Categorification of 1 (and of the Alexander polynomial)

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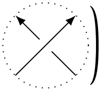
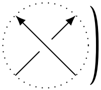
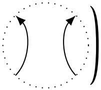
UNIVERSITÉ DE GENÈVE
université PARIS DIDEROT PARIS 7

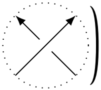
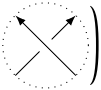
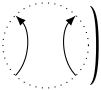
6th SwissMAP General Meeting

HOMFLY-PT polynomial

$aP\Big($  $\Big) - a^{-1}P\Big($  $\Big) = (q - q^{-1})P\Big($  $\Big),$

$$P(\text{unknot}) = 1 \qquad \text{or} \qquad P(\text{unknot}) = \frac{a - a^{-1}}{q - q^{-1}}.$$

HOMFLY-PT polynomial

$aP\Big($  $\Big) - a^{-1}P\Big($  $\Big) = (q-q^{-1})P\Big($  $\Big),$

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$\mathfrak{sl}_N$-invariants $a = q^N$

- General N: Reshetikhin–Turaev $\mathfrak{sl}_N$-invariants ($U_q(\mathfrak{sl}_N)$),
- $N = 2$: Jones polynomial,
- $N = 1$: Trivial invariant P_1 (1 on every link),
- $N = 0$: Alexander polynomial.

Categorification

Philosophy

Promote numerical invariants to (graded) vector space valued invariants.

$\chi \quad \leftrightarrow$ de Rham cohomology de Rahm, '31

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χ	$\leftrightarrow$	de Rham cohomology de Rahm, '31
Jones	$\leftrightarrow$	Khovanov homology Khovanov, '00
$\mathfrak{sl}_N$-invariant $(N \geq 2)$	$\leftrightarrow$	$\mathfrak{sl}_N$-homology Khovanov–Rozansky '08
HOMFLY-PT	$\leftrightarrow$	Triply graded homology Khovanov–Rozansky, '06
Alexander	$\leftrightarrow$	Knot Floer homology Ozsváth–Szabó, '04

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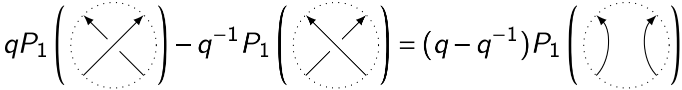
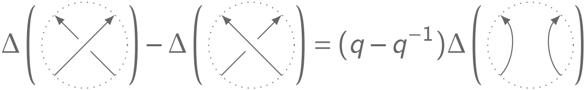
		$\Uparrow$	Rasmussen, '15
HOMFLY-PT	$\leftrightarrow$	Triply graded homology	Khovanov–Rozansky, '06
		$\Downarrow$?	Dunfield–Gukov–Rasmussen, '06
Alexander	$\leftrightarrow$	Knot Floer homology	Ozsváth–Szabó, '04

Today

- The $\mathfrak{gl}_1$ link invariant P_1 $\leftrightarrow$ $\mathfrak{gl}_1$-homology R.–Wagner, '18.

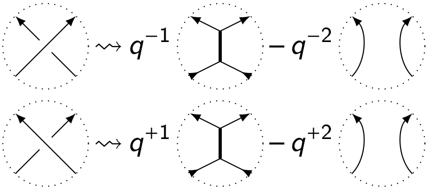


- The Alexander polynomial Δ $\leftrightarrow$ $\mathfrak{gl}_0$-homology R.–Wagner, '19.



$\mathfrak{gl}_1$ invariant

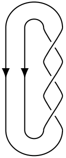
Link diagram $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$-lin. comb. of plane graphs



plane graph $\rightsquigarrow$ element of $\mathbb{N}[q, q^{-1}]$

$$\Gamma \rightsquigarrow (q + q^{-1})^{\#V(\Gamma)/2} = [2]^{\#V(\Gamma)/2}.$$

$\mathfrak{gl}_1$ invariant – Example



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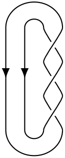
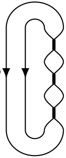


q^{-3} 

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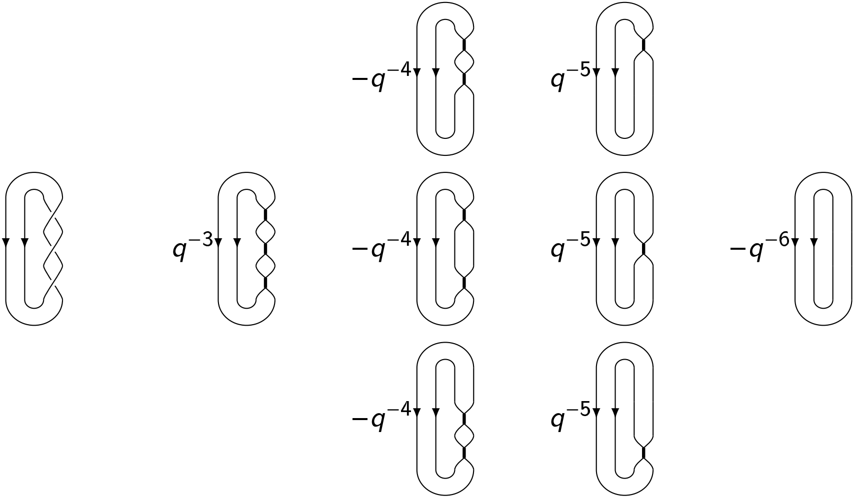

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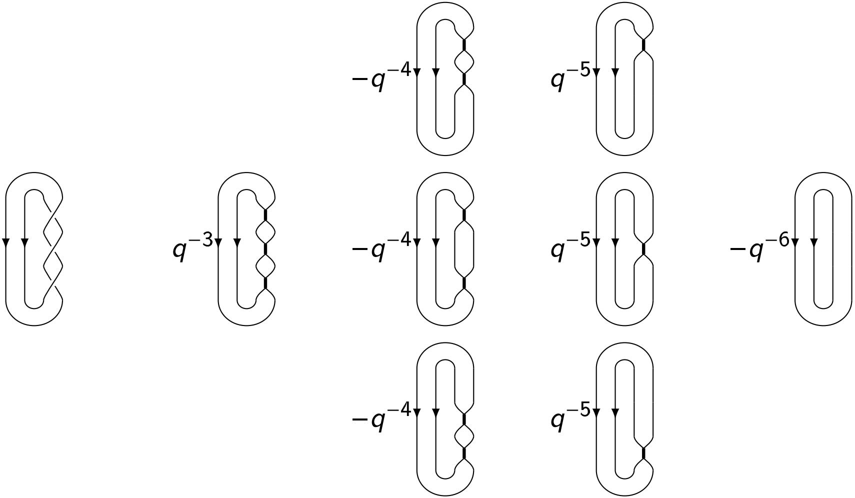
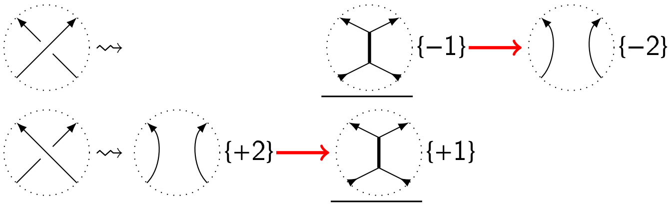


$$1 = q^{-3}[2]^3 - 3q^{-4}[2]^2 + 3q^{-5}[2] - q^{-6}$$

$\mathfrak{gl}_1$-homology

Braid closure diagram $\rightsquigarrow$ hypercube of plane graphs graphs (with shifts)



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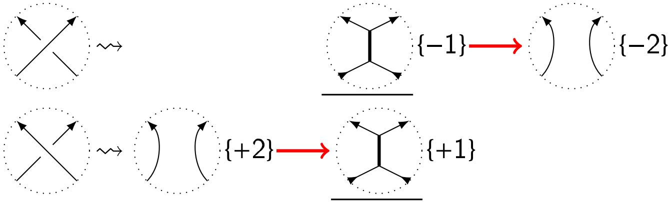
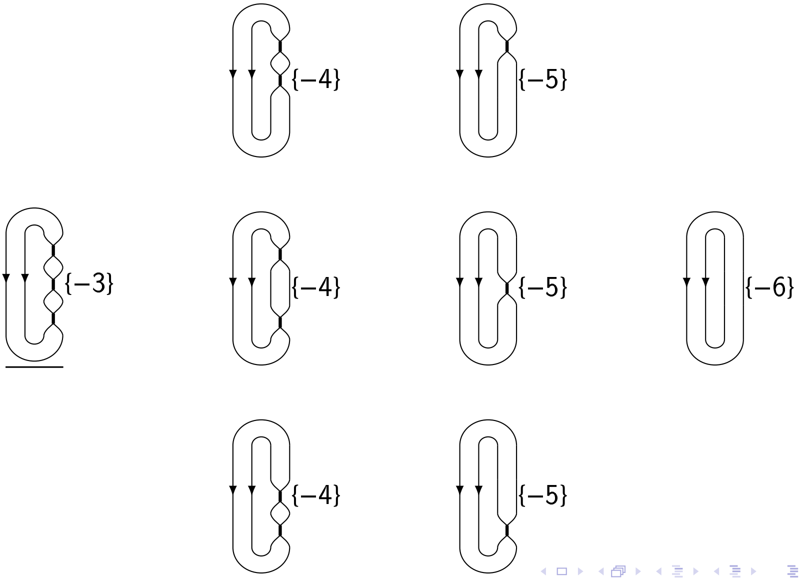


Planar (vinyl) graph $\rightsquigarrow$graded vector space

dimension $[2]^{\#V(\Gamma)/2}$

$\longrightarrow$ $\rightsquigarrow$graded linear map

$\mathfrak{gl}_1$-homology – Example



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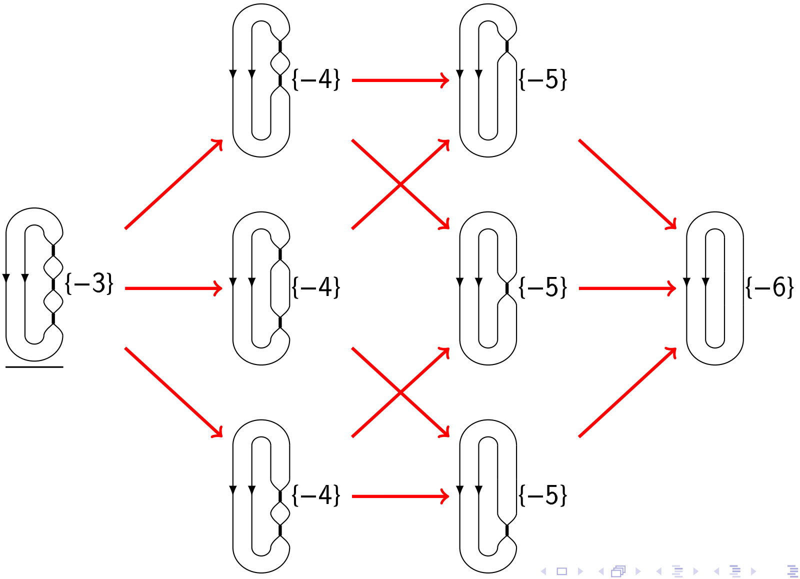
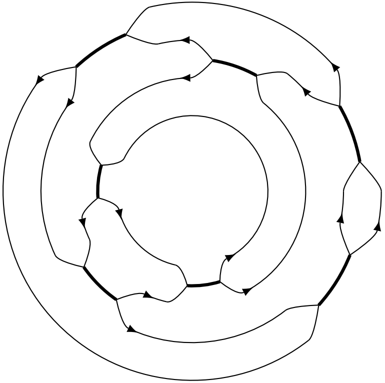
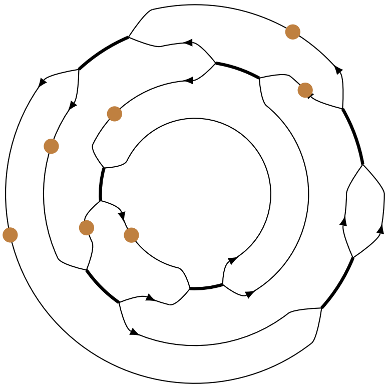


Vinyl graph $\rightsquigarrow$ vector space



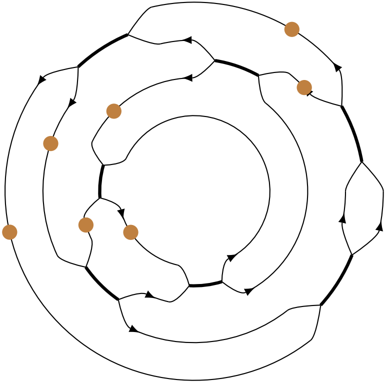
Vinyl graph $\Gamma \circlearrowleft$ index k.

Vinyl graph $\rightsquigarrow$ vector space



Vinyl graph $\Gamma \circlearrowleft$ index k.
Dot configuration d ●.

Vinyl graph ⇝ vector space



Vinyl graph Γ ↻ index k.
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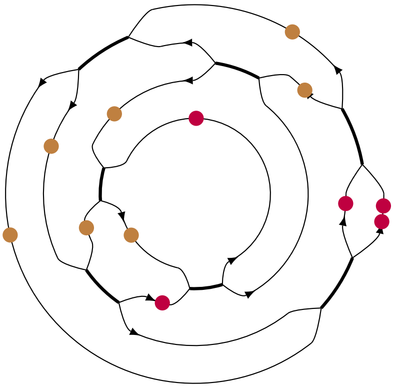
$$D(\Gamma) = \bigoplus_d \mathbb{Q}.$$

Vinyl graph $\rightsquigarrow$ vector space



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Multiplication μ on $D(\Gamma)$.

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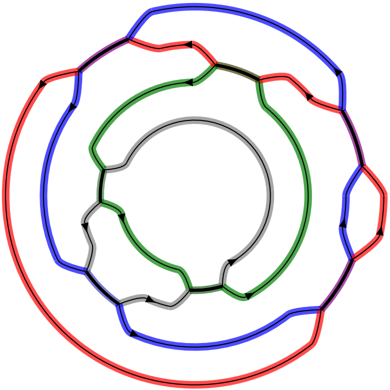

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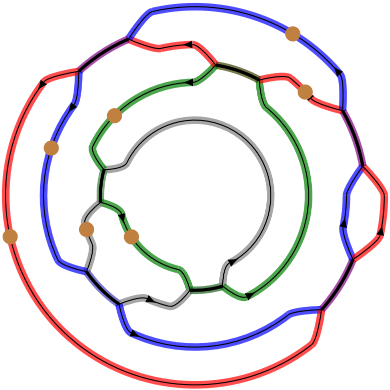
Coloring $c = (C_1, C_2, \ldots, C_k)$
$\Gamma = \bigsqcup_i C_i$.

Vinyl graph $\rightsquigarrow$ vector space



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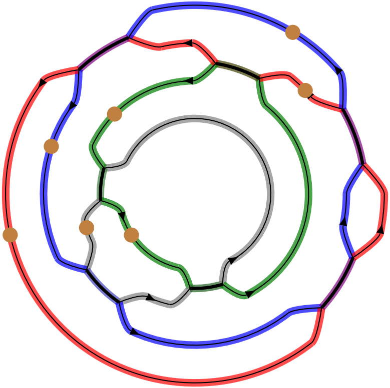
$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{C_i \; C_j} (X_i - X_j)}$$

Vinyl graph $\rightsquigarrow$ vector space



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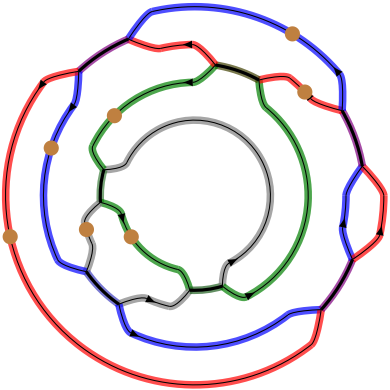
Vinyl graph $\rightsquigarrow$ vector space



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$$\tau(d) = \sum_{c \in \mathrm{col}(\Gamma)} \tau(d,c)$$

Vinyl graph $\rightsquigarrow$ vector space

Proposition (R.–Wagner, '17)

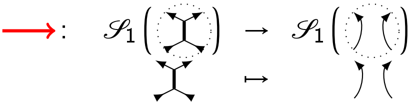
For any dot configuration d, $\tau(d) \in \mathbb{Q}[X_1,\dots,X_k]^{S_k}$.

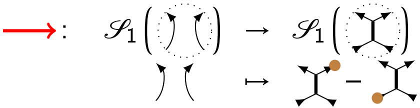
$$\mathscr{S}_1(\Gamma) = D(\Gamma)/\ker(\tau \circ \mu(\_,\_)_{X_\bullet \mapsto 0}).$$

Theorem (R.–Wagner, '18)

For any vinyl graph Γ, $\dim_q \mathscr{S}_1(\Gamma) = [2]^{\# V(\Gamma)/2}$.

linear maps





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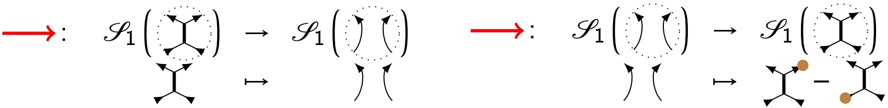

Theorem (R.–Wagner ’18)

1. *These maps in the flattening of the hypercube produces a chain complex. Its homology, denoted* $H_{\mathfrak{gl}_1}$ *is a link invariant which categorifies* P_1.
2. *There is a spectral sequence from the triply graded homology to* $H_{\mathfrak{gl}_1}$.

Examples

1. Trefoil: the Poincaré polynomial is $1 + q^{-4}(t + t^2)$.
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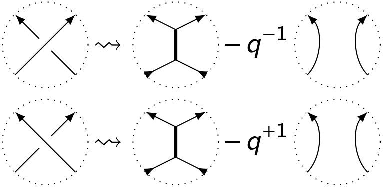
Do I have time?

Alexander polynomial

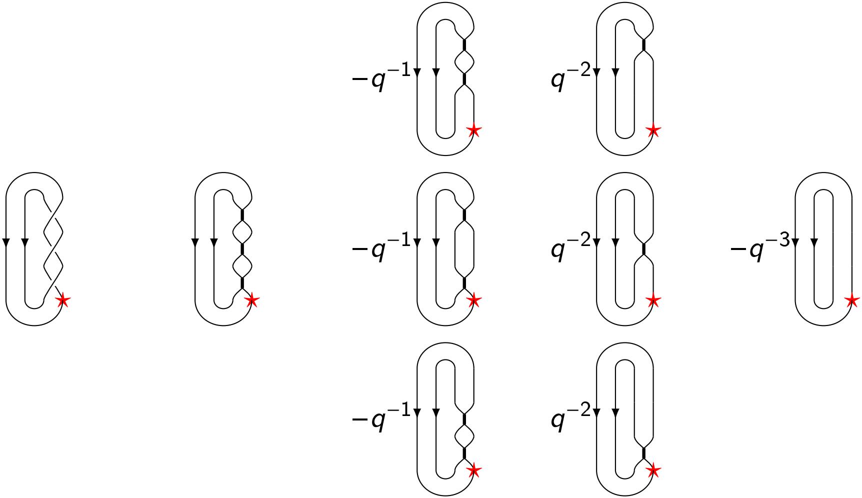
Marked ($\star$) braid closure $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$-lin. comb. of marked plane graphs



Marked plane graph $\rightsquigarrow$ element of $\mathbb{N}[q, q^{-1}]$

$\Gamma \rightsquigarrow$ complicated (comes from $U_q(\mathfrak{gl}(1|1)) - \text{mod}$).

Alexander polynomial – Example



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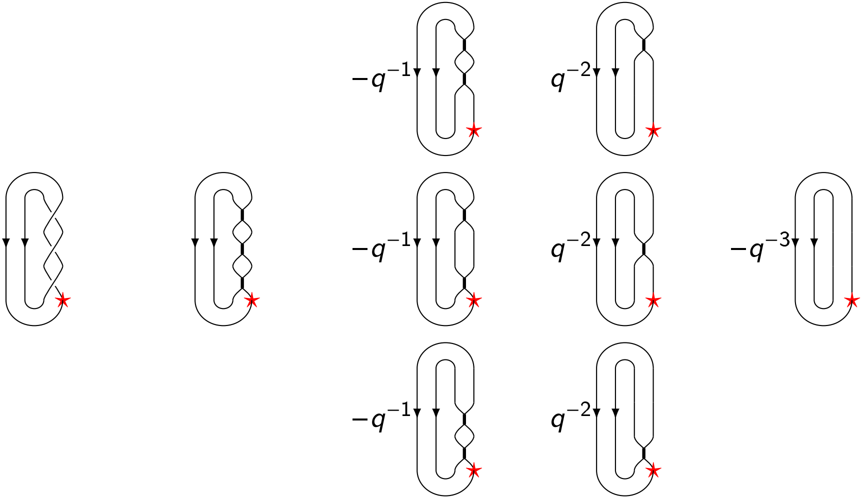

$$q^2 - 1 + q^{-2} \;=\; [2]^2 \;\; -3q^{-1}[2] \;\; +3q^{-2} \;\; 0$$

$\mathfrak{gl}_0$-homology

Same hypercube with a different functor.

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Same hypercube with a different functor.

$$\mathscr{S}_0'(\Gamma_\star) \subseteq \mathscr{S}_1(\Gamma) = \langle \text{at least } k-1 \bullet \text{ at } \star \rangle\{-k+1\}$$

$\longrightarrow$ $\rightsquigarrow$ induced by $\mathscr{S}_1$.

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$\longrightarrow$ $\rightsquigarrow$ induced by $\mathscr{S}_1$.

Theorem (R.–Wagner, '19)

For any right-marked vinyl graph $\Gamma_\star$, $\dim_q \mathscr{S}_0'(\Gamma_\star)$ *is the expected graded dimension.*

Theorem (R.–Wagner '19)

1. *The flattening of the hypercube with $\mathscr{S}_0'$ produces a chain complex. Its homology, denoted $H_{\mathfrak{gl}_0}$ is a knot invariant which categorifies the Alexander polynomial.*
2. *There is a spectral sequence from the reduced triply graded homology to $H_{\mathfrak{gl}_0}$.*

Thank you !!