

# Catégorification de 1 et du polynôme d'Alexander

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- ▶ L'invariant  $gl_1$  des entrelacs  $P_1$  satisfait:

$$qP_1 \left( \text{diagram 1} \right) - q^{-1}P_1 \left( \text{diagram 2} \right) = (q - q^{-1})P_1 \left( \text{diagram 3} \right)$$

The equation shows a relationship between three diagrams enclosed in dotted circles. Diagram 1 (left) is a crossing with two arrows pointing away from the center. Diagram 2 (middle) is a crossing with two arrows pointing towards the center. Diagram 3 (right) consists of two separate curved arrows, one on the left and one on the right, both pointing upwards.

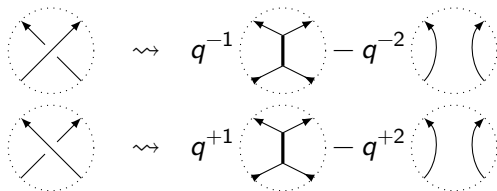
- ▶ Le polynôme d'Alexander  $\Delta$  satisfait:

$$\Delta \left( \text{diagram 1} \right) - \Delta \left( \text{diagram 2} \right) = (q - q^{-1})\Delta \left( \text{diagram 3} \right)$$

The equation shows a relationship between the same three diagrams as above, but with the Alexander polynomial  $\Delta$  instead of  $P_1$ .

# L'invariant $gl_1$

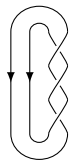
Diagramme de clôture de tresse  $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$ -comb. lin.  
de graphes vinyles



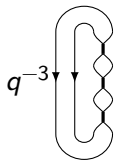
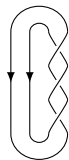
graphes vinyles  $\rightsquigarrow$  élément de  $\mathbb{N}[q, q^{-1}]$

$$\Gamma \rightsquigarrow \langle \Gamma \rangle_1 = (q + q^{-1})^{\#V(\Gamma)/2} = [2]^{\#V(\Gamma)/2}.$$

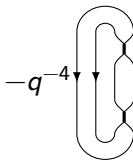
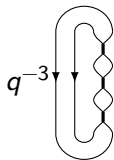
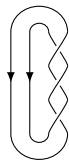
# L'invariant $gl_1$ – Un exemple



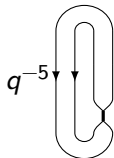
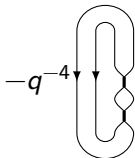
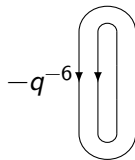
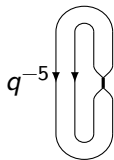
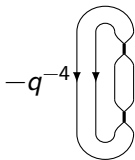
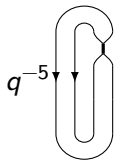
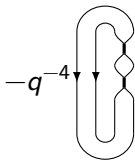
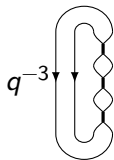
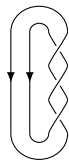
# L'invariant $gl_1$ – Un exemple



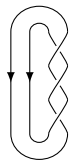
# L'invariant $gl_1$ – Un exemple



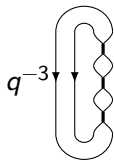
# L'invariant $gl_1$ – Un exemple



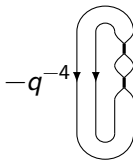
# L'invariant $gl_1$ – Un exemple



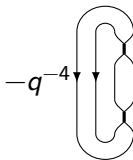
1 =



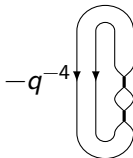
$q^{-3}[2]^3$



$-q^{-4}$

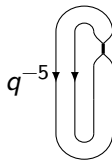


$-q^{-4}$

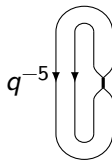


$-q^{-4}$

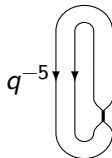
$-3q^{-4}[2]^2$



$q^{-5}$

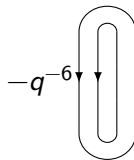


$q^{-5}$



$q^{-5}$

$+3q^{-5}[2]$



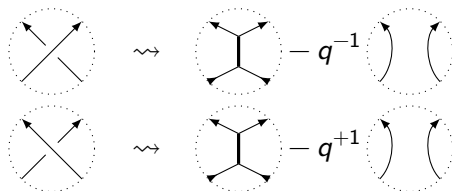
$-q^{-6}$

$-q^{-6}$



# Le polynôme d'Alexander.

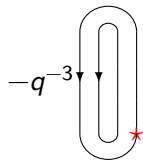
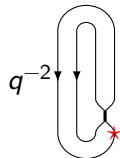
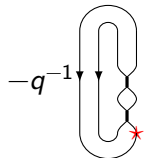
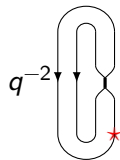
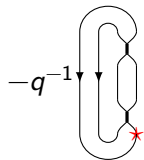
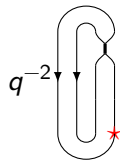
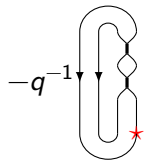
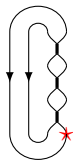
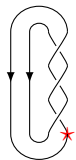
Diagramme de clôture de tresse  $\rightsquigarrow$   $\mathbb{Z}[q, q^{-1}]$ -comb. lin de graphes vinyles



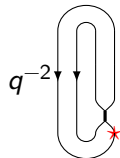
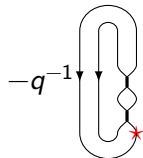
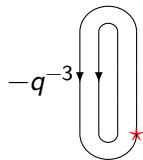
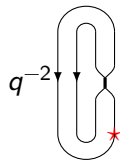
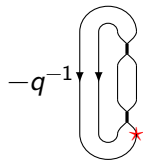
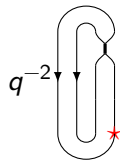
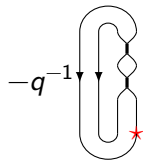
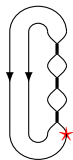
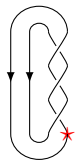
Graphe vinyle  $\rightsquigarrow$  élément de  $\mathbb{N}[q, q^{-1}]$

$\Gamma$   $\rightsquigarrow$   $\langle \Gamma \rangle_0$

# Le polynôme d'Alexander – Un exemple



# Le polynôme d'Alexander – Un exemple



$$q^2 - 1 + q^{-2} =$$

$$[2]^2$$

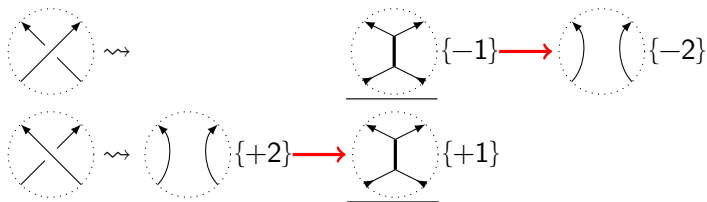
$$-3q^{-1}[2]$$

$$+3q^{-2}$$

$$0$$

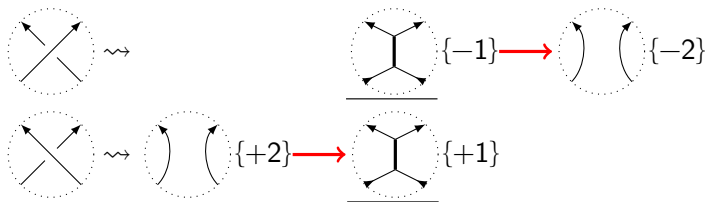
# L'homologie $\mathfrak{gl}_1$

Diagramme de clôture de tresses  $\rightsquigarrow$  hypercube de graphes vinyles



# L'homologie $\mathfrak{gl}_1$

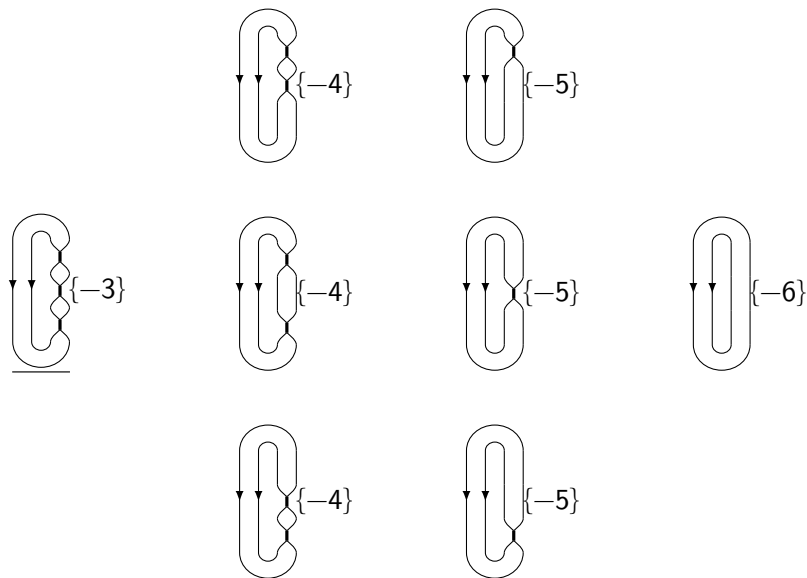
Diagramme de clôture de tresses  $\rightsquigarrow$  hypercube de graphes vinyles



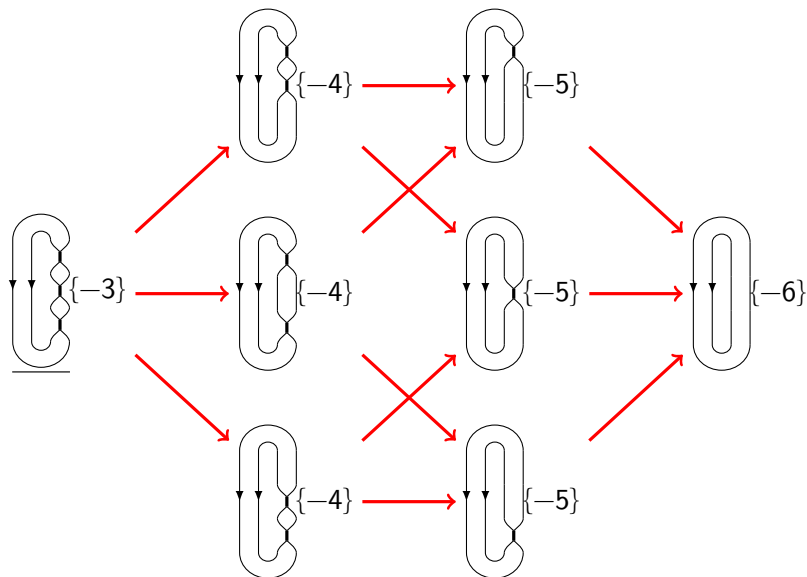
Graphes vinyles  $\rightsquigarrow$  espace vectoriel gradué  
de dimension  $[2]^{\#\mathcal{V}(\Gamma)/2}$

$\longrightarrow$   $\rightsquigarrow$  applications linéaires graduées

# L'homologie $gl_1$ – Un exemple

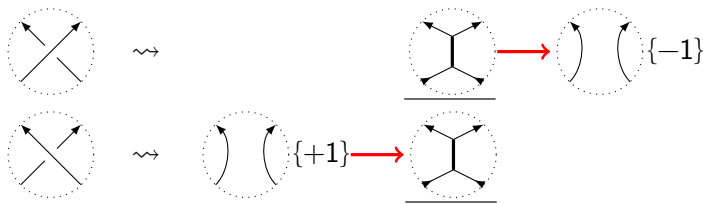


# L'homologie $gl_1$ – Un exemple



# L'homologie $\mathfrak{gl}_0$ – Un exemple

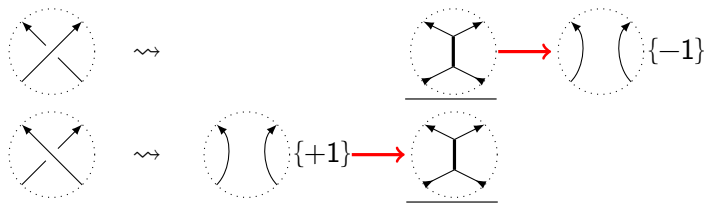
Clôture de tresse pointée ( $\star$ )  $\rightsquigarrow$  hypercube de graphes vinyles pointés





# L'homologie $\mathfrak{gl}_0$ – Un exemple

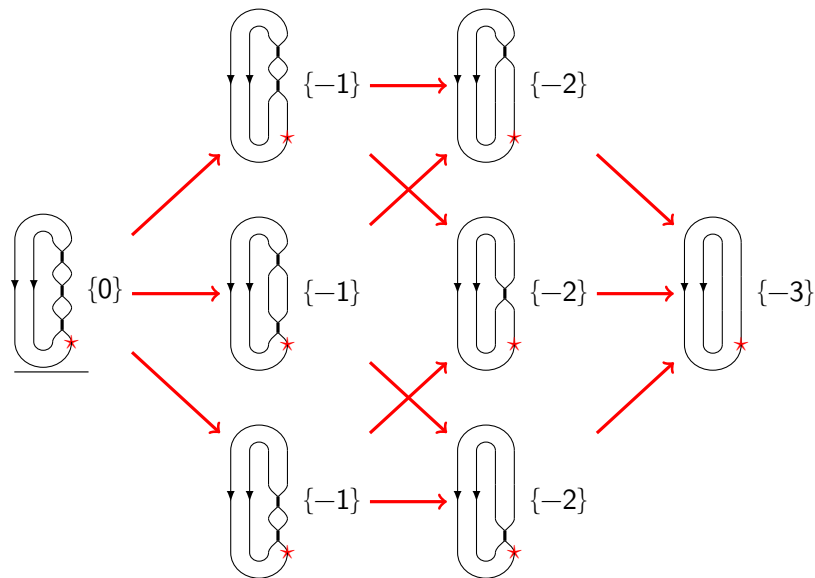
Clôture de tresse pointée ( $\star$ )  $\rightsquigarrow$  hypercube de graphes vinyles pointés



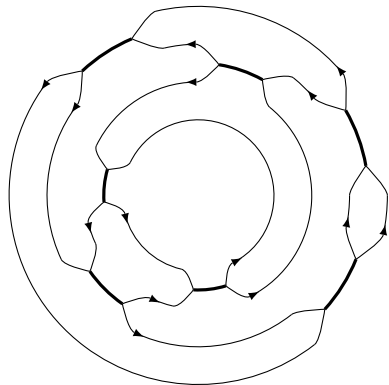
$\mathcal{F}'_0$ : graphe vinyle pointé  $\rightsquigarrow$  espace vectoriel gradué  
de dimension  $\langle \Gamma \rangle_0$

$\longrightarrow$   $\rightsquigarrow$  application linéaire

# L'homologie $g\mathcal{l}_0$ – Un exemple

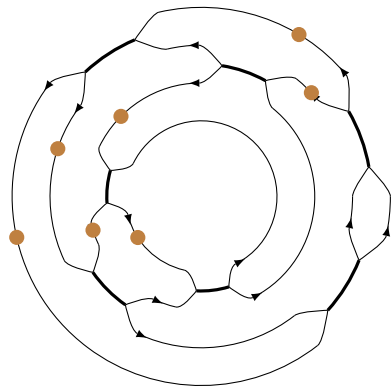


# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle  $\Gamma$   $\circlearrowright$  d'indice  $k$ .

## Graphes vinyles $\rightsquigarrow$ espaces vectoriels

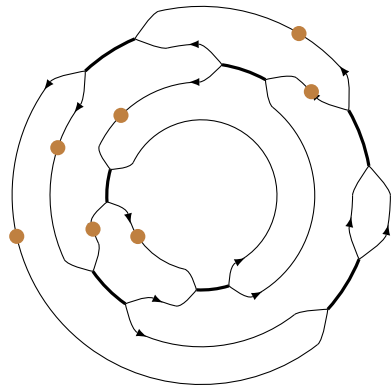


Grphe vinyle  $\Gamma \circlearrowleft$  d'indice  $k$ .

Configuration de points  $d$ .

$$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$$

## Graphes vinyles $\rightsquigarrow$ espaces vectoriels



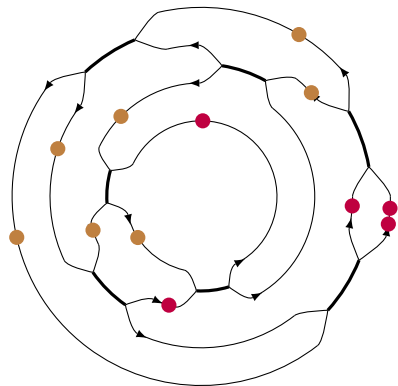
Grphe vinye  $\Gamma$   $\circlearrowright$  d'indice  $k$ .

Configuration de points  $d$   $\bullet$ .

$$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Grphe vinye  $\Gamma$  d'indice  $k$ .

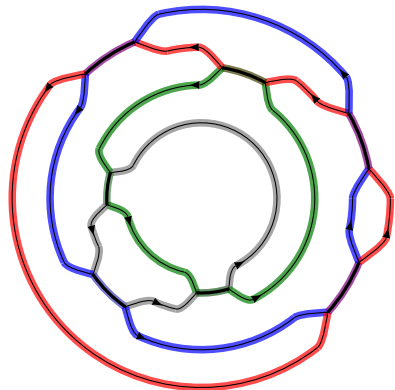
Configuration de points  $d$ .

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$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication  $\mu$  sur  $D(\Gamma)$ .

# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle  $\Gamma$   $\circlearrowright$  d'indice  $k$ .

Configuration de points  $d$   $\bullet$ .

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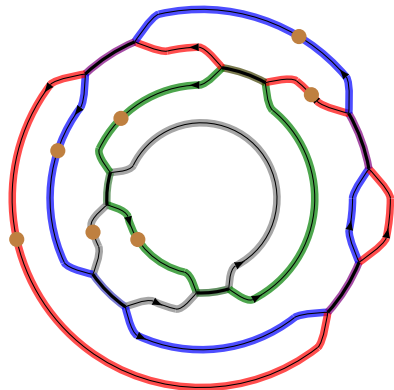
$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication  $\mu$  sur  $D(\Gamma)$ .

Coloriage  $c = (C_1, C_2, \dots, C_k)$

$$\Gamma = \bigsqcup_i C_i.$$

# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Graphe vinyle  $\Gamma$  d'indice  $k$ .

Configuration de points  $d$  ●.

$$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication  $\mu$  sur  $D(\Gamma)$ .

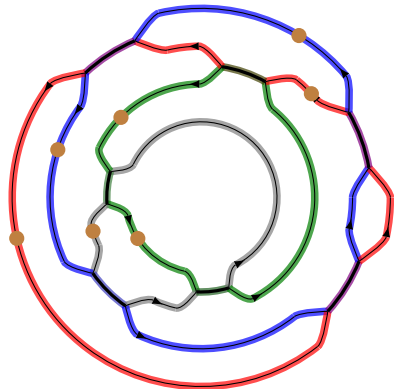
Coloriage  $c = (C_1, C_2, \dots, C_k)$

$$\Gamma = \bigsqcup_i C_i.$$

$$\tau(d, c) = \frac{\prod_{i=1}^k X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{\substack{C_i, C_j \\ \text{Y}}} (X_i - X_j)}$$



# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Grphe vinye  $\Gamma$  d'indice  $k$ .

Configuration de points  $d$  ●.

$$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

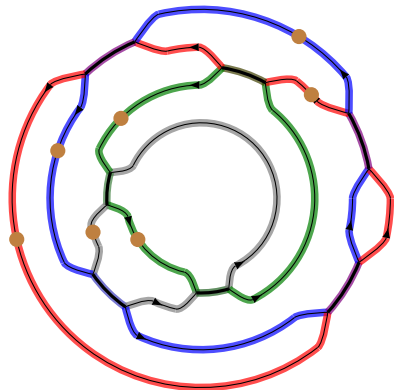
Multiplication  $\mu$  sur  $D(\Gamma)$ .

Coloriage  $c = (C_1, C_2, \dots, C_k)$

$$\Gamma = \bigsqcup_i C_i.$$

$$\tau(d, c) = \frac{\prod_{i=1}^k X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{\substack{C_i, C_j \\ \text{Y}}} (X_i - X_j)} = \frac{-X_1^2 X_2^2 X_3 X_4^2}{(X_1 - X_2)^3 (X_3 - X_4)^2 (X_1 - X_4) (X_2 - X_3)}$$

# Graphes vinyles $\rightsquigarrow$ espaces vectoriels



Grphe vinye  $\Gamma \circlearrowleft$  d'indice  $k$ .

Configuration de points  $d \bullet$ .

$$\deg(d) = 2\#\bullet - \#V(\Gamma)/2$$

$$D(\Gamma) = \bigoplus_d \mathbb{Q}\{\deg(d)\}.$$

Multiplication  $\mu$  sur  $D(\Gamma)$ .

Coloriage  $c = (C_1, C_2, \dots, C_k)$

$$\Gamma = \bigsqcup_i C_i.$$

$$\tau(d, c) = \frac{\prod_{i=1}^k X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{\substack{C_i, C_j \\ \text{Y}}} (X_i - X_j)}$$

$$\tau_\infty(d) = \sum_{c \in \text{col}(\Gamma)} \tau(d, c)$$