Évaluation des mousses $\mathfrak{sl}_N$

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

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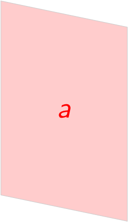
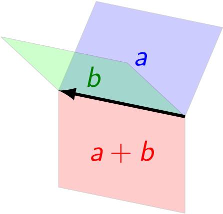
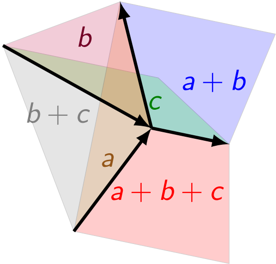
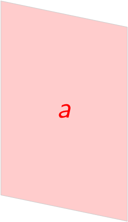
UNIVERSITÉ DE BOURGOGNE

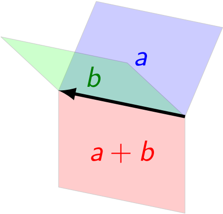
Dijon – 14/03/2017
Grenoble – 17/03/2017

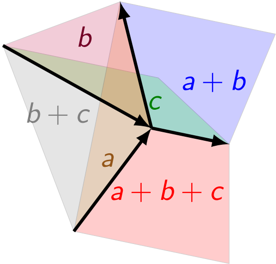

a


a
b
$a + b$


b
$a + b$
c
$b + c$
a
$a + b + c$







Définition (R.–Wagner, '17)

$$\langle F\rangle_N = \sum_c \frac{(-1)^{\sum_{1\leq i<j\leq N}\theta^+_{ij}(F,c)}\prod_f P_f(c(f))}{(-1)^{\sum_{i=1}^{N} i\chi(F_i(c))/2}\prod_{1\leq i<j\leq N}(X_i-X_j)^{\frac{\chi(F_{ij}(c))}{2}}}$$

Définition (crochet de Kauffman, polynôme de Jones)

$$\langle \emptyset \rangle_{\mathrm{K}} = 1 \qquad \left\langle \bigcirc \sqcup L \right\rangle_{\mathrm{K}} = [2]_q \, \langle L \rangle$$

$$\left\langle \times \right\rangle_{\mathrm{K}} = \left\langle \asymp \right\rangle_{\mathrm{K}} - q \left\langle) \quad (\right\rangle_{\mathrm{K}}$$

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$

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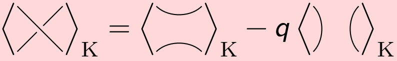

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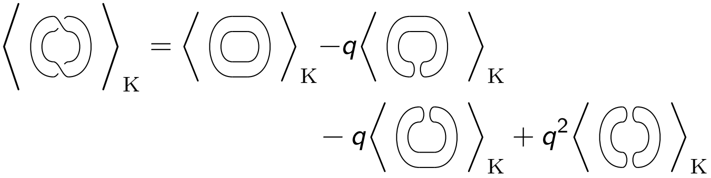

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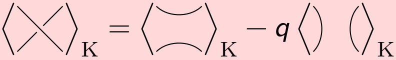
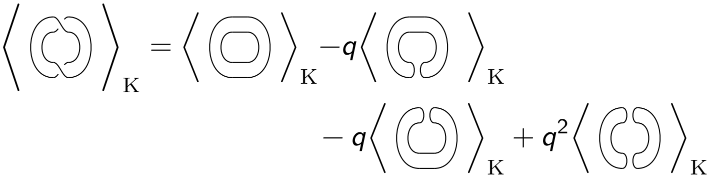
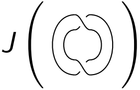


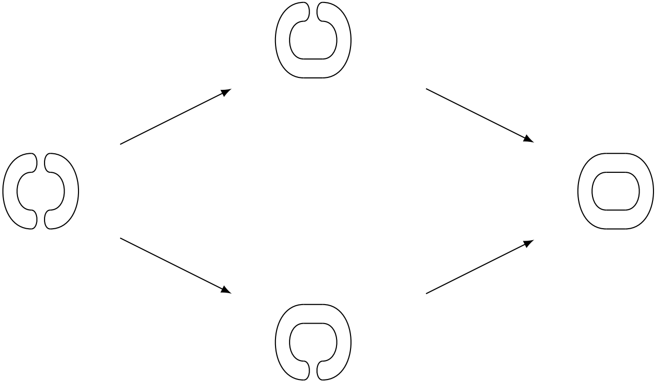
$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$



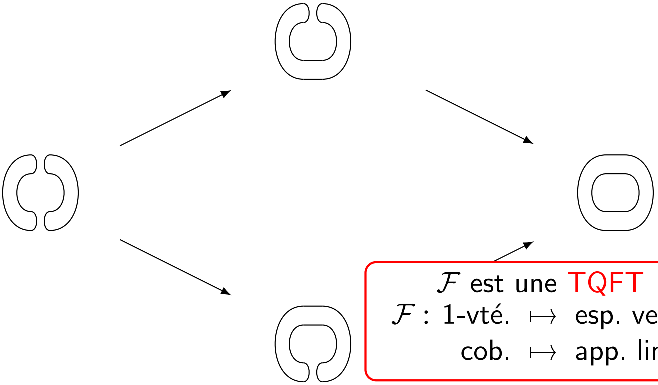


$$= q^6 + q^4 + q^2 + 1$$

Homologie de Khovanov



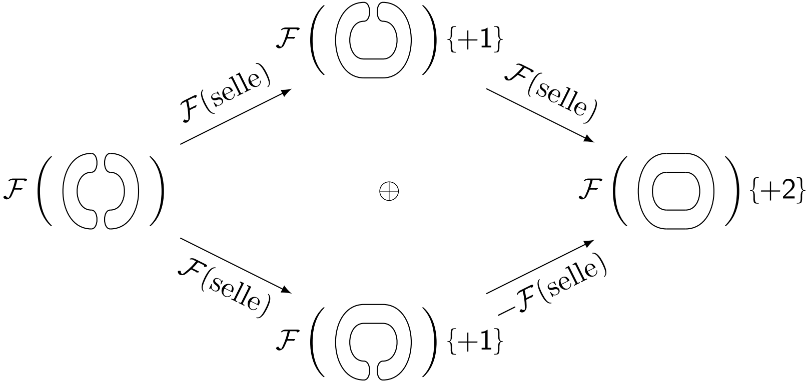
Homologie de Khovanov



$\mathcal{F}$ est une TQFT

$$\begin{aligned} \mathcal{F} : \text{1-vté.} &\mapsto \text{esp. vect.} \\ \text{cob.} &\mapsto \text{app. lin.} \end{aligned}$$

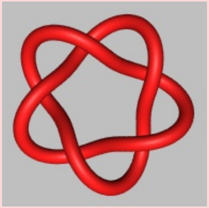
Homologie de Khovanov



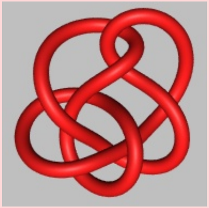
Décaler le degré homologique par $-n_-$, le q-degré par $n_+ - 2n_-$.
Prendre l'homologie.

Proposition (Bar-Natan, '02)

L'homologie est strictement plus puissante que le polynôme de Jones.



5_1



10_{132}

(source `www.colab.sfu.ca/KnotPlot/KnotServer/`*)*

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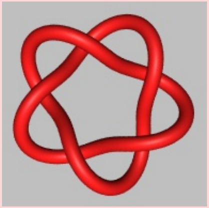
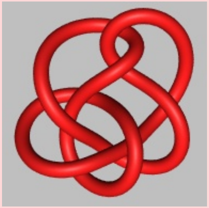


5_1



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Théorème (Kronheimer–Mrowka, '10)

L'homologie de Khovanov détecte le nœud trivial.

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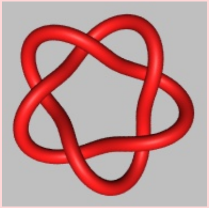

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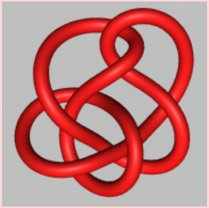


10_{132}

(source `www.colab.sfu.ca/KnotPlot/KnotServer/`*)*

Théorème (Kronheimer–Mrowka, '10)

L'homologie de Khovanov détecte le nœud trivial.

Conj. de Milnor (Kronheimer–Mrowka, '93, Rasmussen '04)

Le genre slice du nœud torique (p, q) est $\frac{(p-1)(q-1)}{2}$.

$$\begin{tikzcd}
\text{Entrelacs} \arrow[r, "\text{homologie de Khovanov}"] \arrow[dr, "\text{polynôme de Jones (invariant } \mathfrak{sl}_2\text{)}"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & \text{Suite de } \mathbb{Z}\text{-mod. gradués} \arrow[d, "\chi_q"] \\
 & \text{Pol. de Laurent}
\end{tikzcd}$$

- Une stratégie pour les croisements
- Une TQFT ad-hoc

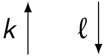
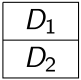
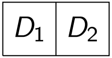
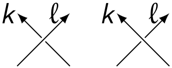
$$\begin{tikzcd}
\text{Entrelacs} \arrow[r, "\text{homologie } \mathfrak{sl}_N"] \arrow[dr, "\text{invariant } \mathfrak{sl}_N"'] \arrow[dr, phantom, shift left=6ex, "\circlearrowright"] & \text{Suite de $\mathbb{Z}$-mod. gradués} \arrow[d, "\chi_q"] \\
& \text{Pol. de Laurent}
\end{tikzcd}$$

- Une stratégie pour les croisements ⇝ complexes de Rickard
- Une TQFT ad-hoc ⇝ évaluation des mousses

L'invariant $\mathfrak{sl}_N$

Proposition (Drinfel'd)

On peut déformer $U(\mathfrak{sl}_N)$ en une algèbre de Hopf $U_q(\mathfrak{sl}_N)$ sur $\mathbb{C}(q)$ de manière à rendre le tressage non trivial.

	$\mathrm{id}_{\wedge_q^k V}$, $\mathrm{id}_{(\wedge_q^\ell V)^*}$
	$f_1 \circ f_2$
	$f_1 \otimes f_2$
	tressage

	évaluation
	co-évaluation

L'invariant $\mathfrak{sl}_N$

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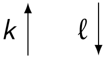
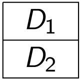
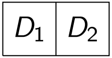
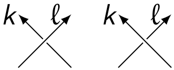
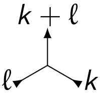
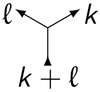
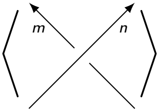
	$\mathrm{id}_{\bigwedge_q^k V}$, $\mathrm{id}_{(\bigwedge_q^\ell V)^*}$
	$f_1 \circ f_2$
	$f_1 \otimes f_2$
	tressage

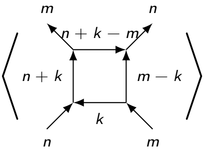
	évaluation
	co-évaluation
	$\bigwedge_q^k V \otimes \bigwedge_q^\ell V \longrightarrow \bigwedge_q^{k+\ell} V$
	$\bigwedge_q^{k+\ell} V \longrightarrow \bigwedge_q^k V \otimes \bigwedge_q^\ell V$

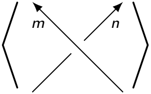
Calcul MOY (Murakami–Ohtsuki–Yamada)

Lusztig ('94) :



$$= \sum_{k=\max(0,m-n)}^{m} (-1)^{m-k} q^{k-m}$$





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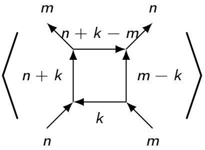
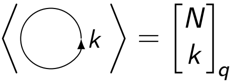
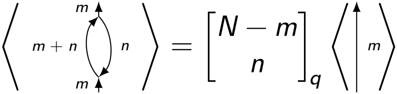
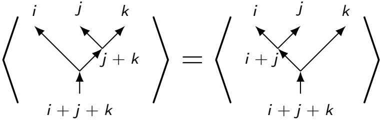
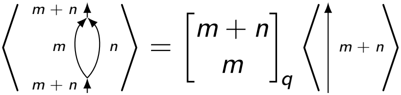
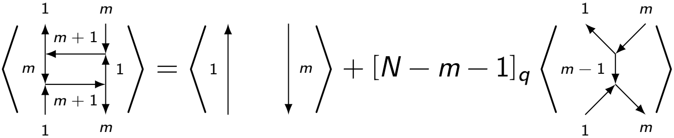


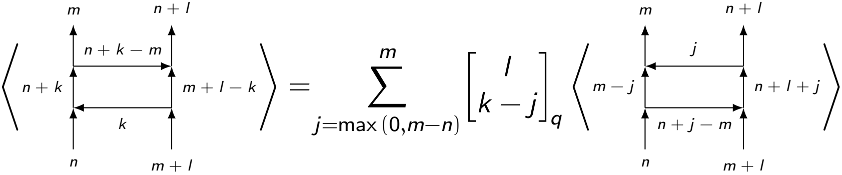

k
=
$\begin{bmatrix}N\\k\end{bmatrix}_q$


m
m + n
n
m
=
$\begin{bmatrix}N-m\\n\end{bmatrix}_q$
m


i
j
k
j + k
i + j + k
=
i
j
k
i + j
i + j + k


m + n
m
n
m + n
=
$\begin{bmatrix}m+n\\m\end{bmatrix}_q$
m + n


1
m
m + 1
m
1
m + 1
1
m
=
1
m
$+ [N-m-1]_q$
1
m
m − 1
1
m


m
n + l
n + k − m
n + k
m + l − k
k
n
m + l
=
$\sum_{j=\max(0,m-n)}^{m} \begin{bmatrix}l\\k-j\end{bmatrix}_q$
m
n + l
j
m − j
n + l + j
n + j − m
n
m + l

$$\text{On veut : } \left\{ \begin{array}{rrcl} \mathcal{F} : & \mathsf{Foam}_N & \longrightarrow & \mathbb{Z}[X_1, \dots, X_N] - \mathsf{mod}_{\mathrm{gr}} \\ & \text{graphe MOY} & \longmapsto & \text{module gradué} \\ & \text{mousse} & \longmapsto & \text{morphisme de module} \end{array} \right.$$

$$\text{On veut : }\left\{\begin{array}{lrcl} \mathcal{F}: & \mathsf{Foam}_N & \longrightarrow & \mathbb{Z}[X_1,\dots,X_N]-\mathsf{mod}_{\mathrm{gr}} \\ & \text{graphe MOY} & \longmapsto & \text{module gradué} \\ & \text{mousse} & \longmapsto & \text{morphisme de module}\end{array}\right.$$

Construction universelle

Une évaluation $\rightsquigarrow$ (Parfois) une TQFT

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Construction universelle

Une évaluation ⇝ (Parfois) une TQFT

Théorème (R.–Wagner, ’17)

L’évaluation définie sur le premier slide induit une TQFT ad-hoc grâce à la Construction universelle.

Construction universelle

(Blanchet, Habbeger, Masbaum, Vogel)

Donné $\tau : \{\text{cobordismes fermés } \emptyset \to \emptyset\} \longrightarrow R$

$$\begin{array}{ccl} \Gamma & \longmapsto & \mathcal{F}(\Gamma) := \bigoplus\limits_{{}_{\emptyset}F_{\Gamma}} R_F \\[2em] {}_{\Gamma_1}G_{\Gamma_2} & \longmapsto & \mathcal{F}(G) : \left(\begin{array}{ccc} \mathcal{F}(\Gamma_1) & \to & \mathcal{F}(\Gamma_2) \\ {}_{\emptyset}F_{\Gamma_1} & \mapsto & {}_{\emptyset}FG_{\Gamma_2} \end{array} \right) \end{array}$$

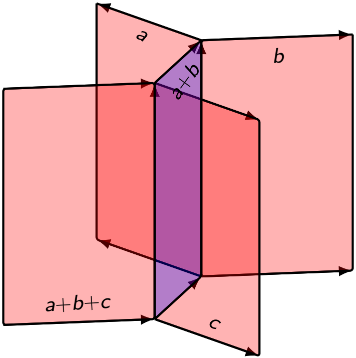
Construction universelle

(Blanchet, Habbeger, Masbaum, Vogel)

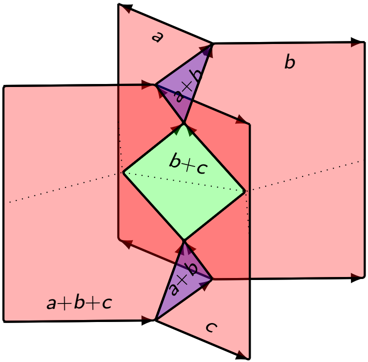
Donné $\tau : \{\text{cobordismes fermés } \emptyset \to \emptyset\} \longrightarrow R$

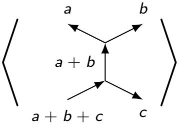
$$\Gamma \longmapsto \mathcal{F}(\Gamma) := \bigoplus_{{}_{\emptyset}F_{\Gamma}} R_F \Big/ \begin{matrix} \sum_i \lambda_i F_i = 0 \text{ si} \\ \sum_i \lambda_i \tau(F_i G) = 0 \text{ pour tout } {}_{\Gamma}G_{\emptyset} \end{matrix}$$

$$[{}_{\Gamma_1}G_{\Gamma_2}] \longmapsto \mathcal{F}(G) \colon \begin{pmatrix} \mathcal{F}(\Gamma_1) & \to & \mathcal{F}(\Gamma_2) \\ [{}_{\emptyset}F_{\Gamma_1}] & \mapsto & [{}_{\emptyset}FG_{\Gamma_2}] \end{pmatrix}$$

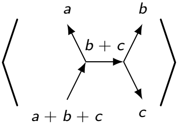

a
b
a+b
a+b+c
c

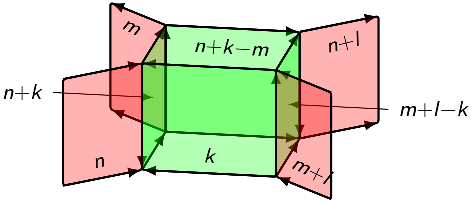
=


a
b
a+b
b+c
a+b
a+b+c
c


a
b
a + b
a + b + c
c

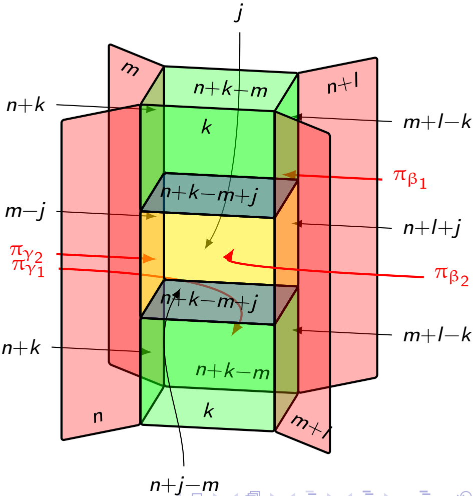
=

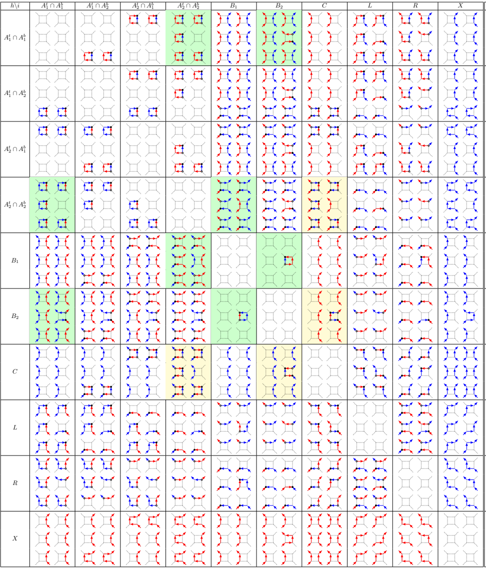

a
b
b + c
a + b + c
c



$$= \sum_{j=\max(0,m-n)}^{m} \sum_{\alpha \in T(k-j,l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1,\beta_2\\ \gamma_1,\gamma_2}} c^{\alpha}_{\beta_1\beta_2} c^{\widehat{\alpha}}_{\gamma_1\gamma_2}$$




$h\backslash i$
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$A_1^t \cap A_1^b$
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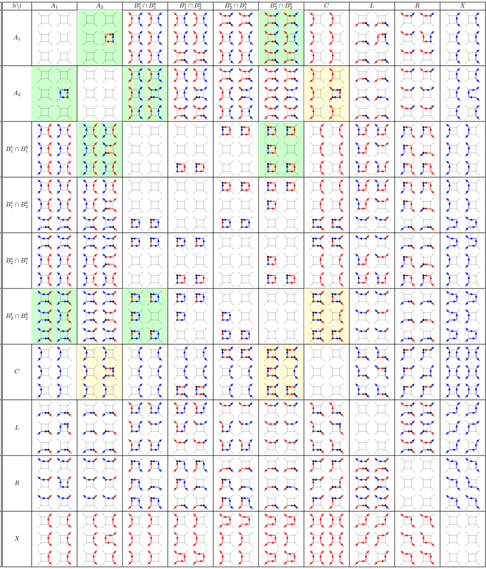
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Proposition

Le module associé par une TQFT à un graphe MOY symétrique a une structure d'algèbre de Frobenius.

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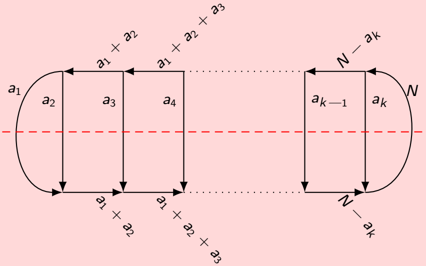
Le module associé par une TQFT à un graphe MOY symétrique a une structure d'algèbre de Frobenius.

Proposition (R.–Wagner, '17)

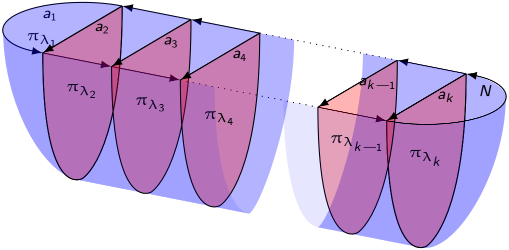
L'algèbre de Frobenius associée à



est isomorphe à l'anneau de cohomologie T-équivariante de

$$\mathfrak{Flag}(\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \cdots \subset \mathbb{C}^{a_1+\cdots+a_{k-1}} \subset \mathbb{C}^{N}).$$

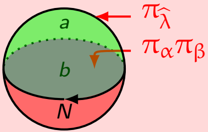
$$\prod_{i=1}^{k} \pi_{\lambda_i}(X_{a_i+1}, \ldots, X_{a_{i+1}}) \mapsto$$



Corollaire (R.–Wagner, '17)

Les coefficients de Littlewood–Richardson peuvent se calculer de la manière suivante :

$$c^{\lambda}_{\alpha\beta} = (-1)^{|\widehat{\lambda}|+N(N+1)/2} \Bigg\langle$$



$$\Bigg\rangle_N$$

$$= (-1)^{N(N+1)/2+|\widehat{\lambda}|} \sum_{\substack{A \sqcup B = \{X_1,\ldots,X_N\} \\ |A|=a, |B|=b}} (-1)^{|B<A|} \frac{a_\alpha(A)\, a_\beta(A)\, a_{\widehat{\lambda}}(B)}{\Delta(X_1,\ldots,X_N)}.$$

Merci !