} = (y_1, \dots, y_b)$ be another alphabet. The following identity holds:*

$$\prod_{i=1}^{a} \prod_{j=1}^{b} (x_i - y_j) = \sum_{\lambda \in T(a,b)} (-1)^{|\lambda^*|} s_\lambda(\mathbf{X}) s_{\lambda^*}(\mathbf{Y}), \tag{18}$$

where $T(a,b)$ denotes the set of partitions with at most a lines and at most b columns, and λ^* denote the transposed of the complementary of λ (it is therefore an element of $T(b,a)$).

Exercise 8. Prove Proposition 3.3 using the Vandermonde determinant on $\mathbf{X} \cup \mathbf{Y}$.

Proposition 3.2. *The Schur polynomials associated with partition with at most a lines form a linear basis of $\mathbb{Q}[\mathbf{X}_a]$. The structural constant for the multiplication in that ring are called the Littlewood–Richardson coefficients (and they do not depend on a) and are denote $c_{\alpha\beta}^{\gamma}$.*

$$s_\alpha s_\beta = \sum_\gamma c_{\alpha\beta}^{\gamma} s_\gamma.$$

The Littlewood–Richardson coefficients are nonnegative integers.

One has $\mathbb{Q}[\mathbf{X},\mathbf{Y}]^{S_{a+b}} \subseteq \mathbb{Q}[\mathbf{X},\mathbf{Y}]^{S_a \times S_b} = \mathbb{Q}[\mathbf{X}]^{S_a} \otimes \mathbb{Q}[\mathbf{Y}]^{S_b}$, so that if P is a symmetric polynomial in $\mathbf{X} \cup \mathbf{Y}$ it can be writtenas a sum of product of symmetric polynomials in $\mathbf{X}$ and symmetric polynmials in $\mathbf{Y}$.

Proposition 3.3. *For λ a partition, one has:*

$$s_\lambda(\mathbf{X} \cup \mathbf{Y}) = \sum_{\alpha,\beta} c_{\alpha\beta}^{\lambda} s_\alpha(\mathbf{X}) s_\beta \mathbf{Y},$$

where the coefficient $c_{\alpha\beta}^{\lambda}$ are still the the Littlewood–Richardson coefficients.

Definition 3.4. Let Γ be a vinyl graph. If e is an edge of Γ, a *decoration* of e is a symmetric polynomial in $\mathtt{t}(e)$ variables. The space of decoration of the edge e is denoted D_e. A *decoration* of Γ is an element of the graded algebra.

$$D_\Gamma := \bigotimes_{e \in E(\Gamma)} D_e,$$

where the tensor product is taken over $\mathbb{Q}$. A decoration P is therefore a sum of pure tensor and we will often assume that P is a pure tensor and extend definition linearly. If P is such a pure tensor, we denote $(P_e)_{e\in E(\Gamma)}$ a collection of decoration of edges of Γ such that $P = \bigotimes_e P_e$.

3.2. **Evaluation.**

Definition 3.5. Let Γ be a vinyl graph of index k and write $\mathbf{X}_k = (x_1,\dots,x_k)$. An *omni-coloring of* Γ is a map $c : E(\Gamma) \to \mathscr{P}(\mathbf{X}_k)$ such that:

- For each $e \in E(\Gamma)$, $\#c(e) = \mathtt{t}(e)$.
- For each generic ray r, $\bigcup_{e\cap r} c(e) = \mathbf{X}_k$.

Note that the definition imply a flow property of omni-colorings at vertices.

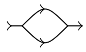
For an omnicoloring c and a decoration P of a vinyl graph Γ, define

$$\tau(P,c) = \frac{\prod_{e\in E(\Gamma)} P_e(c(e))}{\prod_{v\in V_{\mathrm{split}}(V)} \prod_{\substack{i\in c(l(v))\\ j\in c(r(v))}} (x_i - x_j)}.$$

and

$$\tau(P) = \sum_c \tau(P,c).$$

Proposition 3.6. *The quantity $\tau(P)$ is a symmetric polynomial in $\mathbf{X}_k$ of degree* $\deg(P) - 2d(\Gamma)$.

Sketch of the proof. The fact that $\tau(P)$ is symmetric follows from directly from the fact that if $\sigma \in S_k$, $\tau(P,\sigma\cdot c) = \sigma\cdot(P,c)$. The degree statement is obvious since it is true at the level of $\tau(P,c)$. By definition, $\tau(P) \in \mathbb{Q}\left[\mathbf{X}_a, \left(\frac{1}{x_j - x_i}\right)_{1\le i<j\le a}\right]$. By symmetry it is enough to show that $\tau(P)$ can be written without $\frac{1}{x_2-x_1}$. For simplicity consider the case where $k = 2$, in particular Γ is necessarily elementary and is the (circular) concatenation of say m of element of the form:  . There are 2^m omni-colorings for Γ corresponding to choosing either of the two possible coloring for each of these digons and the evaluation is actually the product of evaluation of the closed digon (each carrying its own decoration), so that it is enough to prove it for digon. In that case it is a simple computation the two terms have the same denominator $(x_2 - x_1)$ and the sum of their numerators is obviously anti-symmetric so that we indeed get a polynomial evaluation.

The general case follows the same idea: one gathers omni-colorings according to sub vinyl graph covered by colors x_1 and x_2. In each of these batch contains 2^m omni-colorings with m being precisely the power of $\frac{1}{x_2-x_1}$ in the evaluation for these omni-colorings. Similarly m anti-symmetry features implies that we indeed can get rid of all the $(x_2 - x_1)$ in the denominator.

Exercise 9. Fill in the gaps of this proof. □

3.3. **Universal construction.** Let Γ be a vinyl graph of index k. Recall that D_Γ is the space of decorations of Γ, define the symmetric bilinear form $(-;-)_1$ on D_Γ by:

$$(P;Q) := \tau(PQ)_{|x_1,\dots,x_k\mapsto 0} := \tau_0(PQ). \tag{19}$$

In other words $(P;Q)$ extract the constant coefficient of the polynomial $\tau(PQ)$. If P and Q are homogeneous, then $(P;Q)$, then $(P;Q) = 0$ unless $\deg(P)+\deg(Q) = 2d(\Gamma)$ (and in that case this might be zero as well).

The *radical* (or kernel) of $(-;-)$ is the sub space of $\mathscr{D}_\Gamma$ defined by:

$$\mathrm{rad}_\Gamma = \{Q \in D_\Gamma \text{ s.t. for all } P \in D_\Gamma, (P;Q) = 0\}.$$

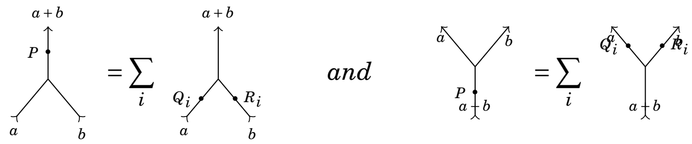
In other words, for each $P \in D_\Gamma$, $(P;-)$ is a linear form and rad_Γ and

$$\mathrm{rad}_\Gamma = \bigcap_{P \in D_\Gamma} \ker(P;-).$$

Define the $\mathfrak{gl}_1$-state space associated with Γ to be: $S_1(\Gamma) := q^{-d(\Gamma)} D_\Gamma / \mathrm{rad}_\Gamma$.

Remark 3.7. The space $S_1(\Gamma)$ has finite dimension and it naturally inherits a structure of Frobenius algebra (with multiplication of degree $d(\Gamma)$ and trace given by τ). The comultiplication is far from easy to understand in general (but this would be interesting).

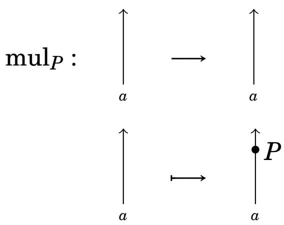
Lemma 3.8. *Let* $\mathbf{X}$ *be a set of* a *variables,* $\mathbf{Y}$ *a set of* b *variables and* $P(\mathbf{X},\mathbf{Y}) = \sum_i Q_i(\mathbf{X}) R_i(\mathbf{Y})$ *is a symmetric polynomial in* $\mathbf{X} \cup \mathbf{Y}$ *and the* Q_i *(resp.* R_i*) are symmetric in* $\mathbf{X}$ *(resp in* $\mathbf{Y}$*), then in* D_Γ*, one has:*

(20) 

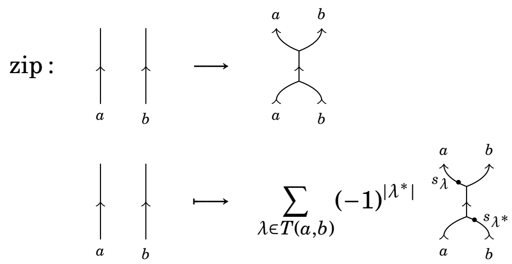
Sketch of the proof. The identity satisfied by P implies that for any decoration S of Γ, $\tau(PS) = \sum_i \tau(Q_i R_i S)$, so that $P - \sum_o Q_i R_i$ is indeed in the radical of $(-;-)$. ☐

Actually the definition of $S_1(-)$ should be extended into a functor from a category where objects are vinyl graphs and morphisms are *vinyl foams*. For simplicity and by demagogy, we decided not to mention those. Instead we will claim that certain definitions induce morphisms between the graded vector spaces $S_1(\Gamma)$.

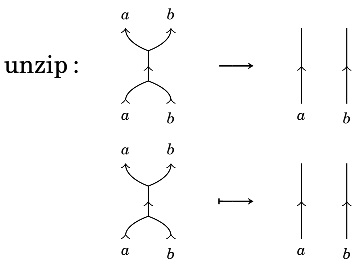
Proposition 3.9. *The following locally defined maps between* D_Γ *induce well-defined morphisms on* $S_1(\Gamma)$.

(21) 

Where P *is a symmetric polynomial in* a *variables.*

(22) 

Notations are introduced in Proposition 3.3.

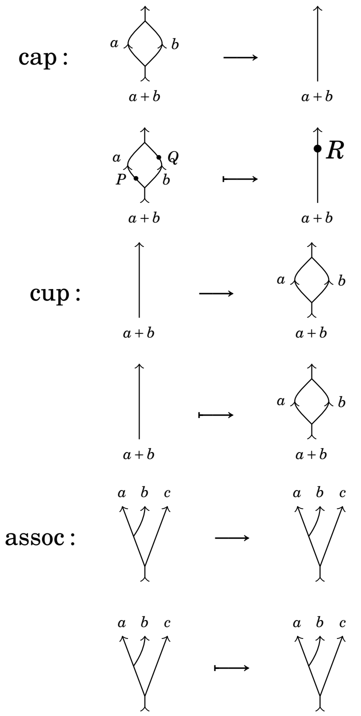
(23) 

Note that it is assumed that that the middle edge carries 1 *as a decoration. This is not a problem because of dot migration (Lemma 3.8).*

(24)

(25)

(26)



with
$$R = \sum_{\mathbf{X} \cup \mathbf{Y} = \mathbf{Z}} \frac{P(\mathbf{X})Q(\mathbf{Y})}{\prod_{\substack{x \in \mathbf{X} \\ y \in \mathbf{Y}}} (x-y)}.$$

Same remark as for (23). *Note that the vertically mirrored morphism exists as-well and is also well-defined.*

Proof. For all these maps, one needs to show that the radical is mapped in the radical.

Polynomial multiplication and associativity are respectively straightforward and trivial.

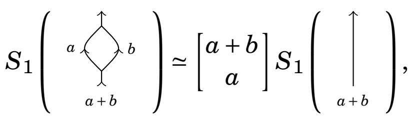
For the zip. Let P_1 be a decoration of Γ_1 such that for all decoration Q_1 of $\Gamma_1, \tau_0(P_1Q_1) = 0$. Denote P_2 the image of P_1 and Q_2 a decoration of Γ_2. Because of dot migration, we can assume that Q_2 is induced by a decoration Q_1 of Γ_1. Let c be an omni-coloring of Γ_2, because of Propistion , if c is not induced by an omni-coloring of Γ_1, then $\tau(P_2Q_2, c) = 0$. Otherwise one has: $\tau(P_2Q_2, c) = \tau(P_1Q_1, c)$. Hence $\tau_0(P_2Q_2) = \tau_0(P_1Q_1)$ and P_2 is indeed in the radical.

Exercise 10. Work out the unzip.
Hint: *Use the decoration appearing in the zip map.*

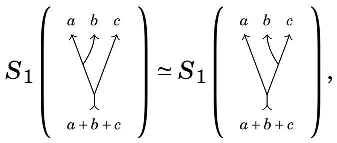
For the cap, one needs to realize that the formula does indeed gives a polynomial. Similarly to the zip/unzip duality, once the cap is done, the cup is almost trivial. Associativity is more or less trivial. □

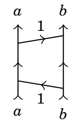
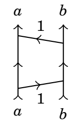
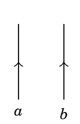
Proposition 3.10. *Using morphisms described in Proposition , one can construct the following isomorphisms:*

(27)



(28)



(29) $S_1\Big($  $\Big) \simeq S_1\Big($  $\Big) \oplus [b-a] S_1\Big($  $\Big)$,

where the Laurent polynomial coefficient indicate direct sums of q-shifted copies of a given space.

Lemma 3.11. *If Γ is a collection of circles, $S_1(\Gamma) \simeq \mathbb{Q}$.*

Corollary 3.12. *For any vinyl graph Γ, $\mathrm{qdim}(S_1(\Gamma)) = w(\Gamma)$. In particular, if Γ is elementary $\mathrm{qdim}(S_1(\Gamma)) = (q+q^{-1})^{\#V(\Gamma)/2}$.*

3.4. $\mathfrak{gl}_0$**-state spaces.**

Definition 3.13. For any pointed vinyl graph Γ of index k. The multiplication by the polynomial x^{k-1} at the edges carrying the base point is an endomorphism $\phi_\star$ of $S_1(\Gamma)$. The $\mathfrak{gl}_0$-state space is the image of $\phi_\star$ shifted in degree by $1-k$ it is denote $S_0(\Gamma)$.

Remark 3.14. Alternatively, one can "push" $\phi_\star$ in the quotient of the universal construction and define it as a quotient of $S_1(\Gamma)$: consider

$$K_\Gamma = \mathrm{rad}_\Gamma = \left\{Q \in D_\Gamma \text{ s.t. for all } P \in D_\Gamma, \tau_0(PQx^{k-1}) = 0\right\}.$$

One clearly has $K_\Gamma \subseteq \mathrm{rad}(-;-)$, so that D_Γ/K_Γ is a quotient of $S_1(\Gamma)$ and $\varphi_\star$ induces a (degree 0) isomorphism between $q^{k-1}D_\Gamma/K_\Gamma$ and $S_0(\Gamma)$.

Lemma 3.15. *The isomorphisms listed in Proposition 3.10 hold when considering $\mathfrak{gl}_0$ state spaces instead of $\mathfrak{gl}_1$'s, provided they occur away from the base point.*

Lemma 3.16. *If Γ is a disjoint union of chain of dumbells, then $S_0(\Gamma) = 0$ unless Γ is connected and in this case $S_0(\Gamma) = q^0\mathbb{Q}$.*

Proof. The first part of the statement follows from degree consideration. For the second part, degree consideration implies that $S_0(\Gamma)$ is either 0 or $\mathbb{Q}$. In order to see that this space in not zero, it is enough to evaluate the decoration which is x^{k-1} at the base point and 1 everywhere else. A computation shows that it is ± 1.

Exercise 11. Do this computation. It is worth noticing that in S_1 multiplying at the base point by x^{k-1} is the same as multiplying by the product of maximal elementary polynomial at all edge intersecting the ray containing the base point (but the one of the base point).

□

Corollary 3.17. *If Γ is not connected, then $S_0(\Gamma) = 0$.*

Exercise 12. Let Γ be an elementary pointed vinyl graph of index k. A state s is a choice for each split vertex of thin edge (left or right). Each state s has a weight $w(s) \in \mathbb{Z}$ which is the number of right minus the number of left chosen. An *M-state*[4] is a state for which the sub-graph of Γ given by edges of thickness 2 and edges of thickness 1 chosen in s is connected. The weight of an M-state is $w_M(s) = w(s) + 1 - k$.

Prove the following identity:

$$\mathrm{qdim}(S_0(\Gamma)) = \sum_{s \text{ M-state}} q^{w_M(s)}.$$

Hint: *One can fist start proving that the right-hand side satisfies some relations and then look at chain of dumbells.*

[4]The letter M refers to Mikhail Khovanov.

4. Homology theories

Similar to Khovanov homology, one can build up homology theory using hypercube of resolutions and the “functors” S_1 and S_0.

4.1. $\mathfrak{gl}_1$**-homology.** Let D be a diagram of a braid closure with n crossings , we construct an oriented hypercube of dimension n using the rules given by Figure 2. The 2^n vertices of this hypercube are graded $\mathbb{Q}$-vector spaces and the $n2^{n-1}$ arrows are degree 0 linear maps.

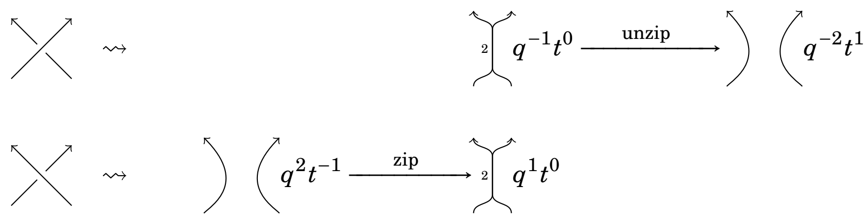


FIGURE 2. A schematic description of the complex $C_{\mathfrak{gl}_1}(D)$: each diagram should be surrounded by $S_1(-)$ (omitted for readability): the letter q encodes internal grading shifts, while t encodes homological grading.

By construction every square in this hypercube commutes so that one can flatten it (by putting signs appropriately, so that every square anti-commutes) to make it a chain complex. This chain complex is denoted $C_{\mathfrak{gl}_1}(D)$.

Theorem 4.1 ([RW20])**.** *The homology of the chain complex $C_{\mathfrak{gl}_1}(D)$ is a brigraded vector space which depends only on the link L represented by D. It is called the symmetric $\mathfrak{gl}_1$-homology of L and it is denoted $H_{\mathfrak{gl}_1}(L)$. Its graded Euler characteristic is the $\mathfrak{gl}_1$ invariant.*

About the proof. One first prove that the homotopy type of $C_{\mathfrak{gl}_1}$ is invariant under braid relation. This is done by giving explicit homotopies. This is very close from what it classically done for Khovanov homology and other variants of that. Invariance by conjugation (first Markov move) is trivial by construction. What remains is invariance by stablisation (aka Reidemeister 1). One immediately sees that this might be problematic since we are then dealing with braids of different indexes. . . and indeed invariance under this move is very difficult to established and so far we do not have a combinatorial explanation of that. The proof relies on the proof of invariance of the triply graded homology and on a spectral sequence which does from that to the symmetric $\mathfrak{gl}_1$-homology. This proof of invariance requires working over a field of characteristic 0). □

4.2. $\mathfrak{gl}_0$**-homology.** Let D be a pointed braid diagram. The presence of the base point enables to use $S_0(-)$ instead of $S_1(-)$ in the above definition above. We shift things a bit differently. This is sum up in Figure 3.

This gives rise to a sub-chain-complex (shifted by q^{1-k}). Denote this chain complex by $C_{\mathfrak{gl}_0}(D)$.

Theorem 4.2 ([RW22])**.** *The homology of the chain complex $C_{\mathfrak{gl}_0}(D)$ is a brigraded vector space which depends only on the marked link $L_\star$ represented by D. It is called the symmetric $\mathfrak{gl}_0$-homology of $L_\star$ and it is denoted $H_{\mathfrak{gl}_0}(L_\star)$. In particular it is a knot invariant. Its grade Euler characteristic is the Alexander polynomial of the knot.*

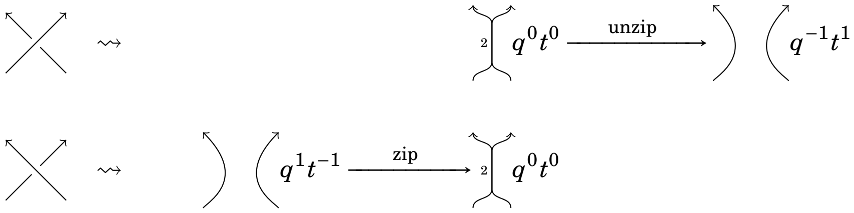


FIGURE 3. A schematic description of the complex $C_{\mathfrak{gl}_0}(D)$: each diagram should be surrounded by $S_0(-)$ (omitted for readability): as before, the letter q encodes internal grading shifts, while t encodes homological grading.

About the proof. The homotopy type of $C_{\mathfrak{gl}_0}$ is invariant under braid relation (away from the base point). The same is homotopy as for the invariance of $C_{\mathfrak{gl}_1}$ still do the job. As before, invariance by conjugation (away form the base point) is trivial by construction. Stabilization (at the base point) is actually very easy (remember that it was difficult for $H_{\mathfrak{gl}_1}$). If D and D' only differs by stabilization at the basepoint, then the chain complexes $C_{\mathfrak{gl}_0}(D)$ and $C_{\mathfrak{gl}_0}(D')$ are isomorphic (and the isomorphism is very easy to write down).

The last thing to do is to be able to move the base point. This turns out to be the difficult point. The proof is still very much combinatorial but frustratingly technical. □

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