Catégorification de 1 et du polynôme d'Alexander

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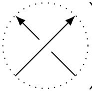
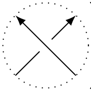
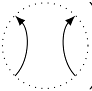
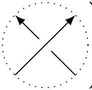
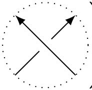
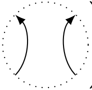
Séminaire de Topologie et Géométrie — Genève

- L'invariant $\mathfrak{gl}_1$ des entrelacs P_1 satisfait:

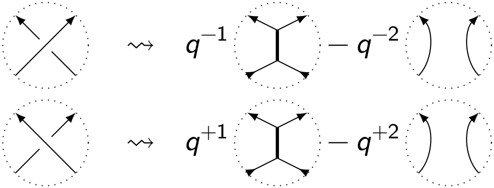
qP_1 () $- q^{-1}P_1$ () $= (q - q^{-1})P_1$ ()

- Le polynôme d'Alexander Δ satisfait:

Δ () $- \Delta$ () $= (q - q^{-1})\Delta$ ()

L'invariant $\mathfrak{gl}_1$

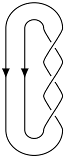
Diagramme de clôture de tresse $\rightsquigarrow$ $\mathbb{Z}[q, q^{-1}]$-comb. lin. de graphes vinyles



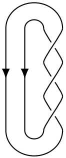
graphes vinyles $\rightsquigarrow$ élément de $\mathbb{N}[q, q^{-1}]$

$$\Gamma \rightsquigarrow \langle \Gamma \rangle_1 = \left(q + q^{-1}\right)^{\#V(\Gamma)/2} = [2]^{\#V(\Gamma)/2}.$$

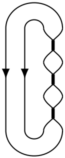
L'invariant $\mathfrak{gl}_1$ – Un exemple



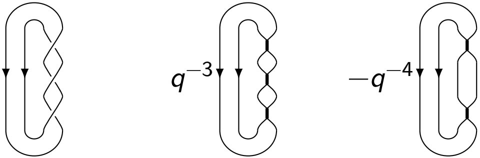
L'invariant $\mathfrak{gl}_1$ – Un exemple



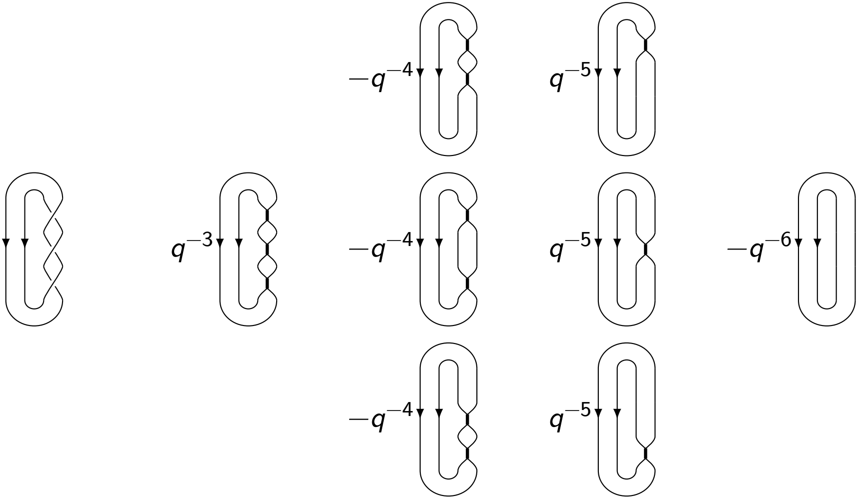
q^{-3}



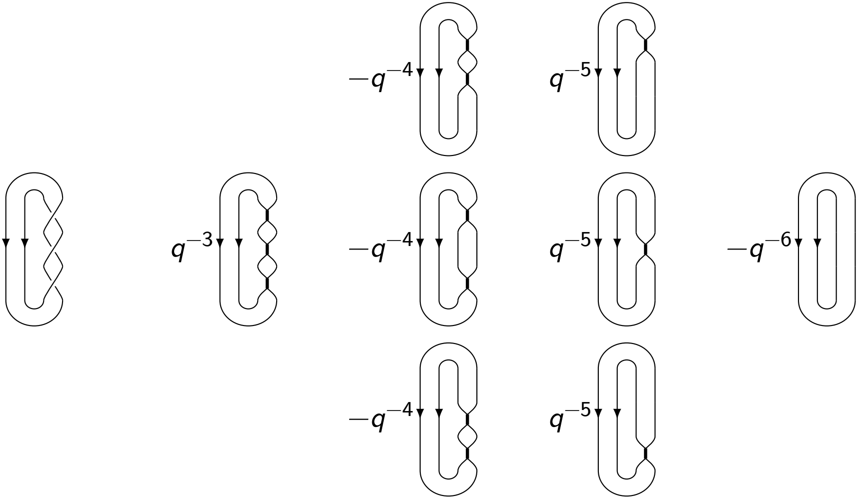
L'invariant $\mathfrak{gl}_1$ – Un exemple



L'invariant $\mathfrak{gl}_1$ – Un exemple



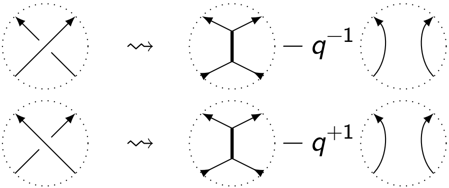
L'invariant $\mathfrak{gl}_1$ – Un exemple



$$1 = q^{-3}[2]^3 \quad -3q^{-4}[2]^2 \quad +3q^{-5}[2] \quad -q^{-6}$$

Le polynôme d'Alexander.

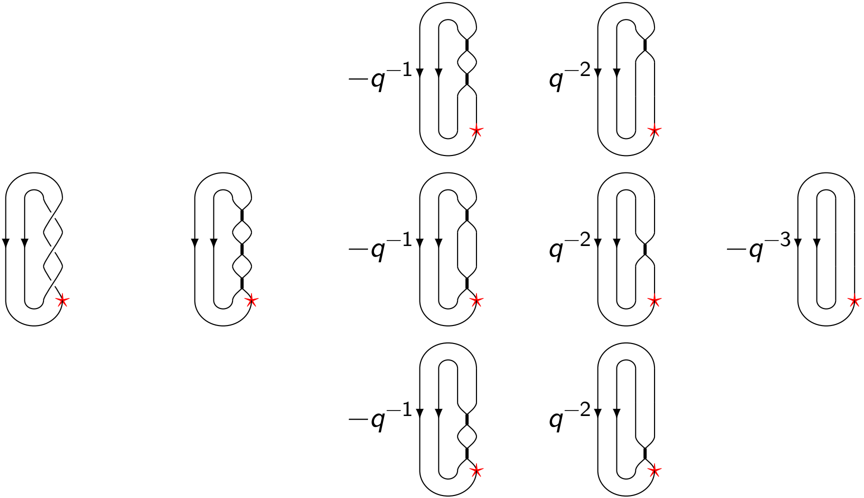
Diagramme de clôture de tresse $\rightsquigarrow$ $\mathbb{Z}[q, q^{-1}]$-comb. lin de graphes vinyles



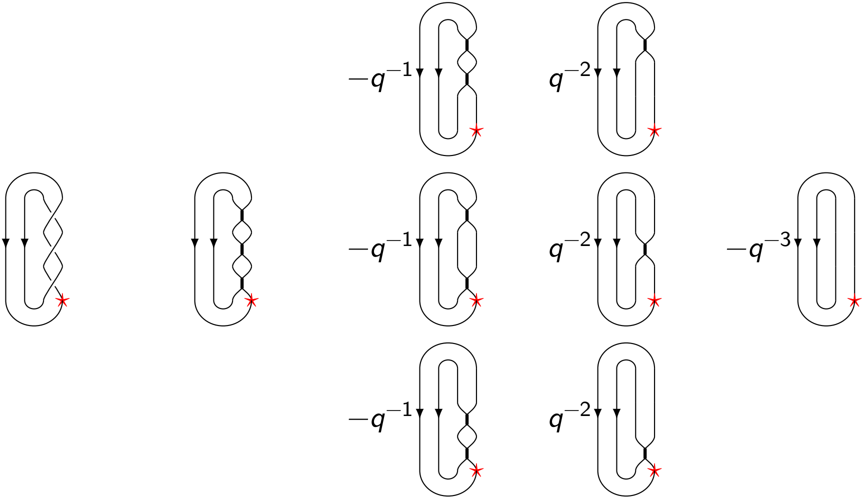
Graphe vinyle $\rightsquigarrow$ élément de $\mathbb{N}[q, q^{-1}]$

Γ $\rightsquigarrow$ $\langle \Gamma \rangle_0$

Le polynôme d'Alexander – Un exemple



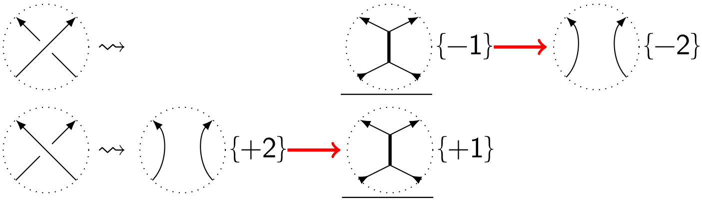
Le polynôme d'Alexander – Un exemple



$$q^2 - 1 + q^{-2} \;=\; [2]^2 \quad -3q^{-1}[2] \quad +3q^{-2} \quad 0$$

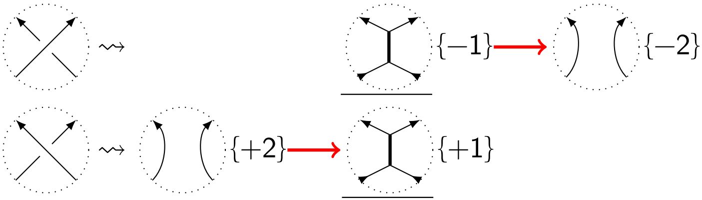
L'homologie $\mathfrak{gl}_1$

Diagramme de clôture de tresses $\rightsquigarrow$ hypercube de graphes vinyles



L'homologie $\mathfrak{gl}_1$

Diagramme de clôture de tresses $\rightsquigarrow$ hypercube de graphes vinyles

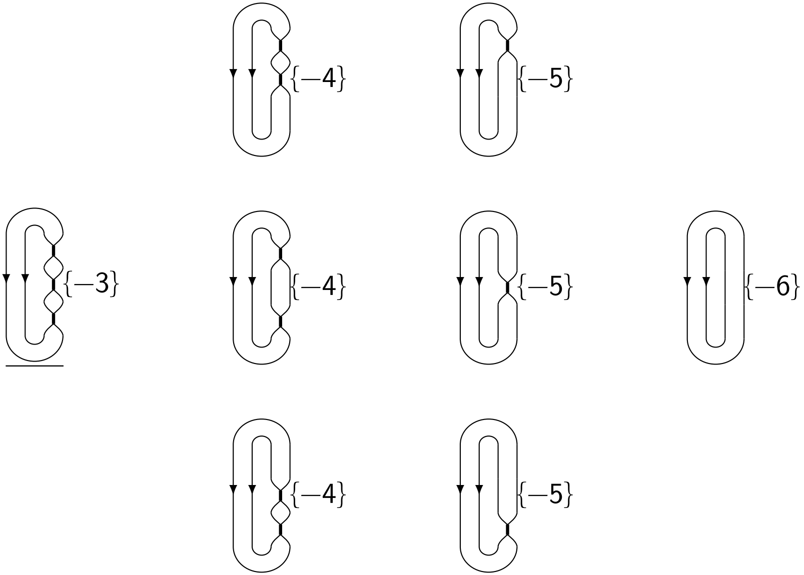


Graphes vinyles $\rightsquigarrow$ espace vectoriel gradué

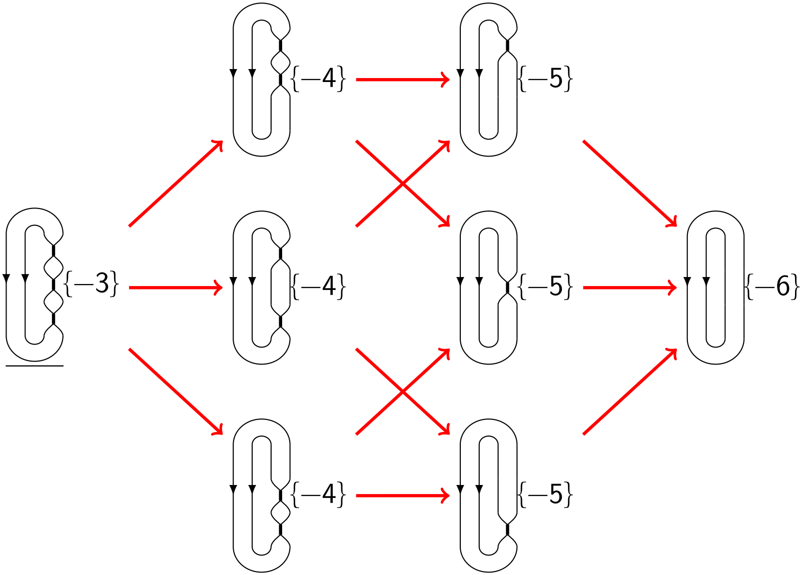
de dimension $[2]^{\#V(\Gamma)/2}$

$\longrightarrow$ $\rightsquigarrow$ applications linéaires graduées

L'homologie $\mathfrak{gl}_1$ – Un exemple

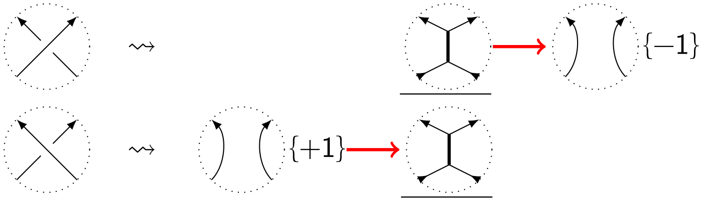


L'homologie $\mathfrak{gl}_1$ – Un exemple



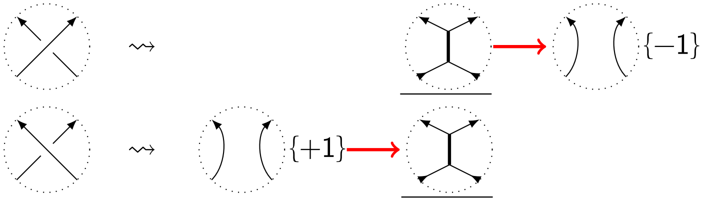
L'homologie $\mathfrak{gl}_0$ – Un exemple

Clôture de tresse pointée ($\star$) $\rightsquigarrow$ hypercube de graphes vinyles pointés



L'homologie $\mathfrak{gl}_0$ – Un exemple

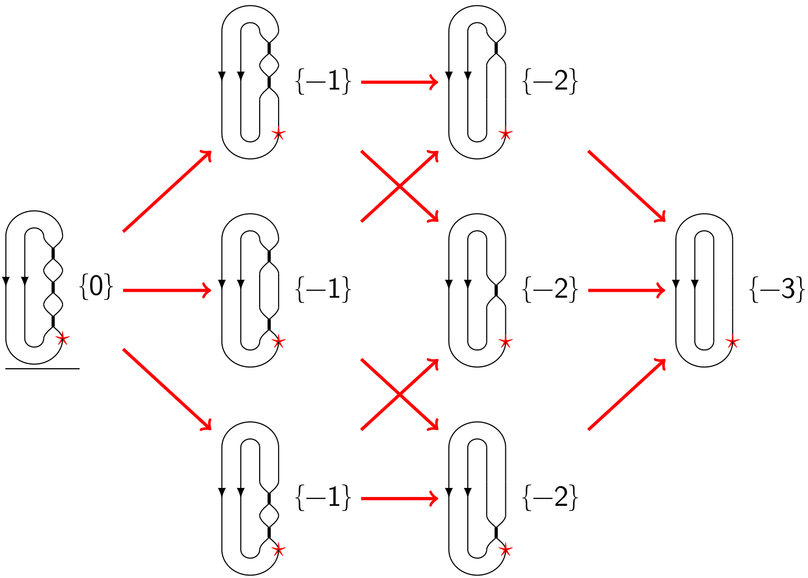
Clôture de tresse pointée ($\star$) $\rightsquigarrow$ hypercube de graphes vinyles pointés



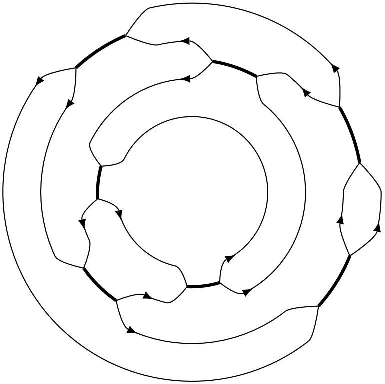
$\mathcal{F}'_0$: graphe vinyle pointé $\rightsquigarrow$ espace vectoriel gradué de dimension $\langle\Gamma\rangle_0$

$\longrightarrow$ $\rightsquigarrow$ application linéaire

L'homologie $\mathfrak{gl}_0$ – Un exemple

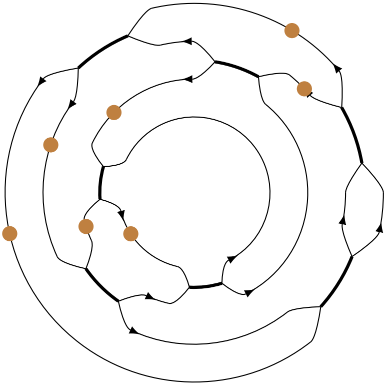


Graphes vinyles $\rightsquigarrow$ espaces vectoriels



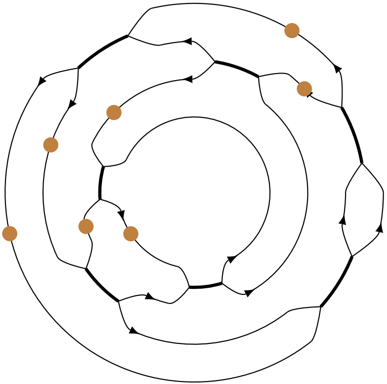
Graphe vinyle Γ $\circlearrowleft$ d'indice k.

Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle Γ ↺ d'indice k.
Configuration de points d ●.
$\deg(d) = 2\#● - \#V(\Gamma)/2$

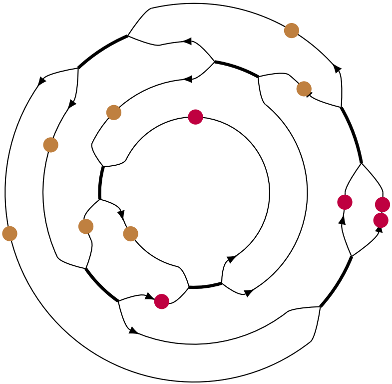
Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle Γ ↺ d'indice k.
Configuration de points d ●.
$\deg(d) = 2\#● - \#V(\Gamma)/2$

$$D(\Gamma) = \bigoplus_{d} \mathbb{Q}\{\deg(d)\}.$$

Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle Γ ↻ d'indice k.

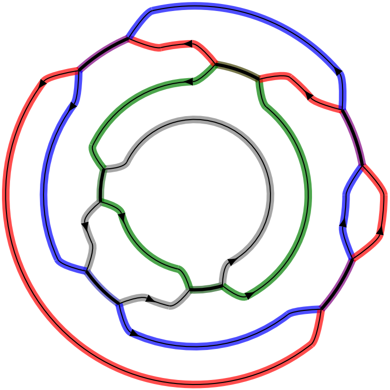
Configuration de points d ●.

$\deg(d) = 2\#● - \#V(\Gamma)/2$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication μ sur $D(\Gamma)$.

Graphes vinyles $\rightsquigarrow$ espaces vectoriels

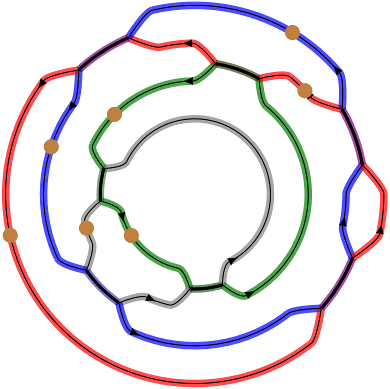


Graphe vinyle Γ $\circlearrowleft$ d'indice k.
Configuration de points d ●.
$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication μ sur $D(\Gamma)$.
Coloriage $c = (C_1, C_2, \ldots, C_k)$
$\Gamma = \bigsqcup_i C_i$.

Graphes vinyles $\rightsquigarrow$ espaces vectoriels



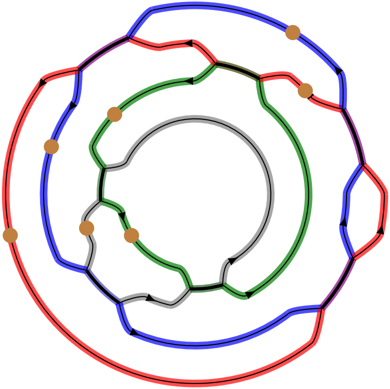
Graphe vinyle Γ $\circlearrowleft$ d'indice k.
Configuration de points d $\bullet$.
$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$

$$D(\Gamma) = \bigoplus_{d} \mathbb{Q}\{\deg(d)\}.$$

Multiplication μ sur $D(\Gamma)$.
Coloriage $c = (C_1, C_2, \ldots, C_k)$
$\Gamma = \bigsqcup_i C_i$.

$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod (X_i - X_j)}$$

Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle Γ ↻ d'indice k.
Configuration de points d $\bullet$.
$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$

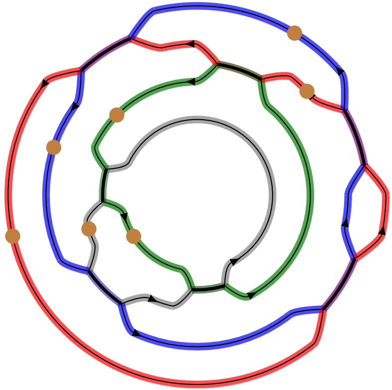
$$D(\Gamma) = \bigoplus_{d} \mathbb{Q}\{\deg(d)\}.$$

Multiplication μ sur $D(\Gamma)$.
Coloriage $c = (C_1, C_2, \ldots, C_k)$
$\Gamma = \bigsqcup_i C_i$.

$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod (X_i - X_j)} = \frac{-X_1{}^2 X_2{}^2 X_3 X_4{}^2}{(X_1 - X_2)^3 (X_3 - X_4)^2 (X_1 - X_4)(X_2 - X_3)}$$

C_i C_j

Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle Γ ↻ d'indice k.
Configuration de points d ●.
$\deg(d) = 2\#● - \#V(\Gamma)/2$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication μ sur $D(\Gamma)$.
Coloriage $c = ({\color{red}C_1}, {\color{blue}C_2}, \ldots, {\color{green}C_k})$
$\Gamma = \bigsqcup_i C_i$.

$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{● \text{ in } C_i\}}}{\prod_{{\color{red}C_i}\ {\color{blue}C_j}} ({\color{red}X_i} - {\color{blue}X_j})}$$

$$\tau_\infty(d) = \sum_{c \in \mathrm{col}(\Gamma)} \tau(d,c)$$